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Research Article

A Local Equivariant Index Theorem for Sub-Signature Operators

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ABSTRACT

In this paper, we prove a local equivariant index theorem for sub-signature operators which generalizes Weiping Zhang's index theorem for sub-signature operators.

1. INTRODUCTION

The Atiyah-Singer index Theorem ([2,3]) gives a cohomological interpretation of the Fredholm index of an elliptic operator. The Atiyah-Bott-Segal-Singer index formula, which called the equivariant index theorem, is a generalization with group action of the Atiyah-Singer index theorem. The first direct proof of this result was given by Patodi, Gilkey, Atiyah-Bott-Patodi partly by using invariant theory [1,12]. This theorem generalizes the Atiyah-Singer index theorem and the Atiyah-Bott fixed point formula for elliptic complexes, which is a generalization of the Lefschetz fixed point formula. In [7], Berline and Vergne gave a heat kernel proof of the Atiyah-Bott-Segal-Singer index formula. Moreover, Lafferty, Yu and Zhang [14] presented a simple and direct geometric proof of the equivariant index theorem for an orientation-preserving isometry on an even dimensional spin manifold by using Clifford asymptotics of heat kernel. Furthermore, Ponge and H. Wang gave a different proof of the equivariant index formula by the Greiner's approach to the heat kernel asymptotics [19]. In [15], in order to prove family rigidity theorems, Liu and Ma proved the equivariant family index formula. In [22], Y. Wang gave another proof of the local equivarint index theorem for a family of Dirac operators by the Greiner's approach to the heat kernel asymptotics. In [21], using the Greiner's approach to the heat kernel asymptotics, Y. Wang proved the equivariant Gauss-Bonnet-Chern formula and gave the variation formulas for the equivariant Ray-Singer metric, which are originally due to J. M. Bismut and W. Zhang [9].

In parallel, Freed [11] considered the case of an orientation reversing involution acting on an odd dimensional spin manifold and gave the associated Lefschetz formulas by the K-theretical way. In [20], Wang constructed an even spectral triple by the Dirac operator and the orientation-reversing involution and computed the Connes-Chern character for this spectral triple. In [16], Liu and Wang proved an equivariant odd index theorem for Dirac operators with involution parity and the Atiyah-Hirzebruch vanishing theorems for odd dimensional spin manifolds. In [24] and [25], Zhang introduced the sub-signature operators and proved a local index formula for these operators. By computing the adiabatic limit of eta-invariants associated to the so-called sub-signature operators, a new proof of the Riemann-Roch-Grothendieck type formula of Bismut-Lott was given in [17] and [10]. The motivation of the present article is to prove a local equivariant index formula for sub-signature operators. As the subsignature operator is locally a twisted Dirac operator, we can obtain our theorem by the proof of equivariant twisted Dirac operators. We give a direct proof of a local equivariant index theorem for subsignature operators by the Volterra calculus, rather than derived from the local equivariant index theorem of twisted Dirac operators. Thus our direct proof of the equivariant index theorem of the subsignature operators using Volterra calculus can be seen as analogous to the works [21,23,26].

This paper is organized as follows: In Section 2, we recall some background on sub-signature operators. In Section 3.1, we prove a local equivariant index formula for sub-signature operators in even dimension. In Section 3.2, we prove a local equivariant odd dimensional index formula for sub-signature operators with an orientation-reversing involution.

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2. THE SUB-SIGNATURE OPERATORS

In this section, we give the standard setup (also see Section 1 in [24]). Let M be an oriented closed manifold of dimension n. Let E be an oriented sub-bundle of the tangent vector bundle TM. Let g^{TM} be a metric on TM. Let g^E be the induced metric on E. Let $E^\perp$ be the sub-bundle of TM orthogonal to E with respect to g^{TM}. Let $g^{E^\perp}$ be the metric on $E^\perp$ induced from g^{TM}. Then (TM, g^{TM}) has the following orthogonal splittings

$$TM = E \oplus E^\perp, \tag{2.1}$$

$$g^{TM} = g^E \oplus g^{E^\perp}. \tag{2.2}$$

Clearly, $E^\perp$ carries a canonically induced orientation. We identify the quotient bundle TM/E with $E^\perp$.

Let $\Omega(M) = \bigoplus_0^n \Omega^i(M) = \bigoplus_0^n \Gamma(\wedge^i(T^*M))$ be the set of smooth sections of $\wedge(T^*M)$. Let $*$ be the Hodge star operator of g^{TM}. Then $\Omega(M)$ inherits the following inner product

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \overline{*\beta}, \quad \alpha, \beta \in \Omega(M). \tag{2.3}$$

We use g^{TM} to identify TM and T^*M. For any $e \in \Gamma(TM)$, let $e\wedge$ and i_e be the standard notation for exterior and interior multiplications on $\Omega(M)$. Let $c(e) = e\wedge - i_e$, $\hat{c}(e) = e\wedge + i_e$ be the Clifford actions on $\Omega(M)$ verifying that

$$c(e)c(e') + c(e')c(e) = -2\langle e, e'\rangle_{g^{TM}}, \tag{2.4}$$

$$\hat{c}(e)\hat{c}(e') + \hat{c}(e')\hat{c}(e) = 2\langle e, e'\rangle_{g^{TM}}, \tag{2.5}$$

$$c(e)\hat{c}(e') + \hat{c}(e')c(e) = 0. \tag{2.6}$$

Denote $k = \dim E$ and we assume k is even. Let $\{f_1, \cdots, f_k\}$ be an oriented (local) orthonormal basis of E. Set

$$\hat{c}(E, g^E) = \hat{c}(f_1)\cdots\hat{c}(f_k), \tag{2.7}$$

where $\hat{c}(E, g^E)$ does not depend on the choice of the orthonormal basis. Let

$$\epsilon = \mathrm{Id}_{\wedge^{even}(T^*M)} - \mathrm{Id}_{\wedge^{odd}(T^*M)}$$

be the Z_2-grading operator of

$$\wedge(T^*M) = \wedge^{even}(T^*M) \oplus \wedge^{odd}(T^*M).$$

Set

$$\tau(M, g^E) = \left(\frac{1}{\sqrt{-1}}\right)^{\frac{k(k+1)}{2}} \epsilon\hat{c}(E, g^E). \tag{2.8}$$

It is easy to check

$$\tau(M, g^E)^2 = 1. \tag{2.9}$$

Let

$$\wedge_\pm(T^*M, g^E) = \left\{\omega \in \wedge^*(T^*M), \tau(M, g^E)\omega = \pm\omega\right\}$$

the (even/odd) eigen-bundles of $\tau(M, g^E)$ and by $\Omega_\pm(M, g^E)$ the corresponding set of smooth sections. Let $\delta = d^*$ be the formal adjoint operator of the exterior differential operator d on $\Omega(M)$ with respect to the inner product (2.3). Set on $\Omega(M) = \Gamma(\Lambda T^*M)$

$$D_E = \frac{1}{2}\left(\hat{c}(E, g^E)(\mathrm{d} + \delta) + (-1)^k(\mathrm{d} + \delta)\hat{c}(E, g^E)\right). \tag{2.10}$$

Then we can check

$$D_E\tau(M, g^E) = -\tau(M, g^E)D_E, \tag{2.11}$$

$$D_E^* = (-1)^{\frac{k(k+1)}{2}} D_E, \tag{2.12}$$

where D_E^* is the formal adjoint operator of D_E with respect to the inner product (2.3). Set

$$\tilde{D}_E = (\sqrt{-1})^{\frac{k(k+1)}{2}} D_E.$$

From (2.11), $\tilde{D}_E$ is a formal self-adjoint first order elliptic differential operator on $\Omega(M)$ interchanging $\Omega_\pm(M, g^E)$.

Definition 2.1. *The sub-signature operator $\tilde{D}_{E,+}$ with respect to (E, g^{TM}) is the restriction of $\tilde{D}_E$ on $\Omega_+(M, g^E)$.*

If we denote the restriction of $\tilde{D}_E$ on $\Omega_\pm(M, g^E)$ by $\tilde{D}_{E,\pm}$, then

$$\tilde{D}^*_{E,\pm} = \tilde{D}_{E,\mp}.$$

Recall that E is the subbundle of TM and that we have the orthogonal decomposition (2.1) of TM and the metric g^{TM}. Let P^E (resp. $P^{E^\perp}$) be the orthogonal projection from TM to E(resp. $E^\perp$). Let ∇^{TM} be the Levi-Civita connection of g^{TM}. We will use the same notation for its lift to $\Omega(M)$. Set

$$\nabla^E = P^E \nabla^{TM} P^E, \tag{2.13}$$

$$\nabla^{E^\perp} = P^{E^\perp} \nabla^{TM} P^{E^\perp}. \tag{2.14}$$

Then ∇^E(resp.$\nabla^{E^\perp}$) is a Euclidean connection on E(resp.$E^\perp$), and we will use the same notation for its lifting on $\Omega(E^*)$(resp. $\Omega(E^{\perp,*})$). Let S be the tensor defined by

$$\nabla^{TM} = \nabla^E + \nabla^{E^\perp} + S.$$

Then S takes values in skew-adjoint endomorphisms of TM, and interchanges E and $E^\perp$. Let $\{e_1, \cdots, e_n\}$ be an oriented (local) orthonormal base of TM. To specify the role of E, set $\{f_1, \cdots, f_k\}$ be an oriented (local) orthonormal basis of E. We will use the greek subscripts for the basis of E. Then by Proposition 1.4 in [24], we have

Proposition 2.2. *The following identity holds,*

$$\tilde{D}_E = (\sqrt{-1})^{\frac{k(k+1)}{2}} \Big(\hat{c}(E, g^E)(\mathrm{d} + \delta) + \frac{1}{2} \sum_i c(e_i)(\nabla^{TM}_{e_i} \hat{c}(E, g^E)) \Big). \tag{2.15}$$

Similar to Lemma 1.1 in [24], we have

Lemma 2.3. *For any $X \in \Gamma(TM)$, the following identity holds,*

$$\nabla^{TM}_X \hat{c}(E, g^E) = -\hat{c}(E, g^E) \sum_\alpha \hat{c}(S(X)f_\alpha)\hat{c}(f_\alpha). \tag{2.16}$$

Let Δ^{TM}, Δ^E be the Bochner Laplacians

$$\Delta^{TM} = \sum_i^n (\nabla^{TM,2}_{e_i} - \nabla^{TM}_{\nabla^{TM}_{e_i} e_i}), \tag{2.17}$$

$$\Delta^E = \sum_i^k (\nabla^{E,2}_{e_i} - \nabla^E_{\nabla^E_{e_i} e_i}). \tag{2.18}$$

Let K be the scalar curvature of (M, g^{TM}). Let R^{TM} (*resp.*, R^E, $R^{E^\perp}$) be the curvature of ∇^{TM} (*resp.*, ∇^E, $\nabla^{E^\perp}$). Let $\{h_1, \cdots, h_{n-k}\}$ be an oriented (local) orthonormal base of $E^\perp$. Now we can state the following Lichnerowicz type formula for $\tilde{D}^2_E$. From Theorem 1.1 in [24], we have

Theorem 2.4. *[24] The following identity holds,*

$$\begin{aligned}
\tilde{D}^2_E = & -\Delta^{TM} + \frac{K}{4} + \frac{1}{8} \sum_{1\le i,j\le n} \sum_{1\le\alpha,\beta\le k} \langle R^E(e_i, e_j)f_\beta, f_\alpha\rangle c(e_i)c(e_j)\hat{c}(f_\alpha)\hat{c}(f_\beta) \\
& + \frac{1}{8} \sum_{1\le i,j\le n} \sum_{1\le s,t\le n-k} \langle R^{E^\perp}(e_i, e_j)h_t, h_s\rangle c(e_i)c(e_j)\hat{c}(h_s)\hat{c}(h_t) + \frac{1}{2}\sum_\alpha \hat{c}\left((\Delta^{TM} - \Delta^E)f_\alpha\right)\hat{c}(f_\alpha) \\
& + \sum_{i,\alpha} \left(\hat{c}(S(e_i)f_\alpha)\hat{c}(f_\alpha)\nabla^{TM}_{e_i} - \hat{c}(S(e_i)\nabla^E_{e_i} f_\alpha)\hat{c}(f_\alpha) + \frac{1}{2}\hat{c}\left(\nabla^E_{(\nabla^{TM}_{e_i} - \nabla^E_{e_i})e_i} f_\alpha\right)\hat{c}(f_\alpha) + \frac{3}{4} \parallel S(e_i)f_\alpha) \parallel^2 \right) \\
& + \frac{1}{4} \sum_{i,\alpha\ne\beta} \hat{c}(S(e_i)f_\alpha)\hat{c}(S(e_i)f_\beta)\hat{c}(f_\alpha)\hat{c}(f_\beta).
\end{aligned} \tag{2.19}$$

3. A LOCAL EQUIVARIANT INDEX THEOREM FOR SUB-SIGNATURE OPERATORS

3.1. A Local Even Dimensional Equivariant Index Theorem for Sub-Signature Operators

Let M be a closed oriented Riemannian manifold of even dimension n and ϕ an orientation-preserving isometry on M. Then the smooth map ϕ induces a map $\tilde{\phi} = \phi^{-1,*} : \wedge T^*_x M \to \wedge T^*_{\phi(x)} M$ on the exterior algebra bundle $\wedge T^*_x M$. Let $\tilde{D}_E$ be the sub-signature operator. We assume that $d\phi$ preserves E and $E^{\perp}$ and their orientations, then $\tilde{\phi}\hat{c}(E,g^E)=\hat{c}(E,g^E)\tilde{\phi}$. Then $\tilde{\phi}\tilde{D}_E=\tilde{D}_E\tilde{\phi}$. We will compute the equivariant index

$$\mathrm{Ind}_{\phi}(\tilde{D}_E^{+})=\mathrm{Tr}(\tilde{\phi}|_{\ker\tilde{D}_E^{+}})-\mathrm{Tr}(\tilde{\phi}|_{\ker\tilde{D}_E^{-}}). \tag{3.1}$$

We recall the Greiner's approach to the heat kernel asymptotics as in [19] and [4,5,13]. Define the operator given by

$$(Q_0u)(x,s)=\int_0^{\infty}e^{-s\tilde{D}_E^2}[u(x,t-s)]dt,\ \ u\in\Gamma_c(M\times\mathbb{R},\wedge T^*M), \tag{3.2}$$

maps u continuously to $D'(M\times\mathbb{R},\wedge T^*M))$ which is the dual space of $\Gamma_c(M\times\mathbb{R},\wedge T^*M))$. We have

$$\left(\tilde{D}_E^2+\frac{\partial}{\partial t}\right)Q_0u=Q_0\left(\tilde{D}_E^2+\frac{\partial}{\partial t}\right)u=u,\ \ \ u\in\Gamma_c(M\times\mathbb{R},\wedge T^*M)). \tag{3.3}$$

Let $(\tilde{D}_E^2+\frac{\partial}{\partial t})^{-1}$ be the Volterra inverse of $\tilde{D}_E^2+\frac{\partial}{\partial t}$ as in [5]. That is

$$\left(\tilde{D}_{E,\pm}+\frac{\partial}{\partial t}\right)^{-1}\left(\tilde{D}_{E,\pm}+\frac{\partial}{\partial t}\right)=I-R_1,\ \ \ \left(\tilde{D}_{E,\pm}+\frac{\partial}{\partial t}\right)\left(\tilde{D}_{E,\pm}+\frac{\partial}{\partial t}\right)^{-1}=I-R_2, \tag{3.4}$$

where R_1,R_2 are smoothing operators. Let

$$(Q_0u)(x,t)=\int_{M\times\mathbb{R}}K_{Q_0}(x,y,t-s)u(y,s)dyds, \tag{3.5}$$

and $k_t(x,y)$ is the heat kernel of $e^{-t\tilde{D}_E^2}$. We get

$$K_{Q_0}(x,y,t)=k_t(x,y)\text{ when }t>0,\ \ \text{when }t<0,\ K_{Q_0}(x,y,t)=0. \tag{3.6}$$

Then Q_0 has the Volterra property, i.e., it has a distribution kernel of the form $K_{Q_0}(x,y,t-s)$ where $K_{Q_0}(x,y,t)$ vanishes on the region $t<0$. The parabolic homogeneity of the heat operator $\tilde{D}_E^2+\frac{\partial}{\partial t}$, i.e. the homogeneity with respect to the dilations of $\mathbb{R}^n\times\mathbb{R}^1$ given by

$$\lambda\cdot(\xi,\tau)=(\lambda\xi,\lambda^2\tau),\qquad(\xi,\tau)\in\mathbb{R}^n\times\mathbb{R}^1,\ \ \lambda\neq0. \tag{3.7}$$

Let $p_2(x,\xi)+p_1(x,\xi)+p_0(x,\xi)$ be the symbol of $\tilde{D}_E^2$, then the symbol of $\tilde{D}_E^2+\frac{\partial}{\partial t}$ is $\sqrt{-1}\tau+p_2(x,\xi)+p_1(x,\xi)+p_0(x,\xi)$, it is homogeneous with respect to (ξ,τ).

In the following, for $g\in S(\mathbb{R}^{n+1})$ and $\lambda\neq0$, we let g_λ be the tempered distribution defined by

$$\left\langle g_\lambda(\xi,\tau),u(\xi,\tau)\right\rangle=|\lambda|^{-(n+2)}\left\langle g(\xi,\tau),u(\lambda^{-1}\xi,\lambda^{-2}\tau)\right\rangle,\ \ u\in S(\mathbb{R}^{n+1}). \tag{3.8}$$

Definition 3.1. *A distribution $g\in S(\mathbb{R}^{n+1})$ is parabolic homogeneous of degree m, $m\in Z$, if for any $\lambda\neq0$, we have $g_\lambda=\lambda^m g$.*

Let $\mathbb{C}_-$ denote the complex halfplane $\{\mathrm{Im}\tau<0\}$ with closure $\overline{\mathbb{C}_-}$. Then:

Lemma 3.2. *[5] Let $q(\xi,\tau)\in C^\infty((\mathbb{R}^n\times\mathbb{R})/0)$ be a parabolic homogeneous symbol of degree m such that:*

(i) q extends to a continuous function on $(\mathbb{R}^n\times\overline{\mathbb{C}_-})\backslash0$ in such way to be holomorphic in the last variable when the latter is restricted to $\mathbb{C}_-$. Then there is a unique $g\in S(\mathbb{R}^{n+1})$ agreeing with q on $\mathbb{R}^{n+1}\backslash0$ so that:
(ii) g is homogeneous of degree m;
(iii) The inverse Fourier transform $\breve{g}(x,t)$ vanishes for $t<0$.

Let U be an open subset of $\mathbb{R}^n$. We define Volterra symbols and Volterra ΨDOs on $U\times\mathbb{R}^{n+1}\backslash0$ as follows.

Definition 3.3. *$S_V^m(U\times\mathbb{R}^{n+1})$, $m\in\mathbb{Z}$, consists in smooth functions $q(x,\xi,\tau)$ on $U\times\mathbb{R}^n\times\mathbb{R}$ with an asymptotic expansion $q\sim\sum_{j\geq0}q_{m-j}$, where:*

(i) $q_l\in C^\infty(U\times[(\mathbb{R}^n\times\mathbb{R})/0]$ is a homogeneous Volterra symbol of degree l, i.e. q_l is parabolic homogeneous of degree l and satisfies the property (i) in Lemma 2.3 with respect to the last $n+1$ variables;
(ii) The sign $\sim$ means that, for any integer N and any compact K, U, there is a constant $C_{NK\alpha\beta k}>0$ such that for $x\in K$ and for $|\xi|+|\tau|^{\frac{1}{2}}>1$ we have

$$|\partial_x^\alpha\partial_\xi^\beta\partial_\tau^k(q-\sum_{j<N}q_{m-j})(x,\xi,\tau)|\leq C_{NK\alpha\beta k}(|\xi|+|\tau|^{\frac{1}{2}})^{m-N-|\beta|-2k}. \tag{3.9}$$

Definition 3.4. *$\Psi_V^m(U\times\mathbb{R})$, $m\in\mathbb{Z}$, consists in continuous operators Q_0 from $C_c^\infty(U_x\times\mathbb{R}_t)$ to $C^\infty(U_x\times\mathbb{R}_t)$ such that:*

(i) Q_0 has the Volterra property;
(ii) $Q_0=q(x,D_x,D_t)+R$ for some symbol q in $S_V^m(U\times\mathbb{R})$ and some smoothing operator R.

In what follows, if Q_0 is a Volterra ΨDO, we let $K_{Q_0}(x,y,t-s)$ denote its distribution kernel, so that the distribution $K_{Q_0}(x,y,t)$ vanishes for $t<0$.

Definition 3.5. *Let $q_m(x,\xi,\tau)\in C^\infty(U\times(\mathbb{R}^{n+1}/0))$ be a homogeneous Volterra symbol of order m and let $g_m\in C^\infty(U)\otimes\mathbb{S}'(\mathbb{R}^{n+1})$ denote its unique homogeneous extension given by Lemma 2.3. Then:*

(i) $\check{q}_m(x,y,t)$ is the inverse Fourier transform of $g_m(x,\xi,\tau)$ in the last $n+1$ variables;
(ii) $q_m(x,D_x,D_t)$ is the operator with kernel $\check{q}_m(x,y-x,t)$.

Proposition 3.6. *([5,13]) The following properties hold.*

1) *Composition. Let $Q_j\in\Psi_V^{m_j}(U\times\mathbb{R})$, $j=1,2$ have symbol q_j and suppose that Q_1 or Q_2 is properly supported. Then Q_1Q_2 is a Volterra ΨDO of order m_1+m_2 with symbol $q_1\circ q_2\sim\sum\frac{1}{\alpha!}\partial_\xi^\alpha q_1 D_x^\alpha q_2$.*
2) *Parametrices. An operator Q is the order m Volterra ΨDO with the paramatrix P then*

$$QP=1-R_1,\quad PQ=1-R_2 \tag{3.10}$$

where $R_1,\ R_2$ are smoothing operators.

Proposition 3.7. *([5,13]) The differential operator $\tilde{D}_E^2+\partial_t$ is invertible and its inverse $(\tilde{D}_E^2+\partial_t)^{-1}$ is a Volterra ΨDO of order -2.*

We denote by M^ϕ the fixed-point set of ϕ, and for $a=0,\cdots,n$, we let $M^\phi=\bigcup_{0\le a\le n}M_a^\phi$, where M_a^ϕ is an a-dimensional submanifold. Given a fixed-point x_0 in a component M_a^ϕ, consider some local coordinates $x=(x^1,\cdots,x^a)$ around x_0. Setting $b=n-a$, we may further assume that over the range of the domain of the local coordinates there is an orthonormal frame $e_1(x),\cdots,e_b(x)$ of N_z^ϕ. This defines fiber coordinates $v=(v_1,\cdots,v_b)$. Composing with the map $(x,v)\in N^\phi(\varepsilon_0)\to\exp_x(v)$ we then get local coordinates $x^1,\cdots,x^a,v^1,\cdots,v^b$ for M near the fixed point x_0. We shall refer to this type of coordinates as *tubular coordinates.* Then $N^\phi(\varepsilon_0)$ is homeomorphic with a tubular neighborhood of M^ϕ. Set $i_{M^\phi}:M^\phi\hookrightarrow M$ be an inclusion map. Since $\mathrm{d}\phi$ preserves E and $E^\perp$, considering the oriented (local) orthonormal basis $\{f_1,\cdots,f_k,h_1,\cdots,h_{n-k}\}$, set

$$\mathrm{d}\phi_{x_0}=\begin{pmatrix}\exp(L_1) & 0\\ 0 & \exp(L_2)\end{pmatrix}, \tag{3.11}$$

where $L_1\in\mathfrak{so}(k)$ and $L_2\in\mathfrak{so}(n-k)$

Let

$$\widehat{A}(R^{M^\phi})=\det{}^{\frac12}\left(\frac{R^{M^\phi}/4\pi}{\sinh(R^{M^\phi}/4\pi)}\right);\quad \nu_\phi(R^{N^\phi}):=\det{}^{-\frac12}(1-\phi^N e^{-\frac{R^{N^\phi}}{2\pi}}). \tag{3.12}$$

The aim of this section is to prove the following result.

Theorem 3.8. *(Local Equivariant Sub-Signature Index Theorem. Even Dimension)*

Let $x_0\in M^\phi$, then

$$\begin{aligned}\lim_{t\to0}\mathrm{Str}\left[\tilde{\phi}(x_0)K_t(x_0,\phi(x_0))\right]=&\left(\frac{1}{\sqrt{-1}}\right)^{\frac{k}{2}}2^{\frac{n}{2}}\left\{\widehat{A}(R^{M^\phi})\nu_\phi(R^{N^\phi})i_{M^\phi}^*\left[\det{}^{\frac12}\left(\cosh\left(\frac{R^E}{4\pi}-\frac{L_1}{2}\right)\right)\right.\right.\\ &\left.\left.\times\det{}^{\frac12}\left(\frac{\sinh\left(\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}\right)}{\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}}\right)\mathrm{Pf}\left(\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}\right)\right]\right\}^{(a,0)}(x_0),\end{aligned} \tag{3.13}$$

where $L_1\in so(k), L_2\in so(n-k)$ and $\mathrm{Pf}\left(\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}\right)$ *denotes the Pfaffian of* $\left(\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}\right)$.

Next we give a detailed proof of Theorem 3.9. Let $Q=(\tilde{D}_E^2+\partial_t)^{-1}$. For $x\in M^\phi$ and $t>0$ set

$$I_Q(x,t):=\widetilde{\phi}(x)^{-1}\int_{N_x^\phi(\varepsilon)}\phi(\exp_x v)K_Q(\exp_x v,\exp_x(\phi'(x)v),t)dv. \tag{3.14}$$

Here we use a trivialization over $\wedge(T^*M)$ about the tubular coordinates. Using the tubular coordinates, we have

$$I_Q(x,t)=\int_{|v|<\varepsilon}\widetilde{\phi}(x,0)^{-1}\widetilde{\phi}(x,v)K_Q(x,v;x,\phi'(x)v;t)dv. \tag{3.15}$$

Let

$$q_{m-j}^{\wedge(T^*M)}(x,v;\xi,v;\tau) := \widetilde{\phi}(x,0)^{-1}\widetilde{\phi}(x,v)q_{m-j}(x,v;\xi,v;\tau). \tag{3.16}$$

We mention the following result

Proposition 3.9. *[19] Let $Q \in \Psi_V^m(M\times\mathbb{R},\wedge(T^*M))$, $m\in\mathbb{Z}$. Uniformly on each component M_a^ϕ*

$$I_Q(x,t) \sim \sum_{j\geq 0} t^{-(\frac{a}{2}+[\frac{m}{2}]+1)} I_Q^j(x) \quad \text{as } t\to 0^+, \tag{3.17}$$

where $I_Q^j(x)$ is defined by

$$I_Q^{(j)}(x) := \sum_{|\alpha|\leq m-[\frac{m}{2}]+2j} \int \frac{v^\alpha}{\alpha!}\left(\partial_v^\alpha q_{2[\frac{m}{2}]-2j+|\alpha|}^{\wedge(T^*M)}\right)^\vee (x,0;0,(1-\phi'(x))v;1)dv. \tag{3.18}$$

Similar to Theorem 1.2 in [15] and Section 2 (d) in [8], we have

$$\begin{aligned}\mathrm{Str}_\tau[\tilde{\phi}\exp(-t\tilde{D}_E^2)] &= (\sqrt{-1})^{\frac{k}{2}}\int_M \mathrm{Str}_\epsilon\left[\hat{c}(E,g^E)k_t(x,\phi(x))\right]\mathrm{d}x\\ &= (\sqrt{-1})^{\frac{k}{2}}\int_M \mathrm{Str}_\epsilon[\hat{c}(E,g^E)K_{(\tilde{D}_E^2+\partial_t)^{-1}}(x,\phi(x),t)]\mathrm{d}x.\end{aligned} \tag{3.19}$$

We will compute the local index in this trivialization. Let (V,q) be a finite dimensional real vector space equipped with a quadratic form. Let $C(V,q)$ be the associated Clifford algebra, i.e., the associative algebra generated by V with the relations $v\cdot w + w\cdot v = -2q(v,w)$ for $v,w\in V$. Let $\{e_1,\cdots,e_n\}$ be an orthomormal basis of (V,q), let $C(V,q)\hat{\otimes}C(V,-q)$ be the grading tensor product of $C(V,q)$ and $C(V,-q)$, and $\wedge^*V\hat{\otimes}\wedge^*V$ be the grading tensor product of $\wedge^*V$ and $\wedge^*V$. Define the symbol map:

$$\sigma : C(V,q)\hat{\otimes}C(V,-q) \to \wedge^*V\hat{\otimes}\wedge^*V; \tag{3.20}$$

where $\sigma(c(e_{j_1})\cdots c(e_{j_l})\otimes 1) = e^{j_1}\wedge\cdots\wedge e^{j_1}\otimes 1$, $\sigma(1\otimes\hat{c}(e_{j_1})\cdots\hat{c}(e_{j_l})) = 1\otimes\hat{e}^{j_1}\wedge\cdots\wedge\hat{e}^{j_1}$. Using the interior multiplication $\iota(e_j):\wedge^*V\to\wedge^{*-1}V$ and the exterior multiplication $\varepsilon(e_j):\wedge^*V\to\wedge^{*+1}V$, we define representations of $C(V,q)$ and $C(V,-q)$ on the exterior algebra:

$$c: \mathrm{C}(V,q)\to \mathrm{End}\wedge V,\ e_j\mapsto c(e_j):\ \varepsilon(e_j)-\iota(e_j); \tag{3.21}$$

$$\hat{c}: \mathrm{C}(V,-q)\to \mathrm{End}\wedge V,\ e_j\mapsto \hat{c}(e_j):\ \varepsilon(e_j)+\iota(e_j). \tag{3.22}$$

The tensor product of these representations yields an isomorphism of superalgebras

$$c\otimes\hat{c}: C(V,q)\hat{\otimes}C(V,-q)\to \mathrm{End}\wedge V \tag{3.23}$$

which we will also denote by c. We obtain a supertrace (i.e., a linear functional vanishing on supercommutators) on $C(V,q)\hat{\otimes}C(V,-q)$ by setting $\mathrm{Str}(a) = \mathrm{Str}_{\mathrm{End}\wedge V}[c(a)]$ for $a\in C(V,q)\hat{\otimes}(V,-q)$, where $\mathrm{Str}_{\mathrm{End}\wedge V}$ is the canonical supertrace on $\mathrm{End}V$.

Lemma 3.10. *For $1\leq i_1<\cdots<i_p\leq n$, $1\leq j_1<\cdots<j_q\leq n$, when $p=q=n$,*

$$\mathrm{Str}[c(e_{i_1})\cdots c(e_{i_n})\hat{c}(e_{i_1})\cdots\hat{c}(e_{i_n})] = (-1)^{\frac{n(n+1)}{2}}2^n \tag{3.24}$$

and otherwise equals zero.

We will also denote the volume element in $\wedge V\hat{\otimes}\wedge V$ by $\omega = e^1\wedge\cdots\wedge e^n\wedge\hat{e}^1\wedge\cdots\wedge\hat{e}^n$. For $a\in\wedge V\hat{\otimes}\wedge V$, let $\mathrm{T}a$ be the coefficient of ω. The linear functional $\mathrm{T}:\wedge V\hat{\otimes}\wedge V\to \mathrm{R}$ is called the Berezin trace. Then for a $a\in C(V,q)\hat{\otimes}(V,.q)$, we have $\mathrm{Str}_s(a) = (-1)^{\frac{n(n+1)}{2}}2^n(\mathrm{T}\sigma)(a)$. We define the Getzler order as follows:

$$\deg\partial_j = \frac{1}{2}\deg\partial_t = -\deg x^j = 1,\quad \deg c(e_j)=1,\ \ \deg\hat{c}(e_j)=0. \tag{3.25}$$

Let $Q\in\Psi_V^*(\mathbb{R}^n\times\mathbb{R},\wedge^*T^*M)$ have symbol

$$q(x,\xi,\tau)\sim\sum_{k\leq m'} q_k(x,\xi,\tau), \tag{3.26}$$

where $q_k(x,\xi,\tau)$ is an order k symbol. Then taking components in each subspace $\wedge^jT^*M\otimes\wedge^lT^*M$ of $\wedge T^*M\otimes\wedge T^*M$ and using Taylor expansions at $x=0$ give formal expansions

$$\sigma[q(x,\xi,\tau)]\sim\sum_{j,k}\sigma[q_k(x,\xi,\tau)]^{(j,l)}\sim\sum_{j,k,\alpha}\frac{x^\alpha}{\alpha!}\sigma[\partial_x^\alpha q_k(0,\xi,\tau)]^{(j,l)}. \tag{3.27}$$

The symbol $\frac{x^\alpha}{\alpha!}\sigma[\partial_x^\alpha q_k(0,\xi,\tau)]^{(j,l)}$ is the Getzler homogeneous of $k+j-|\alpha|$. Therefore, we can expand $\sigma[q(x,\xi,\tau)]$ as

$$\sigma[q(x,\xi,\tau)] \sim \sum_{j\geq 0} q_{(m-j)}(x,\xi,\tau), \qquad q_{(m)} \neq 0, \tag{3.28}$$

where $q_{(m-j)}$ is a Getzler homogeneous symbol of degree $m-j$.

Definition 3.11. *The integer m is called as the Getzler order of Q. The symbol $q_{(m)}$ is the principal Getzler homogeneous symbol of Q. The operator $Q_{(m)} = q_{(m)}(x, D_x, D_t)$ is called the model operator of Q.*

Let $e_1,\ldots,e_n$ be an oriented orthonormal basis of $T_{x_0}M$ such that $e_1,\cdots,e_a$ span $T_{x_0}M^\phi$ and $e_{a+1},\cdots,e_n$ span $N^\phi_{x_0}$. This provides us with normal coordinates $(x_1,\cdots,x_n)\to \exp_{x_0}(x^1e_1+\cdots+x^ne_n)$. Moreover using parallel translation enables us to construct a synchronous local oriented tangent frame $e_1(x),...,e_n(x)$ such that $e_1(x),\cdots,e_a(x)$ form an oriented frame of TM^ϕ_a and $e_{a+1}(x),\cdots,e_n(x)$ form an (oriented) frame N^τ (when both frames are restricted to M^ϕ). This gives rise to trivializations of the tangent and exterior algebra bundles. Write

$$\phi'(0) = \begin{pmatrix} 1 & 0 \\ 0 & \phi^N \end{pmatrix} = \exp(A_{ij}), \tag{3.29}$$

where $A_{ij}\in \mathfrak{so}(n)$.

Let $\wedge(n) = \wedge^*\mathbb{R}^n$ be the exterior algebra of $\mathbb{R}^n$. We shall use the following gradings on $\wedge(n)\hat{\otimes}\wedge(n)$,

$$\wedge(n)\hat{\otimes}\wedge(n) = \bigoplus_{\substack{1\leq k_1,k_2\leq a\\ 1\leq \bar{l}_1,\bar{l}_2\leq b}} \wedge^{k_1,\bar{l}_1}(n)\hat{\otimes}\wedge^{k_2,\bar{l}_2}(n), \tag{3.30}$$

where $\wedge^{k,\bar{l}}(n)$ is the space of forms $dx^{i_1}\wedge\cdots\wedge dx^{i_{k+\bar{l}}}$ with $1\leq i_1<\cdots<i_k\leq a$ and $a+1\leq i_{k+1}<\cdots<i_{k+\bar{l}}\leq n$. Given a form $\omega\in\wedge(n)\hat{\otimes}\wedge(n)$, denote by $\omega^{(k_1,\bar{l}_1),(k_2,\bar{l}_2)}$ its component in $\wedge(n)^{(k_1,\bar{l}_1)}\hat{\otimes}\wedge^{(k_2,\bar{l}_2)}(n)$. We denote by $|\omega|^{(a,0),(a,0)}$ the Berezin integral $|\omega^{(*,0),(*,0)}|^{(a,0),(a,0)}$ of its component $\omega^{(*,0),(*,0)}$ in $\wedge^{(*,0),(*,0)}(n)$.

Let $A\in Cl(V,q)\hat{\otimes}Cl(V,-q)$, then

$$\begin{aligned}\mathrm{Str}[\widetilde{\phi}A] &= (-1)^{\frac{n}{2}}2^n(-\frac{1}{4})^{\frac{b}{2}}det(1-\phi^N)|\sigma(A)|^{((a,0),(a,0))}\\ &+(-1)^{\frac{n}{2}}2^n\sum_{0\leq l_1<b,0\leq l_2\leq b}|\sigma(\widetilde{\phi})^{((0,l_1),(0,l_2))}\sigma(A)^{((a,b-l_1),(a,b-l_2))}|^{(n,n)}.\end{aligned} \tag{3.31}$$

In order to calculate $\mathrm{Str}[\widetilde{\phi}A]$, we need to consider the representation of $|\sigma(\widetilde{\phi})^{((0,b),(0,l_2))}\sigma(A)^{((a,0),(a,b-l_2))}|^{(n,n)}$. Let the matrix ϕ^N equal

$$\phi^N = \begin{pmatrix} A_{\frac{a}{2}+1} & & & \\ & \ddots & 0 & \\ & & \ddots & \\ & 0 & & \ddots \\ & & & A_{\frac{n}{2}}\end{pmatrix}, \; A_{\frac{a}{2}+1} = \begin{pmatrix} \cos\theta_{\frac{a}{2}+1} & \sin\theta_{\frac{a}{2}+1} \\ -\sin\theta_{\frac{a}{2}+1} & \cos\theta_{\frac{a}{2}+1}\end{pmatrix}, \; A_{\frac{n}{2}} = \begin{pmatrix} \cos\theta_{\frac{n}{2}} & \sin\theta_{\frac{n}{2}} \\ -\sin\theta_{\frac{n}{2}} & \cos\theta_{\frac{n}{2}}\end{pmatrix}. \tag{3.32}$$

From Lemma 3.2 in [26], then

Lemma 3.12. *We have*

$$\begin{aligned}\tilde{\phi} = \left(\frac{1}{2}\right)^{\frac{n-a}{2}} \prod_{j=\frac{a}{2}+1}^{n} &\Big[(1+\cos\theta_j) - (1-\cos\theta_j)c(e_{2j-1})c(e_{2j})\hat{c}(e_{2j-1})\hat{c}(e_{2j})\\ &+\sin\theta_j\left(c(e_{2j-1})c(e_{2j}) - \hat{c}(e_{2j-1})\hat{c}(e_{2j})\right)\Big].\end{aligned} \tag{3.33}$$

Then we obtain

$$\begin{aligned}\sigma(\tilde{\phi})^{((0,b),(0,l_2))} &= \left(\frac{1}{2}\right)^{\frac{n-a}{2}}\sigma\left\{\prod_{j=\frac{a}{2}+1}^{n}\left[-(1-\cos\theta_j)c(e_{2j-1})c(e_{2j})\hat{c}(e_{2j-1})\hat{c}(e_{2j}) + \sin\theta_j\left(c(e_{2j-1})c(e_{2j})\right)\right]\right\}^{((0,b),(0,l_2))}\\ &= \left(\frac{1}{2}\right)^{\frac{n-a}{2}} e^{a+1}\wedge\cdots\wedge e^n\sigma\left\{\prod_{j=\frac{a}{2}+1}^{n}\left[-(1-\cos\theta_j)\hat{c}(e_{2j-1})\hat{c}(e_{2j}) + \sin\theta_j\right]\right\}^{(0,l_2)}\end{aligned}$$

$$=\left(\frac{1}{2}\right)^{\frac{n-a}{2}} e^{a+1}\wedge\cdots\wedge e^{n}\sigma\left\{\prod_{j=\frac{a}{2}+1}^{n}2\sin\frac{\theta_j}{2}\left[\cos\frac{\theta_j}{2}-\sin\frac{\theta_j}{2}\hat{c}(e_{2j-1})\hat{c}(e_{2j})\right]\right\}^{(0,l_2)}$$

$$=\left(\frac{1}{2}\right)^{\frac{n-a}{2}} e^{a+1}\wedge\cdots\wedge e^{n}\det{}^{\frac{1}{2}}(1-\phi^N)\sigma\left[\exp\left(-\frac{1}{4}\sum_{1\le i,j\le n}A_{ij}\hat{c}(e_i)\hat{c}(e_j)\right)\right]^{(0,l_2)}$$

$$=\left(\frac{1}{2}\right)^{\frac{n-a}{2}} e^{a+1}\wedge\cdots\wedge e^{n}\det{}^{\frac{1}{2}}(1-\phi^N)\sigma\left[\exp\left(-\frac{1}{4}\sum_{1\le i,j\le k}(L_1)_{ij}\hat{c}(f_i)\hat{c}(f_j)\right.\right.$$

$$\left.\left.-\frac{1}{4}\sum_{1\le i,j\le n-k}(L_2)_{k+i,k+j}\hat{c}(h_i)\hat{c}(h_j)\right)\right]^{(0,l_2)}. \tag{3.34}$$

Next we calculate $|\sigma(A)|^{((a,0),(a,b-l_2))}$. In the following, we shall use the following "curvature forms": $R' := (R_{i,j})_{1\le i,j\le a}$, $R'' := (R_{a+i,a+j})_{1\le i,j\le b}$. Let

$$\dot{R}=\frac{1}{4}\sum_{1\le\alpha,\beta\le k}\langle R^E f_\alpha,f_\beta\rangle\hat{c}(f_\alpha)\hat{c}(f_\beta),$$

$$\ddot{R}=\frac{1}{4}\sum_{1\le s,t\le n-k}\langle R^{E^\perp} h_s,h_t\rangle\hat{c}(h_s)\hat{c}(h_t);$$

and

$$\tilde{\dot{R}}=\frac{1}{4}\sum_{1\le\alpha,\beta\le k}\langle (R^E-L_1)f_\alpha,f_\beta\rangle\hat{c}(f_\alpha)\hat{c}(f_\beta),$$

$$\tilde{\ddot{R}}=\frac{1}{4}\sum_{1\le s,t\le n-k}\langle (R^{E^\perp}-L_2)h_s,h_t\rangle\hat{c}(h_s)\hat{c}(h_t).$$

By (2.19), let $F=\tilde{D}_E^2$, we get

Proposition 3.13. *The model operator of F is*

$$F_{(2)}=-\sum_{r=1}^{n}\left(\partial_r+\frac{1}{8}\sum_{1\le i,j,l\le n}\langle R^{TM}(e_i,e_j)e_l,e_r\rangle y_l e^i\wedge e^j\right)^2+\frac{1}{8}\sum_{1\le i,j\le n}\sum_{1\le\alpha,\beta\le k}\langle R^E(e_i,e_j)f_\beta,f_\alpha\rangle e^i\wedge e^j\hat{c}(f_\alpha)\hat{c}(f_\beta)$$

$$+\frac{1}{8}\sum_{1\le i,j\le n}\sum_{1\le s,t\le n-k}\langle R^{E^\perp}(e_i,e_j)h_t,h_s\rangle e^i\wedge e^j\hat{c}(h_s)\hat{c}(h_t). \tag{3.35}$$

From the representation of $F_{(2)}$, we get the model operator of $\frac{\partial}{\partial t}+\tilde{D}_E^2$ is $\frac{\partial}{\partial t}+F_{(2)}$. And we have

$$\left(\frac{\partial}{\partial t}+F_{(2)}\right)K_{Q_{(-2)}}(x,y,t)=0. \tag{3.36}$$

Similar to Lemma 2.9 in [19], we get

Lemma 3.14. *Let $Q\in\Psi^{(-2)}(\mathbb{R}^n\times\mathbb{R},\wedge(T^*M))$ be a parametrix for $(F_{(2)}+\partial_t)^{-1}$. Then*

(1) Q has Getzler order -2 and its model operator is $(F_{(2)}+\partial_t)^{-1}$.
(2) For all $t>0$,

$$(\sqrt{-1})^{\frac{k}{2}}\hat{c}(E,g^E)I_{(F_{(2)}+\partial_t)^{-1}}(0,t)=(\sqrt{-1})^{\frac{k}{2}}\hat{c}(E,g^E)\frac{(4\pi t)^{-\frac{a}{2}}}{\det^{\frac{1}{2}}(1-\phi^N)}\det{}^{\frac{1}{2}}\left(\frac{\frac{tR'}{2}}{\sinh\left(\frac{tR'}{2}\right)}\right)\det{}^{-\frac{1}{2}}(1-\phi^N e^{-tR''})\exp\left(t(\tilde{\dot{R}}+\tilde{\ddot{R}})\right). \tag{3.37}$$

Similar to Lemma 3.6 in [22]. we have

Lemma 3.15. *$Q\in\Psi_V^*(\mathbb{R}^n\times\mathbb{R},\wedge(T^*M))$ has the Getzler order m and model operator $Q_{(m)}$. Then as $t\to 0^+$*

(1) $\sigma[I_Q(0,t)]^{(j,l)}=O(t^{\frac{j-m-a-1}{2}})$, if $m-j$ is odd.
(2) $\sigma[I_Q(0,t)]^{(j,l)}=O(t^{\frac{j-m-a-2}{2}})I_{Q(m)}(0,1)^{(j,l)}+O(t^{\frac{j-m-a}{2}})$, if $m-j$ is even.

In particular, for $m=-2$ and $j=a$ and a is even we get

$$\sigma[I_Q(0,t)]^{((a,0),(a,b-l_2))} = I_{Q(-2)}(0,1)^{((a,0),(a,b-l_2))} + O(t^{\frac{1}{2}}). \tag{3.38}$$

With all these preparations, we are going to prove the local even dimensional equivariant index theorem for sub-signature operators. Substituting (3.34), (3.37) into (3.31), we obtain

$$\begin{aligned}
&\lim_{t\to 0} \mathrm{Str}_\varepsilon\left[\tilde{\phi}(x_0)(\sqrt{-1})^{\frac{k}{2}}\hat{c}(E,g^E)I_{(F+\partial_t)^{-1}}(x_0,t)\right] \\
&= (-1)^{\frac{n}{2}}2^n\left(\frac{1}{2}\right)^{\frac{n-a}{2}}(4\pi)^{-\frac{a}{2}}(\sqrt{-1})^{\frac{k}{2}}\left|\widehat{A}(R^{M^\phi})\nu_\phi(R^{N^\phi})\sigma\left[\hat{c}(f_1)\cdots\hat{c}(f_k)\exp(\tilde{\tilde{R}}+\tilde{\tilde{\tilde{R}}})\right]\right|^{((a,0),n)} \\
&= \left(\frac{1}{\sqrt{-1}}\right)^{\frac{k}{2}}2^{\frac{n}{2}}\left\{\widehat{A}(R^{M^\phi})\nu_\phi(R^{N^\phi})i^*_{M^\phi}\left[\det{}^{\frac{1}{2}}\left(\cosh\left(\frac{R^E}{4\pi}-\frac{L_1}{2}\right)\right)\right.\right. \\
&\quad\left.\left.\times\det{}^{\frac{1}{2}}\left(\frac{\sinh\left(\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}\right)}{\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}}\right)\mathrm{Pf}\left(\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}\right)\right]\right\}^{(a,0)}(x_0).
\end{aligned} \tag{3.39}$$

Where we have used the algebraic result of Proposition 3.13 in [6], and the Berezin integral in the right hand side of (3.39) is the application of the following lemma.

Lemma 3.16. *Let $L_1\in so(k), L_2\in so(n-k)$, we have*

$$\begin{aligned}
\left|\sigma\left[\hat{c}(f_1)\cdots\hat{c}(f_k)\exp(\tilde{\tilde{R}}+\tilde{\tilde{\tilde{R}}})\right]\right|^{(n)} &= (-1)^{\frac{n-k}{2}}\det{}^{\frac{1}{2}}\left(\cosh\left(\frac{R^E-L_1}{2}\right)\right) \\
&\quad\times\det{}^{\frac{1}{2}}\left(\frac{\sinh\left(\frac{R^{E^\perp}-L_2}{2}\right)}{(R^{E^\perp}-L_2)/2}\right)\mathrm{Pf}\left(\frac{R^{E^\perp}-L_2}{2}\right).
\end{aligned} \tag{3.40}$$

Proof. In order to compute this differential form, we make use of the Chern root algorithm (see [22]). Assume that $n=\dim M$ and $k=\dim E$ are both even integers. As in [7], let $L_1\in so(k), L_2\in so(n-k)$, we write

$$R^E-L_1=\begin{pmatrix}\begin{pmatrix}0 & -\theta_1\\ \theta_1 & 0\end{pmatrix} & & 0\\ & \ddots & \\ 0 & & \begin{pmatrix}0 & -\theta_{-\frac{k}{2}}\\ \theta_{-\frac{k}{2}} & 0\end{pmatrix}\end{pmatrix},\quad R^{E^\perp}-L_2=\begin{pmatrix}\begin{pmatrix}0 & -\hat{\theta}_1\\ \hat{\theta}_1 & 0\end{pmatrix} & & 0\\ & \ddots & \\ 0 & & \begin{pmatrix}0 & -\hat{\theta}_{\frac{n-k}{2}}\\ \hat{\theta}_{\frac{n-k}{2}} & 0\end{pmatrix}\end{pmatrix}. \tag{3.41}$$

Then we obtain

$$\begin{aligned}
\frac{1}{4}\sum_{1\le\alpha,\beta\le k}\langle(R^E-L_1)f_\alpha,f_\beta\rangle\hat{c}(f_\alpha)\hat{c}(f_\beta) &= \frac{1}{2}\sum_{1\le\alpha<\beta\le k}\langle(R^E-L_1)f_\alpha,f_\beta\rangle\hat{c}(f_\alpha)\hat{c}(f_\beta) \\
&= \frac{1}{2}\sum_{1\le j\le\frac{k}{2}}\theta_j\hat{c}(f_{2j-1})\hat{c}(f_{2j});
\end{aligned} \tag{3.42}$$

$$\begin{aligned}
\frac{1}{4}\sum_{1\le s,t\le n-k}\langle(R^{E^\perp}-L_2)h_s,h_t\rangle\hat{c}(h_s)\hat{c}(h_t) &= \frac{1}{2}\sum_{1\le s<t\le n-k}\langle(R^{E^\perp}-L_2)h_s,h_t\rangle\hat{c}(h_s)\hat{c}(h_t) \\
&= \frac{1}{2}\sum_{1\le l\le\frac{n-k}{2}}\hat{\theta}_l\hat{c}(h_{2l-1})\hat{c}(h_{2l}).
\end{aligned} \tag{3.43}$$

Then the left hand side of (3.40) is

$$\begin{aligned}
&\left|\sigma\left(\hat{c}(f_1)\cdots\hat{c}(f_k)\exp(\tilde{\tilde{R}}+\tilde{\tilde{\tilde{R}}})\right)\right|^{(n)} \\
&= \left|\sigma\left(\hat{c}(f_1)\cdots\hat{c}(f_k)\prod_{1\le j\le\frac{k}{2}}\exp\left(\frac{1}{2}\theta_j\hat{c}(f_{2j-1})\hat{c}(f_{2j})\right)\prod_{1\le l\le\frac{n-k}{2}}\exp\left(\frac{1}{2}\hat{\theta}_l\hat{c}(h_{2l-1})\hat{c}(h_{2l})\right)\right)\right|^{(n)}
\end{aligned}$$

$$= \left|\sigma\left(\hat{c}(f_1)\cdots\hat{c}(f_k)\prod_{1\le j\le \frac{k}{2}}\left[\cos\frac{\theta_j}{2}-\sin\frac{\theta_j}{2}\hat{c}(f_{2j-1})\hat{c}(f_{2j})\right]\prod_{1\le l\le \frac{n-k}{2}}\left[\cos\frac{\hat{\theta}_l}{2}-\sin\frac{\hat{\theta}_l}{2}\hat{c}(h_{2l-1})\hat{c}(h_{2l})\right]\right)\right|^{(n)}$$

$$= (-1)^{\frac{n-k}{2}}\prod_{1\le j\le \frac{k}{2}}\cos\frac{\theta_j}{2}\prod_{1\le l\le \frac{n-k}{2}}\sin\frac{\hat{\theta}_l}{2}. \tag{3.44}$$

Now we consider the right hand side of (3.40),

$$(R^E-L_1)^{2p} = (-1)^p\begin{pmatrix}\begin{pmatrix}\theta_1^{2p} & 0\\ 0 & \theta_1^{2p}\end{pmatrix} & & 0\\ & \ddots & \\ 0 & & \begin{pmatrix}\theta_{\frac{k}{2}}^{2p} & 0\\ 0 & \theta_{\frac{k}{2}}^{2p}\end{pmatrix}\end{pmatrix}. \tag{3.45}$$

Then

$$\det{}^{\frac{1}{2}}\left(\cosh\left(\frac{R^E-L_1}{2}\right)\right) = \prod_{j=1}^{\frac{k}{2}}\left(\sum_{p=0}^{\infty}\left(\frac{\theta_j}{2}\right)^{2p}\frac{(-1)^p}{(2p)!}\right) = \prod_{j=1}^{\frac{k}{2}}\cosh\frac{\sqrt{-1}\theta_j}{2} = \prod_{j=1}^{\frac{k}{2}}\frac{e^{\frac{\sqrt{-1}\theta_j}{2}}+e^{-\frac{\sqrt{-1}\theta_j}{2}}}{2} = \prod_{j=1}^{\frac{k}{2}}\cos\frac{\theta_j}{2}. \tag{3.46}$$

Similarly, we have

$$\det{}^{\frac{1}{2}}\left(\frac{\sinh\left(\frac{R^{E^\perp}-L_2}{2}\right)}{(R^{E^\perp}-L_2)/2}\right) = \prod_{j=1}^{\frac{n-k}{2}}\frac{\sin\frac{\hat{\theta}_j}{2}}{\frac{\hat{\theta}_j}{2}}. \tag{3.47}$$

On the other hand,

$$\mathrm{Pf}\left(\frac{R^{E^\perp}-L_2}{2}\right) = T\left(\exp\left(\sum_{s<t}\langle\frac{R^{E^\perp}-L_2}{2}h_s,h_t\rangle h^s\wedge h^t\right)\right) = T\left(\exp\left(\sum_{1\le j\le\frac{n-k}{2}}\frac{\hat{\theta}_j}{2}h^{2j-1}\wedge h^{2j}\right)\right) = \prod_{j=1}^{\frac{n-k}{2}}\frac{\hat{\theta}_j}{2}. \tag{3.48}$$

Combining these equations, the proof of lemma 3.17 is complete. □

To summarize, we have proved Theorem 3.9.

3.2. The Local Odd Dimensional Equivariant Index Theorem for Sub-Signature Operators

In this section, we give a proof of a local odd dimensional equivariant index theorem for sub-signature operators. Let M be an odd dimensional oriented closed Riemannian manifold. Using (2.19) in Section 2, we may define the sub-signature operators $\tilde{D}_E$. Let γ be an orientation reversing involution isometric acting on M. Let $d\gamma$ preserve $E, E^\perp$ and preserve the orientation of E, then $\tilde{\gamma}\hat{\tau}(E,g^E) = \hat{\tau}(E,g^E)\tilde{\gamma}$, where $\tilde{\gamma}$ is the lift on the exterior algebra bundle $\wedge T^*M$ of $d\gamma$. There exists a self-adjoint lift $\widetilde{\gamma}: \Gamma(M;\wedge(T^*M)) \to \Gamma(M;\wedge(T^*M))$ of $d\gamma$ satisfying

$$\widetilde{\gamma}^2 = 1;\ \ \tilde{D}_E\widetilde{\gamma} = -\widetilde{\gamma}\tilde{D}_E. \tag{3.49}$$

Now the $+1$ and -1 eigenspaces of $\widetilde{\gamma}$ give a splitting

$$\Gamma(M;\wedge(T^*M)) \cong \Gamma^+(M;\wedge(T^*M))\oplus\Gamma^-(M;\wedge(T^*M))) \tag{3.50}$$

then the sub-signature operator interchanges $\Gamma^+(M;\wedge(T^*M))$ and $\Gamma^-(M;\wedge(T^*M))$, and $\hat{c}(E,g^E)$ preserves $\Gamma^+(M;\wedge(T^*M))$ and $\Gamma^-(M;\wedge(T^*M))$.

Denotes by $\tilde{D}_E^+$ the restriction of $\tilde{D}_E$ to $\Gamma^+(M,\wedge(T^*M))$. We assume $\dim E = k$ is even, then $(\tilde{D}_E)\hat{c}(E,g^E) = \hat{c}(E,g^E)(\tilde{D}_E)$ and $\hat{c}(E,g^E)$ is a linear map from $\ker\tilde{D}_E^\pm$ to $\ker\tilde{D}_E^\pm$.

The purpose of this section is to compute

$$\mathrm{ind}_{\hat{c}(E,g^E)}[(\tilde{D}_E^+)] = \mathrm{Tr}(\hat{c}(E,g^E)|_{\ker\tilde{D}_E^+}) - \mathrm{Tr}(\hat{c}(E,g^E)|_{\ker\tilde{D}_E^+}). \tag{3.51}$$

By the Mckean-Singer formula, we have

$$\begin{aligned}\mathrm{ind}_{\hat{c}(E,g^E)}(\tilde{D}_E^+) &= \int_M (\sqrt{-1})^{\frac{k}{2}}\mathrm{Tr}[\tilde{\gamma}\hat{c}(E,g^E)k_t(x,\gamma(x))]dx \\ &= \int_M (\sqrt{-1})^{\frac{k}{2}}\mathrm{Tr}[\tilde{\gamma}\hat{c}(E,g^E)K_{(F+\partial_t)^{-1}}(x,\gamma(x),t)]dx.\end{aligned} \tag{3.52}$$

Let

$$R^E - L_1 = \begin{pmatrix} \begin{pmatrix} 0 & -\theta_1 \\ \theta_1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} 0 & -\theta_{-\frac{k}{2}} \\ \theta_{-\frac{k}{2}} & 0 \end{pmatrix} \end{pmatrix}, R^{E^\perp} - L_2 = \begin{pmatrix} \begin{pmatrix} 0 & -\hat{\theta}_1 \\ \hat{\theta}_1 & 0 \end{pmatrix} & & & 0 \\ & \ddots & & \\ 0 & & \begin{pmatrix} 0 & -\hat{\theta}_{\frac{n-k-1}{2}} \\ \hat{\theta}_{\frac{n-k-1}{2}} & 0 \end{pmatrix} & \\ & & & 0 \end{pmatrix}; \tag{3.53}$$

and

$$\mathrm{Pf}\left(\frac{R^{E^\perp} - L_2}{2}\right) = \prod_{j=1}^{\frac{n-k-1}{2}} \frac{\hat{\theta}_j}{2}. \tag{3.54}$$

Similar to Theorem 3.9, we get the main Theorem in this section.

Theorem 3.17. *(Local odd dimensional equivariant index Theorem for sub-signature operators)*

Let $x_0 \in M^\gamma$, then

$$\begin{aligned}\lim_{t\to 0}\mathrm{Tr}\left[\tilde{\gamma}(x_0)\hat{c}(E,g^E)I_{(F+\partial_t)^{-1}}(x_0,t)\right] = -\left(\frac{1}{\sqrt{-1}}\right)^{\frac{k}{2}-1} 2^{\frac{n}{2}} \Bigg\{\widehat{A}(R^{M^\gamma})\nu_\phi(R^{N^\gamma})i^*_{M^\gamma}\Bigg[\det{}^{\frac{1}{2}}\left(\cosh\left(\frac{R^E}{4\pi} - \frac{L_1}{2}\right)\right) \\ \times \det{}^{\frac{1}{2}}\left(\frac{\sinh\left(\frac{R^{E^\perp}}{4\pi} - \frac{L_2}{2}\right)}{\frac{R^{E^\perp}}{4\pi} - \frac{L_2}{2}}\right)\mathrm{Pf}\left(\frac{R^{E^\perp}}{4\pi} - \frac{L_2}{2}\right)\Bigg]\Bigg\}^{(a,0)}(x_0).\end{aligned} \tag{3.55}$$

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

AUTHORS' CONTRIBUTION

KB and YW contributed in study conceptualization and writing (review and editing) the manuscript. JW and YW contributed in data curation, formal analysis and writing (original draft). YW contributed in funding acquisition and project administration, supervised the project, formal analysis and writing (original draft) the manuscript.

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