

Research Article

Statistical de Rham Hodge Operators and the Kastler-Kalau-Walze Type Theorem for Manifolds With Boundary

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*Keywords*Statistical de Rham Hodge operators
Lichnerowicz type formulas
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noncommutative residue**ABSTRACT**

In this paper, we give the Lichnerowicz type formulas for statistical de Rham Hodge operators. Moreover, Kastler-Kalau-Walze type theorems for statistical de Rham Hodge operators on compact manifolds with (respectively without) boundary are proved.

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The noncommutative residue which is found in [5,19] plays a prominent role in noncommutative geometry. For this reason, it has been studied extensively by geometers. Connes derived a conformal 4-dimensional Polyakov action analogy by the noncommutative residue in [2]. Connes proved that the noncommutative residue on a compact manifold M coincided with the Dixmier's trace on pseudodifferential operators of order $-\dim M$ in [3].

On the other hand, Connes had also observed that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-Hilbert action, which is called the Kastler-Kalau-Walze type theorem now. Kastler gave a brute-force proof of this theorem in [8]. Kalau and Walze [7] proved this theorem in the normal coordinates system simultaneously. And then, for the Dirac operator D , Ackermann proved that the Wodzicki residue $\text{Wres}(D^{-2})$ in turn was essentially the second coefficient of the heat kernel expansion of D^2 in [1].

On the other hand, Wang generalized the Connes' results to the case of manifolds with boundary in [13,14], and proved the Kastler-Kalau-Walze type theorems for the Dirac operator and the signature operator on lower-dimensional manifolds with boundary [15]. In [15,16], Wang computed $\widetilde{\text{Wres}}[\pi^+ D^{-1} \circ \pi^+ D^{-1}]$ and $\widetilde{\text{Wres}}[\pi^+ D^{-2} \circ \pi^+ D^{-2}]$ about symmetric operators and under the circumstances the boundary term vanished. Moreover, Wang got a nonvanishing boundary term for $\widetilde{\text{Wres}}[\pi^+ D^{-1} \circ \pi^+ D^{-3}]$ in [17] and gave a theoretical explanation for the gravitational action on boundary. In others words, Wang provided a kind of method to study the Kastler-Kalau-Walze type theorem for manifolds with boundary.

In [6], Iochum and Levy computed heat kernel coefficients for Dirac operators with one-form perturbations and proved that there were no tadpoles for compact spin manifolds without boundary. Recently, we studied the Lichnerowicz-type formulas for modified Novikov operators. We proved Kastler-Kalau-Walze-type theorems for modified Novikov operators on compact manifolds with (respectively without) boundary in [18]. In [10], Barbara Opozda introduced statistical de Rham Hodge operators on manifolds with statistical structure. The aim of this paper is to prove the Kastler-Kalau-Walze type theorem for statistical de Rham Hodge operators on manifolds without (with) boundary, and also give the Lichnerowicz formulas about statistical de Rham Hodge operators.

The paper is organized as follows: in Section 2, we give the definition of statistical de Rham Hodge operators and their Lichnerowicz formulas. We also give the basic facts and formulas about the noncommutative residue for manifolds with boundary. In Section 3 and in Section 4, we give some expressions and symbols of operators associate with statistical de Rham Hodge operators. Moreover, we also prove the Kastler-Kalau-Walze type theorems for statistical de Rham Hodge operators on 4-dimensional and 6-dimensional manifolds with boundary.

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2. STATISTICAL de RHAM HODGE OPERATORS AND THEIR LICHNEROWICZ FORMULAS

In this section, we prepare some basic notions about statistical de Rham Hodge operators. For details of the geometry of statistical manifolds, see [10].

Let M be a n -dimensional ($n \geq 3$) Riemannian manifold with a positive definite Riemannian metric g , and $\hat{\nabla}$ be a connection. We assume that M is oriented. Let ∇^L be the Levi-Civita connection for g and Vol_M be the volume form determined by g .

For all $X, Y, Z \in T_xM, x \in M$, if $\hat{\nabla}$ is satisfying the following Codazzi condition:

$$(\hat{\nabla}_X g)(Y, Z) = (\hat{\nabla}_Y g)(X, Z), \tag{2.1}$$

we call a structure $(g, \hat{\nabla})$ is a statistical structure, and a connection $\hat{\nabla}$ is a statistical connection for g . Moreover, we define a statistical manifold with statistical structure by $(M, g, \hat{\nabla})$.

We suppose that $(M, g, \hat{\nabla})$ is statistical manifold. Let K be tensor field, where $K : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM)$ and $\Gamma(TM)$ denotes the algebra of smooth vector fields on M . Let K be the difference tensor between $\hat{\nabla}$ and ∇^L , that is

$$\hat{\nabla}_X Y = \nabla_X^L Y + K_X Y. \tag{2.2}$$

Let $K(X, Y)$ stand for $K_X Y$. The de Rham derivative d is a differential operator on $C^\infty(M; \wedge^* T^*M)$. Then we have the de Rham coderivative $\delta = d^*$ and the symmetric operator $D = d + \delta$. The standard Hodge Laplacian is defined by

$$\Delta = \delta d + d\delta. \tag{2.3}$$

For a statistical manifold $(M, g, \hat{\nabla})$, we will investigate a (Lichnerowicz) Laplacian relative to the connection $\hat{\nabla}$. If f is a function, then we set

$$\Delta^{\hat{\nabla}} f = -\text{div}^{\hat{\nabla}} \text{grad} f. \tag{2.4}$$

We now extend the definition (2.4) and set $E = \text{trace}_g K(\cdot, \cdot)$. For any differential form v , we set

$$\Delta^{\hat{\nabla}} v = (\delta - l(E))dv + d(\delta - l(E))v, \tag{2.5}$$

where $l(E)$ is the contraction operator.

By the above definition and Lemma 6.4 in [10], we have

$$\Delta^{\hat{\nabla}} = (\delta - l(E) + d)^2. \tag{2.6}$$

For $v \in \Gamma(TM)$, we define the generalized statistical de Rham Hodge operators by

$$\begin{aligned} D_i &= d + \delta + \lambda_i l(v), \quad (i = 1, 2); \\ D_i^* &= d + \delta + \lambda_i \varepsilon(v^*), \quad (i = 1, 2), \end{aligned} \tag{2.7}$$

where λ_i is a real number and $v^* = g(v, \cdot)$.

In the local coordinates $\{x_i; 1 \leq i \leq n\}$ and the fixed orthonormal frame $\{e_1, \dots, e_n\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\nabla^L(e_1, \dots, e_n) = (e_1, \dots, e_n)(\omega_{s,t}). \tag{2.8}$$

Let $\varepsilon(e_j^*)$, $l(e_j)$ be the exterior and interior multiplications respectively and $c(e_j)$ be the Clifford action. Write

$$c(e_j) = \varepsilon(e_j^*) - l(e_j), \quad \bar{c}(e_j) = \varepsilon(e_j^*) + l(e_j). \tag{2.9}$$

Moreover, we assume that ∂_i is a natural local frame on TM and $(g^{ij})_{1 \leq i, j \leq n}$ is the inverse matrix associated with the metric matrix $(g_{ij})_{1 \leq i, j \leq n}$ on M . The statistical de Rham Hodge operators D_i and D_i^* ($i = 1, 2$) are defined by

$$D_i = d + \delta + \lambda_i l(v) = \sum_{i=1}^n c(e_i) \left[e_i + \frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i) [\bar{c}(e_s) \bar{c}(e_t) - c(e_s) c(e_t)] \right] + \lambda_i l(v); \tag{2.10}$$

$$D_i^* = d + \delta + \lambda_i \varepsilon(v^*) = \sum_{i=1}^n c(e_i) \left[e_i + \frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i) [\bar{c}(e_s) \bar{c}(e_t) - c(e_s) c(e_t)] \right] + \lambda_i \varepsilon(v^*). \tag{2.11}$$

Let $g^{ij} = g(dx_i, dx_j)$, $\xi = \sum_j \xi_j dx_j$ and $\nabla_{\partial_i}^L \partial_j = \sum_k \Gamma_{ij}^k \partial_k$, we denote that

$$\begin{aligned} \sigma_i &= -\frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i) c(e_s) c(e_t); \quad a_i = \frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i) \bar{c}(e_s) \bar{c}(e_t); \\ \xi^j &= g^{ij} \xi_i; \quad \Gamma^k = g^{ij} \Gamma_{ij}^k; \quad \sigma^j = g^{ij} \sigma_i; \quad a^j = g^{ij} a_i. \end{aligned} \tag{2.12}$$

Then the statistical de Rham Hodge operators D_i and D_i^* can be written as

$$D_i = \sum_{i=1}^n c(e_i)[e_i + a_i + \sigma_i] + \lambda_i l(v), \quad (i = 1, 2); \tag{2.13}$$

$$D_i^* = \sum_{i=1}^n c(e_i)[e_i + a_i + \sigma_i] + \lambda_i \varepsilon(v^*), \quad (i = 1, 2). \tag{2.14}$$

On the other hand, we recall some basic facts and formulas about Boutet de Monvel’s calculus and the definition of the noncommutative residue for manifolds with boundary (for details see Section 2 in [15]).

Let $U \subset M$ be a collar neighborhood of ∂M which is diffeomorphic with $\partial M \times [0, 1]$. $f \in C^\infty([0, 1])$ means that there is $\tilde{f} \in C^\infty((-\varepsilon, 1))$ such that $\tilde{f}|_{[0,1]} = f$ for a small positive number ε . Let $h(x_n) \in C^\infty([0, 1])$ and $h(x_n) > 0$. By the definition of $h(x_n) \in C^\infty([0, 1])$ and $h(x_n) > 0$, there exists $\tilde{h} \in C^\infty((-\varepsilon, 1))$ such that $\tilde{h}|_{[0,1]} = h$ and $\tilde{h} > 0$ for some sufficiently small $\varepsilon > 0$. Then there exists a metric \widehat{g} on $\widehat{M} = M \cup_{\partial M} \partial M \times (-\varepsilon, 0]$ which has the form on $U \cup_{\partial M} \partial M \times (-\varepsilon, 0]$

$$\widehat{g} = \frac{1}{\tilde{h}(x_n)} g^{\partial M} + dx_n^2, \tag{2.15}$$

such that $\widehat{g}|_M = g$. We fix a metric \widehat{g} on the \widehat{M} such that $\widehat{g}|_M = g$.

Let

$$F : L^2(\mathbf{R}_t) \rightarrow L^2(\mathbf{R}_\lambda); F(u)(\lambda) = \int e^{-i\lambda t} u(t) dt$$

denote the Fourier transformation and $\varphi(\overline{\mathbf{R}^+}) = r^+ \varphi(\mathbf{R})$ (similarly define $\varphi(\overline{\mathbf{R}^-})$), where $\varphi(\mathbf{R})$ denotes the Schwartz space and

$$r^+ : C^\infty(\mathbf{R}) \rightarrow C^\infty(\overline{\mathbf{R}^+}); f \rightarrow f|_{\overline{\mathbf{R}^+}}; \overline{\mathbf{R}^+} = \{x \geq 0; x \in \mathbf{R}\}.$$

We define $H^+ = F(\varphi(\overline{\mathbf{R}^+}))$; $H_0^- = F(\varphi(\overline{\mathbf{R}^-}))$ which are orthogonal to each other. We have the following property: $h \in H^+$ (H_0^-) if and only if $h \in C^\infty(\mathbf{R})$ which has an analytic extension to the lower (upper) complex half-plane $\{\text{Im}\xi < 0\}$ ($\{\text{Im}\xi > 0\}$) such that for all nonnegative integer l ,

$$\frac{d^l h}{d\xi^l}(\xi) \sim \sum_{k=1}^\infty \frac{d^l}{d\xi^l} \left(\frac{c_k}{\xi^k} \right),$$

as $|\xi| \rightarrow +\infty, \text{Im}\xi \leq 0$ ($\text{Im}\xi \geq 0$).

Let H' be the space of all polynomials and $H^- = H_0^- \oplus H'$; $H = H^+ \oplus H^-$. Denote by π^+ (π^-) respectively the projection on H^+ (H^-). For calculations, we take $H = \tilde{H} = \{\text{rational functions having no poles on the real axis}\}$ (\tilde{H} is a dense set in the topology of H). Then on \tilde{H} ,

$$\pi^+ h(\xi_0) = \frac{1}{2\pi i} \lim_{u \rightarrow 0^-} \int_{\Gamma^+} \frac{h(\xi)}{\xi_0 + iu - \xi} d\xi, \tag{2.16}$$

where Γ^+ is a Jordan close curve included $\text{Im}(\xi) > 0$ surrounding all the singularities of h in the upper half-plane and $\xi_0 \in \mathbf{R}$. Similarly, define π^- on \tilde{H} ,

$$\pi^- h = \frac{1}{2\pi} \int_{\Gamma^+} h(\xi) d\xi. \tag{2.17}$$

So, $\pi^-(H^-) = 0$. For $h \in H \cap L^1(\mathbf{R})$, $\pi^- h = \frac{1}{2\pi} \int_{\mathbf{R}} h(v) dv$ and for $h \in H^+ \cap L^1(\mathbf{R})$, $\pi^- h = 0$.

Next we give the basic notions of Laplace type operators. Let M be a smooth compact oriented Riemannian n -dimensional manifolds without boundary and V' be a vector bundle on M . Any differential operator P of Laplace type has locally the form

$$P = -(g^{ij} \partial_i \partial_j + A^i \partial_i + B), \tag{2.18}$$

where A^i and B are smooth sections of the endomorphism $\text{End}(V')$ on M . If P is a Laplace type operator with the form (2.18), then there is a unique connection ∇ on V' and a unique endomorphism E' such that

$$P = -\left[g^{ij} (\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\nabla_{\partial_i}^L \partial_j}) + E' \right], \tag{2.19}$$

where ∇^L is the Levi-Civita connection on M . Moreover (with local frames of T^*M and V'), $\nabla_{\partial_i} = \partial_i + \omega_i$ and E' are related to g^{ij} , A^i and B through

$$\omega_i = \frac{1}{2} g_{ij} (A^i + g^{kl} \Gamma_{kl}^j \text{id}), \tag{2.20}$$

$$E' = B - g^{ij} (\partial_i (\omega_j) + \omega_i \omega_j - \omega_k \Gamma_{ij}^k), \tag{2.21}$$

where Γ_{kl}^j is the Christoffel coefficient of ∇^L .

By the above definitions, we establish the main theorem in this section. One has the following Lichneriwicz formulas.

Theorem 2.1. *The following equalities hold:*

$$D_1 D_2 = -\left[g^{ij}(\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\nabla_{\partial_i}^L \partial_j}) \right] - \frac{1}{8} \sum_{ijkl} R_{ijkl} \bar{c}(e_i) \bar{c}(e_j) c(e_k) c(e_l) + \frac{1}{4} s$$

$$+ \frac{1}{4} \sum_i [\lambda_2 c(e_i) l(v) + \lambda_1 l(v) c(e_i)]^2 - \frac{1}{2} [\lambda_1 \nabla_{e_j}^{TM}(l(v)) c(e_j) - \lambda_2 c(e_j) \nabla_{e_j}^{TM}(l(v))], \tag{2.22}$$

$$D_2^* D_1^* = -\left[g^{ij}(\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\nabla_{\partial_i}^L \partial_j}) \right] - \frac{1}{8} \sum_{ijkl} R_{ijkl} \bar{c}(e_i) \bar{c}(e_j) c(e_k) c(e_l) + \frac{1}{4} s + \frac{1}{4} \sum_i [\lambda_1 c(e_i) \varepsilon(v^*)$$

$$+ \lambda_2 \varepsilon(v^*) c(e_i)]^2 - \frac{1}{2} [\lambda_2 \nabla_{e_j}^{TM}(\varepsilon(v^*)) c(e_j) - \lambda_1 c(e_j) \nabla_{e_j}^{TM}(\varepsilon(v^*))], \tag{2.23}$$

$$D_2^* D_1 = -\left[g^{ij}(\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\nabla_{\partial_i}^L \partial_j}) \right] - \frac{1}{8} \sum_{ijkl} R_{ijkl} \bar{c}(e_i) \bar{c}(e_j) c(e_k) c(e_l) + \frac{1}{4} s + \lambda_1 \lambda_2 \varepsilon(v^*) l(v)$$

$$+ \frac{1}{4} \sum_i [\lambda_1 c(e_i) l(v) + \lambda_2 \varepsilon(v^*) c(e_i)]^2 - \frac{1}{2} [\lambda_2 \nabla_{e_j}^{TM}(\varepsilon(v^*)) c(e_j) - \lambda_1 c(e_j) \nabla_{e_j}^{TM}(l(v))], \tag{2.24}$$

where s is the scalar curvature.

Proof. By (2.13), we note that

$$D_1 D_2 = (d + \delta)^2 + \lambda_2 (d + \delta) l(v) + \lambda_1 l(v) (d + \delta) + \lambda_1 \lambda_2 [l(v)]^2. \tag{2.25}$$

By [20], the local expression of $(d + \delta)^2$ is

$$(d + \delta)^2 = -\Delta_0 - \frac{1}{8} \sum_{ijkl} R_{ijkl} \bar{c}(\tilde{e}_i) \bar{c}(\tilde{e}_j) c(\tilde{e}_k) c(\tilde{e}_l) + \frac{1}{4} s. \tag{2.26}$$

By [20] and [1], we have

$$-\Delta_0 = \Delta = -g^{ij}(\nabla_i^L \nabla_j^L - \Gamma_{ij}^k \nabla_k^L). \tag{2.27}$$

$$\lambda_2 (d + \delta) l(v) + \lambda_1 l(v) (d + \delta) = \sum_{ij} g^{ij} [\lambda_2 c(\partial_i) l(v) + \lambda_1 l(v) c(\partial_i)] \partial_j$$

$$+ \sum_{ij} g^{ij} [\lambda_1 l(v) c(\partial_i) (\sigma_i + a_i) + \lambda_2 c(\partial_i) \partial_j (l(v)) + \lambda_2 c(\partial_i) (\sigma_i + a_i) l(v)]; \tag{2.28}$$

$$[l(v)]^2 = 0. \tag{2.29}$$

then we obtain

$$D_1 D_2 = - \sum_{ij} g^{ij} [\partial_i \partial_j + 2\sigma_i \partial_j + 2a_i \partial_j - \Gamma_{ij}^k \partial_k + (\partial_i \sigma_j) + (\partial_i a_j) + \sigma_i \sigma_j + \sigma_i a_j + a_i \sigma_j + a_i a_j - \Gamma_{ij}^k \sigma_k$$

$$- \Gamma_{ij}^k a_k] + \sum_{ij} g^{ij} [\lambda_2 c(\partial_i) l(v) + \lambda_1 l(v) c(\partial_i)] \partial_j - \frac{1}{8} \sum_{ijkl} R_{ijkl} \bar{c}(e_i) \bar{c}(e_j) c(e_k) c(e_l) + \frac{1}{4} s$$

$$+ \sum_{ij} g^{ij} [\lambda_2 c(\partial_i) \partial_j (l(v)) + \lambda_2 c(\partial_i) (\sigma_i + a_i) l(v) + \lambda_1 l(v) c(\partial_i) (\sigma_i + a_i)]. \tag{2.30}$$

Similarly, we have

$$D_2^* D_1^* = - \sum_{ij} g^{ij} [\partial_i \partial_j + 2\sigma_i \partial_j + 2a_i \partial_j - \Gamma_{ij}^k \partial_k + (\partial_i \sigma_j) + (\partial_i a_j) + \sigma_i \sigma_j + \sigma_i a_j + a_i \sigma_j + a_i a_j - \Gamma_{ij}^k \sigma_k - \Gamma_{ij}^k a_k]$$

$$+ \sum_{ij} g^{ij} [\lambda_1 c(\partial_i) \varepsilon(v^*) + \lambda_2 \varepsilon(v^*) c(\partial_i)] \partial_j - \frac{1}{8} \sum_{ijkl} R_{ijkl} \bar{c}(e_i) \bar{c}(e_j) c(e_k) c(e_l) + \frac{1}{4} s$$

$$+ \sum_{ij} g^{ij} [\lambda_1 c(\partial_i) \partial_j (\varepsilon(v^*)) + \lambda_1 c(\partial_i) (\sigma_i + a_i) \varepsilon(v^*) + \lambda_2 \varepsilon(v^*) c(\partial_i) (\sigma_i + a_i)]. \tag{2.31}$$

and

$$\begin{aligned}
 D_2^* D_1 &= - \sum_{i,j} g^{ij} \left[\partial_i \partial_j + 2\sigma_i \partial_j + 2a_i \partial_j - \Gamma_{ij}^k \partial_k + (\partial_i \sigma_j) + (\partial_i a_j) + \sigma_i \sigma_j + \sigma_i a_j + a_i \sigma_j + a_i a_j - \Gamma_{ij}^k \sigma_k - \Gamma_{ij}^k a_k \right] \\
 &\quad + \sum_{i,j} g^{ij} \left[\lambda_1 c(\partial_i) l(v) + \lambda_2 \varepsilon(v^*) c(\partial_i) \right] \partial_j - \frac{1}{8} \sum_{ijkl} R_{ijkl} \bar{c}(e_i) \bar{c}(e_j) c(e_k) c(e_l) + \frac{1}{4} s + \lambda_1 \lambda_2 \\
 &\quad \times \varepsilon(v^*) l(v) + \sum_{i,j} g^{ij} \left[\lambda_1 c(\partial_i) \partial_j (l(v)) + \lambda_1 c(\partial_i) (\sigma_i + a_i) l(v) + \lambda_2 \varepsilon(v^*) c(\partial_i) (\sigma_i + a_i) \right].
 \end{aligned} \tag{2.32}$$

By (2.18), (2.20) and (2.30) we have

$$\begin{aligned}
 (\omega_i)_{D_1 D_2} &= \sigma_i + a_i - \frac{1}{2} \left[\lambda_2 c(\partial_i) l(v) + \lambda_1 l(v) c(\partial_i) \right], \\
 E'_{D_1 D_2} &= \sum_{i,j} g^{ij} \left[\partial_i (\sigma_j + a_j) + \sigma_i \sigma_j + \sigma_i a_j + a_i \sigma_j - \Gamma_{ij}^k \sigma_k - \Gamma_{ij}^k a_k + a_i a_j - \lambda_2 c(\partial_i) \partial_j (l(v)) \right. \\
 &\quad \left. - c(\partial_i) (\sigma_j + a_j) \lambda_2 l(v) - \lambda_1 l(v) c(\partial_i) (\sigma_j + a_j) \right] + \frac{1}{8} \sum_{ijkl} R_{ijkl} \bar{c}(e_i) \bar{c}(e_j) c(e_k) c(e_l) - \frac{1}{4} s - \lambda_1 \lambda_2 [l(v)]^2 \\
 &\quad - \sum_{i,j} g^{ij} \left[\partial_i (\sigma_j + a_j) - \frac{1}{2} \partial_i [c(\partial_j) \lambda_2 l(v) + \lambda_1 l(v) c(\partial_j)] \right] + \left[\sigma_i + a_i - \frac{1}{2} [\lambda_2 c(\partial_i) l(v) + \lambda_1 l(v) c(\partial_i)] \right] \\
 &\quad \times \left[\sigma_j + a_j - \frac{1}{2} [\lambda_2 c(\partial_j) l(v) + \lambda_1 l(v) c(\partial_j)] \right] - \left[\sigma_k + a_k - \frac{1}{2} [c(\partial_k) \lambda_2 l(v) + \lambda_1 l(v) c(\partial_k)] \right] \Gamma_{ij}^k.
 \end{aligned} \tag{2.34}$$

Let $c(Y)$ denote the Clifford action, where Y is a smooth vector field on M . Since E' is globally defined on M , taking normal coordinates at x_0 , we have $\sigma^i(x_0) = 0, a^i(x_0) = 0, \partial^j [c(\partial_j)](x_0) = 0, \Gamma^k(x_0) = 0, g^{ij}(x_0) = \delta_i^j$. By (2.21), then we have

$$\begin{aligned}
 E'_{D_1 D_2}(x_0) &= \frac{1}{8} \sum_{ijkl} R_{ijkl} \bar{c}(e_i) \bar{c}(e_j) c(e_k) c(e_l) - \frac{1}{4} s - \frac{1}{4} \sum_i \left[\lambda_2 c(e_i) l(v) + \lambda_1 l(v) c(e_i) \right]^2 \\
 &\quad + \frac{1}{2} \left[\lambda_1 \nabla_{e_j}^{TM} (l(v)) c(e_j) - \lambda_2 c(e_j) \nabla_{e_j}^{TM} (l(v)) \right].
 \end{aligned} \tag{2.35}$$

Similarly, we have

$$\begin{aligned}
 E'_{D_2^* D_1^*}(x_0) &= \frac{1}{8} \sum_{ijkl} R_{ijkl} \bar{c}(e_i) \bar{c}(e_j) c(e_k) c(e_l) - \frac{1}{4} s - \frac{1}{4} \sum_i \left[\lambda_1 c(e_i) \varepsilon(v^*) + \lambda_2 \varepsilon(v^*) c(e_i) \right]^2 \\
 &\quad + \frac{1}{2} \left[\lambda_2 \nabla_{e_j}^{TM} (\varepsilon(v^*)) c(e_j) - \lambda_1 c(e_j) \nabla_{e_j}^{TM} (\varepsilon(v^*)) \right].
 \end{aligned} \tag{2.36}$$

$$\begin{aligned}
 E'_{D_2^* D_1}(x_0) &= \frac{1}{8} \sum_{ijkl} R_{ijkl} \bar{c}(e_i) \bar{c}(e_j) c(e_k) c(e_l) - \frac{1}{4} s - \frac{1}{4} \sum_i \left[\lambda_1 c(e_i) l(v) + \lambda_2 \varepsilon(v^*) c(e_i) \right]^2 \\
 &\quad + \frac{1}{2} \left[\lambda_2 \nabla_{e_j}^{TM} (\varepsilon(v^*)) c(e_j) - \lambda_1 c(e_j) \nabla_{e_j}^{TM} (l(v)) \right] - \lambda_1 \lambda_2 \varepsilon(v^*) l(v).
 \end{aligned} \tag{2.37}$$

which, together with (2.19), we complete the proof. □

The noncommutative residue of a generalized laplacian Δ is expressed by [1], as

$$(n - 2)\phi(\Delta) = (4\pi)^{-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) \widehat{\text{res}}(\Delta^{-\frac{n}{2}+1}), \tag{2.38}$$

where $\phi(\Delta)$ denotes the integral over the diagonal part of the second coefficient of the heat kernel expansion of Δ . Since $D_1 D_2, D_2^* D_1^*$ and $D_2^* D_1$ are generalized laplacian operators, we have

$$\text{Wres}(D_1 D_2)^{-\frac{n-2}{2}} = \frac{(n - 2)(4\pi)^{\frac{n}{2}}}{\left(\frac{n}{2} - 1\right)!} \int_M \text{trace}\left(\frac{1}{6} s + E'_{D_1 D_2}\right) d\text{Vol}_M, \tag{2.39}$$

where Wres is the noncommutative residue.

Similarly, we have

$$\text{Wres}(D_2^* D_1^*)^{-\frac{n-2}{2}} = \frac{(n - 2)(4\pi)^{\frac{n}{2}}}{\left(\frac{n}{2} - 1\right)!} \int_M \text{trace}\left(\frac{1}{6} s + E'_{D_2^* D_1^*}\right) d\text{Vol}_M, \tag{2.40}$$

$$\text{Wres}(D_2^* D_1)^{-\frac{n-2}{2}} = \frac{(n - 2)(4\pi)^{\frac{n}{2}}}{\left(\frac{n}{2} - 1\right)!} \int_M \text{trace}\left(\frac{1}{6} s + E'_{D_2^* D_1}\right) d\text{Vol}_M. \tag{2.41}$$

By Theorem 2.1 and its proof, we have

Theorem 2.2. For even n -dimensional compact oriented manifolds without boundary, the following equalities holds:

$$\text{Wres}(D_1 D_2)^{-\frac{n-2}{2}} = \frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2}-1)!} \int_M 2^n \left(-\frac{1}{12}s - \frac{1}{4}(\lambda_1^2 + \lambda_2^2)|v|^2 \right) d\text{Vol}_M, \tag{2.42}$$

$$\text{Wres}(D_2^* D_1^*)^{-\frac{n-2}{2}} = \frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2}-1)!} \int_M 2^n \left(-\frac{1}{12}s - \frac{1}{4}(\lambda_1^2 + \lambda_2^2)|v^*|^2 \right) d\text{Vol}_M, \tag{2.43}$$

$$\begin{aligned} \text{Wres}(D_2^* D_1)^{-\frac{n-2}{2}} &= \frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2}-1)!} \int_M \left[2^n \left(-\frac{1}{12}s - \frac{\lambda_1^2 + \lambda_2^2 - 2n\lambda_1\lambda_2 + 4\lambda_1\lambda_2}{8}|v|^2 \right) \right. \\ &\quad \left. + \frac{1}{2} \text{tr}[\lambda_2 \nabla_{e_j}^{TM}(\varepsilon(v^*))c(e_j) - \lambda_1 c(e_j) \nabla_{e_j}^{TM}(l(v))] \right] d\text{Vol}_M, \end{aligned} \tag{2.44}$$

where s is the scalar curvature.

3. A KASTLER-KALAU-WALZE TYPE THEOREM FOR 4-DIMENSIONAL MANIFOLDS WITH BOUNDARY

In this section, we prove the Kastler-Kalau-Walze type theorem for 4-dimensional oriented compact manifold with boundary about statistical de Rham Hodge Operators.

Denote by M a n -dimensional manifold with boundary ∂M . We assume that M is compact and oriented. Let \mathcal{B} be Boutet de Monvel’s algebra, we now recall the main theorem in [4,15].

Theorem 3.1. [4](Fedosov-Golse-Leichtnam-Schrohe) Let X and ∂X be connected, $\dim X = n \geq 3$, $A = \begin{pmatrix} \pi^+ P + G & K \\ T & S \end{pmatrix} \in \mathcal{B}$, and denote by p, b and s the local symbols of P, G and S respectively. Define:

$$\widetilde{\text{Wres}}(A) = \int_X \int_S \text{trace}_E [p_{-n}(x, \xi)] \sigma(\xi) dx + 2\pi \int_{\partial X} \int_{S'} \{ \text{trace}_E [(\text{tr} b_{-n})(x', \xi')] + \text{trace}_F [s_{1-n}(x', \xi')] \} \sigma(\xi') dx'. \tag{3.1}$$

Then

- a) $\widetilde{\text{Wres}}([A, B]) = 0$, for any $A, B \in \mathcal{B}$;
- b) It is a unique continuous trace on $\mathcal{B}/\mathcal{B}^{-\infty}$.

Definition 3.2. [15] Lower dimensional volume of spin manifolds with boundary is defined by

$$\text{Vol}_n^{(p_1, p_2)} M := \widetilde{\text{Wres}}[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}]. \tag{3.2}$$

By (2.1.4)-(2.1.8) in [15], we get

$$\widetilde{\text{Wres}}[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}] = \int_M \int_{|\xi|=1} \text{trace}_{\wedge^* T^* M} [\sigma_{-n}(D^{-p_1-p_2})] \sigma(\xi) dx + \int_{\partial M} \Phi, \tag{3.3}$$

and

$$\begin{aligned} \Phi &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \times \text{trace}_{\wedge^* T^* M} [\partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k \sigma_r^+(D^{-p_1})(x', 0, \xi', \xi_n) \\ &\quad \times \partial_{x'}^\alpha \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l(D^{-p_2})(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx', \end{aligned} \tag{3.4}$$

where the sum is taken over $r + l - k - |\alpha| - j - 1 = -n, r \leq -p_1, l \leq -p_2$.

For any fixed point $x_0 \in \partial M$, we choose the normal coordinates U of x_0 in ∂M (not in M) and compute $\Phi(x_0)$ in the coordinates $\tilde{U} = U \times [0, 1) \subset M$ and the metric $\frac{1}{h(x_n)} g^{\partial M} + dx_n^2$. The dual metric of g^M on \tilde{U} is $h(x_n) g^{\partial M} + dx_n^2$. Write $g_{ij}^M = g^M(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$; $g_M^{ij} = g^M(dx_i, dx_j)$, then

$$[g_{ij}^M] = \begin{bmatrix} \frac{1}{h(x_n)} [g_{ij}^{\partial M}] & 0 \\ 0 & 1 \end{bmatrix}; \quad [g_M^{ij}] = \begin{bmatrix} h(x_n) [g_{\partial M}^{ij}] & 0 \\ 0 & 1 \end{bmatrix}, \tag{3.5}$$

and

$$\partial_{x_i} g_{ij}^{\partial M}(x_0) = 0, 1 \leq i, j \leq n-1; \quad g_{ij}^M(x_0) = \delta_{ij}. \tag{3.6}$$

We will give following three lemmas as computation tools.

Lemma 3.3. [15] *With the metric g^M on M near the boundary*

$$\partial_{x_j}(|\xi|_{g^M}^2)(x_0) = \begin{cases} 0 & \text{if } j < n, \\ h'(0)|\xi'|_{g^{\partial M}}^2 & \text{if } j = n; \end{cases} \tag{3.7}$$

$$\partial_{x_j}[c(\xi)](x_0) = \begin{cases} 0 & \text{if } j < n, \\ \partial_{x_n}(c(\xi'))(x_0) & \text{if } j = n, \end{cases} \tag{3.8}$$

where $\xi = \xi' + \xi_n dx_n$.

Lemma 3.4. [15] *With the metric g^M on M near the boundary*

$$\omega_{s,t}(e_i)(x_0) = \begin{cases} \omega_{n,i}(e_i)(x_0) = \frac{1}{2}h'(0) & \text{if } s = n, t = i, i < n; \\ \omega_{i,n}(e_i)(x_0) = -\frac{1}{2}h'(0) & \text{if } s = i, t = n, i < n; \\ \omega_{s,t}(e_i)(x_0) = 0 & \text{other cases} \end{cases}, \tag{3.9}$$

where $(\omega_{s,t})$ denotes the connection matrix of Levi-Civita connection ∇^L .

Lemma 3.5. [15] *When $i < n$, then*

$$\Gamma_{ii}^n(x_0) = \frac{1}{2}h'(0); \Gamma_{ni}^i(x_0) = -\frac{1}{2}h'(0); \Gamma_{in}^i(x_0) = -\frac{1}{2}h'(0),$$

in other cases, $\Gamma_{st}^i(x_0) = 0$.

By (3.3) and (3.4), we firstly compute

$$\widetilde{\text{Wres}}[\pi^+ D_2^{-1} \circ \pi^+ D_1^{-1}] = \int_M \int_{|\xi|=1} \text{trace}_{\wedge^* T^* M}[\sigma_{-4}(D_1 D_2)^{-1}] \sigma(\xi) dx + \int_{\partial M} \Phi_1, \tag{3.10}$$

where

$$\begin{aligned} \Phi_1 &= \int_{|\xi|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \times \text{trace}_{\wedge^* T^* M}[\partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k \sigma_r^+(D_2^{-1})(x', 0, \xi', \xi_n) \\ &\quad \times \partial_{x'}^\alpha \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l(D_1^{-1})(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx', \end{aligned} \tag{3.11}$$

and the sum is taken over $r+l-k-j-|\alpha| = -3, r \leq -1, l \leq -1$.

Locally we can use Theorem 2.2 (2.42) to compute the interior of $\widetilde{\text{Wres}}[\pi^+ D_2^{-1} \circ \pi^+ D_1^{-1}]$, we have

$$\int_M \int_{|\xi|=1} \text{trace}_{\wedge^* T^* M}[\sigma_{-4}(D_1 D_2)^{-1}] \sigma(\xi) dx = 32\pi^2 \int_M \left(-\frac{4}{3}s - 4(\lambda_1^2 + \lambda_2^2)|v|^2 \right) d\text{Vol}_M. \tag{3.12}$$

So we only need to compute $\int_{\partial M} \Phi_1$. Let us now turn to compute the symbols of some operators. By (2.10)-(2.14), then we have the following symbols of some operators.

Lemma 3.6. *The following identities hold:*

$$\begin{aligned} \sigma_1(D_j) &= \sigma_1(D_j^*) = ic(\xi), (j = 1, 2); \\ \sigma_0(D_j) &= \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(e_i)c(e_i)\bar{c}(e_s)\bar{c}(e_t) - \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(e_i)c(e_i)c(e_s)c(e_t) + \lambda_j l(v), (j = 1, 2); \\ \sigma_0(D_j^*) &= \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(e_i)c(e_i)\bar{c}(e_s)\bar{c}(e_t) - \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(e_i)c(e_i)c(e_s)c(e_t) + \lambda_j \varepsilon(v^*), (j = 1, 2). \end{aligned} \tag{3.13}$$

Write

$$D_x^\alpha = (-i)^{|\alpha|} \partial_x^\alpha; \sigma(D) = p_1 + p_0; \sigma(D^{-1}) = \sum_{j=1}^{\infty} q_{-j}. \tag{3.14}$$

By the composition formula of pseudodifferential operators, we have

$$\begin{aligned} 1 &= \sigma(D \circ D^{-1}) \\ &= \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^\alpha [\sigma(D)] D_x^\alpha [\sigma(D^{-1})] \end{aligned}$$

$$\begin{aligned}
 &= (p_1 + p_0)(q_{-1} + q_{-2} + q_{-3} + \dots) + \sum_j (\partial_{\xi_j} p_1 + \partial_{\xi_j} p_0)(D_{x_j} q_{-1} + D_{x_j} q_{-2} + D_{x_j} q_{-3} + \dots) \\
 &= p_1 q_{-1} + (p_1 q_{-2} + p_0 q_{-1} + \sum_j \partial_{\xi_j} p_1 D_{x_j} q_{-1}) + \dots,
 \end{aligned} \tag{3.15}$$

so

$$q_{-1} = p_1^{-1}; \quad q_{-2} = -p_1^{-1}[p_0 p_1^{-1} + \sum_j \partial_{\xi_j} p_1 D_{x_j} (p_1^{-1})]. \tag{3.16}$$

By Lemma 3.6, we have some symbols of operators.

Lemma 3.7. *The following identities hold:*

$$\begin{aligned}
 \sigma_{-1}(D_j^{-1}) &= \sigma_{-1}((D_j^*)^{-1}) = \frac{ic(\xi)}{|\xi|^2}, \quad (j = 1, 2); \\
 \sigma_{-2}(D_j^{-1}) &= \frac{c(\xi)\sigma_0(D_j)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2) \right], \quad (j = 1, 2); \\
 \sigma_{-2}((D_j^*)^{-1}) &= \frac{c(\xi)\sigma_0(D_j^*)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2) \right], \quad (j = 1, 2).
 \end{aligned} \tag{3.17}$$

From the remark above, now we can compute Φ_1 (see formula (3.11) for the definition of Φ_1). We use tr as shorthand of trace. Since $n = 4$, then $\text{tr}_{\wedge^* T^* M}[\text{id}] = 16$, since the sum is taken over $r + l - k - j - |\alpha| = -3$, $r \leq -1$, $l \leq -1$, we have the following five cases:

case 1) I) $r = -1$, $l = -1$, $k = j = 0$, $|\alpha| = 1$.

By (3.11), we get

$$\text{case 1) I) } = - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(D_2^{-1}) \times \partial_{x_i}^\alpha \partial_{\xi_n} \sigma_{-1}(D_1^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.18}$$

By Lemma 3.3, for $i < n$, then

$$\partial_{x_i} \left(\frac{ic(\xi)}{|\xi|^2} \right) (x_0) = \frac{i\partial_{x_i}[c(\xi)](x_0)}{|\xi|^2} - \frac{ic(\xi)\partial_{x_i}(|\xi|^2)(x_0)}{|\xi|^4} = 0, \tag{3.19}$$

so case 1) I) vanishes.

case 1) II) $r = -1$, $l = -1$, $k = |\alpha| = 0$, $j = 1$.

By (3.11), we get

$$\text{case 1) II) } = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(D_2^{-1}) \times \partial_{\xi_n}^2 \sigma_{-1}(D_1^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.20}$$

By Lemma 3.7, we have

$$\partial_{\xi_n}^2 \sigma_{-1}(D_1^{-1})(x_0) = i \left(-\frac{6\xi_n c(dx_n) + 2c(\xi')}{|\xi|^4} + \frac{8\xi_n^2 c(\xi)}{|\xi|^6} \right); \tag{3.21}$$

$$\partial_{x_n} \sigma_{-1}(D_2^{-1})(x_0) = \frac{i\partial_{x_n} c(\xi')(x_0)}{|\xi|^2} - \frac{ic(\xi)|\xi'|^2 h'(0)}{|\xi|^4}. \tag{3.22}$$

By (2.16), (2.17) and the Cauchy integral formula we have

$$\begin{aligned}
 \pi_{\xi_n}^+ \left[\frac{c(\xi)}{|\xi|^4} \right] (x_0)|_{|\xi'|=1} &= \pi_{\xi_n}^+ \left[\frac{c(\xi') + \xi_n c(dx_n)}{(1 + \xi_n^2)^2} \right] \\
 &= \frac{1}{2\pi i} \lim_{u \rightarrow 0^-} \int_{\Gamma^+} \frac{\frac{c(\xi') + \eta_n c(dx_n)}{(\eta_n + i)^2 (\xi_n + iu - \eta_n)}}{(\eta_n - i)^2} d\eta_n \\
 &= -\frac{(i\xi_n + 2)c(\xi') + ic(dx_n)}{4(\xi_n - i)^2}.
 \end{aligned} \tag{3.23}$$

Similarly, we have

$$\pi_{\xi_n}^+ \left[\frac{i\partial_{x_n} c(\xi')}{|\xi|^2} \right] (x_0)|_{|\xi'|=1} = \frac{\partial_{x_n} [c(\xi')](x_0)}{2(\xi_n - i)}. \tag{3.24}$$

then

$$\pi_{\xi_n}^+ \partial_{x_n} \sigma_{-1} (D_2^{-1})|_{|\xi'|=1} = \frac{\partial_{x_n} [c(\xi')] (x_0)}{2(\xi_n - i)} + ih'(0) \left[\frac{(i\xi_n + 2)c(\xi') + ic(dx_n)}{4(\xi_n - i)^2} \right]. \tag{3.25}$$

By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, we have the equalities:

$$\begin{aligned} \text{tr}[c(\xi')c(dx_n)] &= 0; \quad \text{tr}[c(dx_n)^2] = -16; \quad \text{tr}[c(\xi')^2](x_0)|_{|\xi'|=1} = -16; \\ \text{tr}[\partial_{x_n} c(\xi')c(dx_n)] &= 0; \quad \text{tr}[\partial_{x_n} c(\xi')c(\xi')](x_0)|_{|\xi'|=1} = -8h'(0); \quad \text{tr}[\bar{c}(e_i)\bar{c}(e_j)c(e_k)c(e_l)] = 0 (i \neq j). \end{aligned} \tag{3.26}$$

By (3.26) and a direct computation, we have

$$\begin{aligned} h'(0)\text{tr} \left[\frac{(i\xi_n + 2)c(\xi') + ic(dx_n)}{4(\xi_n - i)^2} \times \left(\frac{6\xi_n c(dx_n) + 2c(\xi')}{(1 + \xi_n^2)^2} - \frac{8\xi_n^2 [c(\xi') + \xi_n c(dx_n)]}{(1 + \xi_n^2)^3} \right) \right] (x_0)|_{|\xi'|=1} \\ = -16h'(0) \frac{-2i\xi_n^2 - \xi_n + i}{(\xi_n - i)^4(\xi_n + i)^3}. \end{aligned} \tag{3.27}$$

Similarly, we have

$$-i\text{tr} \left[\left(\frac{\partial_{x_n} [c(\xi')] (x_0)}{2(\xi_n - i)} \right) \times \left(\frac{6\xi_n c(dx_n) + 2c(\xi')}{(1 + \xi_n^2)^2} - \frac{8\xi_n^2 [c(\xi') + \xi_n c(dx_n)]}{(1 + \xi_n^2)^3} \right) \right] (x_0)|_{|\xi'|=1} = -8ih'(0) \frac{3\xi_n^2 - 1}{(\xi_n - i)^4(\xi_n + i)^3}. \tag{3.28}$$

Then

$$\begin{aligned} \text{case 1) II) } &= - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{4ih'(0)(\xi_n - i)^2}{(\xi_n - i)^4(\xi_n + i)^3} d\xi_n \sigma(\xi') dx' \\ &= -4ih'(0)\Omega_3 \int_{\Gamma^+} \frac{1}{(\xi_n - i)^2(\xi_n + i)^3} d\xi_n dx' \\ &= -4ih'(0)\Omega_3 2\pi i \left[\frac{1}{(\xi_n + i)^3} \right] \Big|_{\xi_n=i} dx' \\ &= -\frac{3}{2}\pi h'(0)\Omega_3 dx', \end{aligned}$$

where Ω_3 is the canonical volume of S^3 .

case 1) III) $r = -1, l = -1, j = |\alpha| = 0, k = 1$.

By (3.11), we get

$$\text{case 1) III) } = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1} (D_2^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-1} (D_1^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.29}$$

By Lemma 3.7, we have

$$\partial_{\xi_n} \partial_{x_n} \sigma_{-1} (D_1^{-1})(x_0)|_{|\xi'|=1} = -ih'(0) \left[\frac{c(dx_n)}{|\xi|^4} - 4\xi_n \frac{c(\xi') + \xi_n c(dx_n)}{|\xi|^6} \right] - \frac{2\xi_n i \partial_{x_n} c(\xi')(x_0)}{|\xi|^4}; \tag{3.30}$$

$$\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1} (D_2^{-1})(x_0)|_{|\xi'|=1} = -\frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2}. \tag{3.31}$$

Similar to case 1) II), we have

$$\text{tr} \left\{ \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2} \times ih'(0) \left[\frac{c(dx_n)}{|\xi|^4} - 4\xi_n \frac{c(\xi') + \xi_n c(dx_n)}{|\xi|^6} \right] \right\} = 8h'(0) \frac{i - 3\xi_n}{(\xi_n - i)^4(\xi_n + i)^3} \tag{3.32}$$

and

$$\text{tr} \left[\frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2} \times \frac{2\xi_n i \partial_{x_n} c(\xi')(x_0)}{|\xi|^4} \right] = \frac{-8ih'(0)\xi_n}{(\xi_n - i)^4(\xi_n + i)^2}. \tag{3.33}$$

So we have

$$\begin{aligned} \text{case 1) III) } &= - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{h'(0)4(i - 3\xi_n)}{(\xi_n - i)^4(\xi_n + i)^3} d\xi_n \sigma(\xi') dx' - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{h'(0)4i\xi_n}{(\xi_n - i)^4(\xi_n + i)^2} d\xi_n \sigma(\xi') dx' \\ &= -h'(0)\Omega_3 \frac{2\pi i}{3!} \left[\frac{4(i - 3\xi_n)}{(\xi_n + i)^3} \right]^{(3)} \Big|_{\xi_n=i} dx' + h'(0)\Omega_3 \frac{2\pi i}{3!} \left[\frac{4i\xi_n}{(\xi_n + i)^2} \right]^{(3)} \Big|_{\xi_n=i} dx' \\ &= \frac{3}{2}\pi h'(0)\Omega_3 dx'. \end{aligned} \tag{3.34}$$

case 2) $r = -2, l = -1, k = j = |\alpha| = 0$.

By (3.11), we get

$$\text{case 2)} = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \sigma_{-2}(D_2^{-1}) \times \partial_{\xi_n} \sigma_{-1}(D_1^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.35}$$

By Lemma 3.7, we have

$$\sigma_{-2}(D_2^{-1})(x_0) = \frac{c(\xi)\sigma_0(D_2)(x_0)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n) [\partial_{x_n} [c(\xi')](x_0) |\xi|^2 - c(\xi)h'(0) |\xi|_{\partial M}^2], \tag{3.36}$$

where

$$\sigma_0(D_2)(x_0) = \frac{1}{4} \sum_{s,t,i} \omega_{s,t}(e_i)(x_0) c(e_i) \bar{c}(e_s) \bar{c}(e_t) - \frac{1}{4} \sum_{s,t,i} \omega_{s,t}(e_i)(x_0) c(e_i) c(e_s) c(e_t) + \lambda_2 l(v). \tag{3.37}$$

Write

$$A(x_0) = \frac{1}{4} \sum_{s,t,i} \omega_{s,t}(e_i)(x_0) c(e_i) \bar{c}(e_s) \bar{c}(e_t); B(x_0) = -\frac{1}{4} \sum_{s,t,i} \omega_{s,t}(e_i)(x_0) c(e_i) c(e_s) c(e_t). \tag{3.38}$$

Then

$$\begin{aligned} \pi_{\xi_n}^+ \sigma_{-2}(D_2^{-1})(x_0)|_{|\xi'|=1} &= \pi_{\xi_n}^+ \left[\frac{c(\xi)A(x_0)c(\xi)}{(1+\xi_n^2)^2} \right] + \pi_{\xi_n}^+ \left[\lambda_2 \frac{c(\xi)l(v)(x_0)c(\xi)}{(1+\xi_n^2)^2} \right] \\ &+ \pi_{\xi_n}^+ \left[\frac{c(\xi)B(x_0)c(\xi) + c(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0)}{(1+\xi_n^2)^2} - h'(0) \frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n^2)^3} \right]. \end{aligned} \tag{3.39}$$

By direct calculations, we have

$$\begin{aligned} \pi_{\xi_n}^+ \left[\frac{c(\xi)A(x_0)c(\xi)}{(1+\xi_n^2)^2} \right] &= \pi_{\xi_n}^+ \left[\frac{c(\xi')A(x_0)c(\xi')}{(1+\xi_n^2)^2} \right] + \pi_{\xi_n}^+ \left[\frac{\xi_n c(\xi')A(x_0)c(dx_n)}{(1+\xi_n^2)^2} \right] + \pi_{\xi_n}^+ \left[\frac{\xi_n c(dx_n)A(x_0)c(\xi')}{(1+\xi_n^2)^2} \right] + \pi_{\xi_n}^+ \left[\frac{\xi_n^2 c(dx_n)A(x_0)c(dx_n)}{(1+\xi_n^2)^2} \right] \\ &= -\frac{c(\xi')A(x_0)c(\xi')(2+i\xi_n)}{4(\xi_n-i)^2} + \frac{ic(\xi')A(x_0)c(dx_n)}{4(\xi_n-i)^2} + \frac{ic(dx_n)A(x_0)c(\xi')}{4(\xi_n-i)^2} + \frac{-i\xi_n c(dx_n)A(x_0)c(dx_n)}{4(\xi_n-i)^2}. \end{aligned} \tag{3.40}$$

Since

$$c(dx_n)A(x_0) = -\frac{1}{4} h'(0) \sum_{i=1}^{n-1} c(e_i) \bar{c}(e_i) c(e_n) \bar{c}(e_n), \tag{3.41}$$

by the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, we have the equalities:

$$\text{tr}[c(e_i) \bar{c}(e_i) c(e_n) \bar{c}(e_n)] = 0 \quad (i < n); \quad \text{tr}[Ac(dx_n)] = 0; \quad \text{tr}[\bar{c}(\xi')c(dx_n)] = 0; \tag{3.42}$$

Since

$$\partial_{\xi_n} \sigma_{-1}(D_1^{-1}) = \partial_{\xi_n} q_{-1}(x_0)|_{|\xi'|=1} = i \left[\frac{c(dx_n)}{1+\xi_n^2} - \frac{2\xi_n c(\xi') + 2\xi_n^2 c(dx_n)}{(1+\xi_n^2)^2} \right]. \tag{3.43}$$

By (3.40), (3.42) and (3.43), we have

$$\begin{aligned} \text{tr}[\pi_{\xi_n}^+ \left[\frac{c(\xi)A(x_0)c(\xi)}{(1+\xi_n^2)^2} \right] \times \partial_{\xi_n} \sigma_{-1}(D_1^{-1})(x_0)]|_{|\xi'|=1} &= \frac{1}{2(1+\xi_n^2)^2} \text{tr}[c(\xi')A(x_0)] + \frac{i}{2(1+\xi_n^2)^2} \text{tr}[c(dx_n)A(x_0)] \\ &= \frac{1}{2(1+\xi_n^2)^2} \text{tr}[c(\xi')A(x_0)]. \end{aligned} \tag{3.44}$$

We note that $i < n, \int_{|\xi'|=1} \{\xi_{i_1} \xi_{i_2} \cdots \xi_{i_{d+1}}\} \sigma(\xi') = 0$, so $\text{tr}[c(\xi')A(x_0)]$ has no contribution for computing case 2).

By direct calculations, we have

$$\pi_{\xi_n}^+ \left[\frac{c(\xi)B(x_0)c(\xi) + c(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0)}{(1+\xi_n^2)^2} \right] - h'(0) \pi_{\xi_n}^+ \left[\frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n^2)^3} \right] := P_1 - P_2, \tag{3.45}$$

where

$$\begin{aligned} P_1 &= \frac{-1}{4(\xi_n-i)^2} [(2+i\xi_n)c(\xi')b_0^2(x_0)c(\xi') + i\xi_n c(dx_n)b_0^2(x_0)c(dx_n) \\ &+ (2+i\xi_n)c(\xi')c(dx_n)\partial_{x_n}c(\xi') + ic(dx_n)b_0^2(x_0)c(\xi') + ic(\xi')b_0^2(x_0)c(dx_n) - i\partial_{x_n}c(\xi')] \end{aligned} \tag{3.46}$$

and

$$P_2 = \frac{h'(0)}{2} \left[\frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3} [ic(\xi') - c(dx_n)] \right]. \tag{3.47}$$

By (3.43) and (3.46), we have

$$\text{tr}[P_1 \times \partial_{\xi_n} \sigma_{-1}(D_1^{-1})]_{|\xi'|=1} = \frac{-6ih'(0)}{(1 + \xi_n^2)^2} + 2h'(0) \frac{\xi_n^2 - i\xi_n - 2}{(\xi_n - i)(1 + \xi_n^2)^2}. \tag{3.48}$$

By (3.43) and (3.47), we have

$$\begin{aligned} \text{tr}[P_2 \times \partial_{\xi_n} \sigma_{-1}(D_1^{-1})]_{|\xi'|=1} &= \frac{i}{2} h'(0) \frac{-i\xi_n^2 - \xi_n + 4i}{4(\xi_n - i)^3 (\xi_n + i)^2} \text{tr}[\text{id}] \\ &= 8ih'(0) \frac{-i\xi_n^2 - \xi_n + 4i}{4(\xi_n - i)^3 (\xi_n + i)^2}. \end{aligned} \tag{3.49}$$

By (3.48) and (3.49), we have

$$\begin{aligned} -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[(P_1 - P_2) \times \partial_{\xi_n} \sigma_{-1}(D_1^{-1})](x_0) d\xi_n \sigma(\xi') dx' &= -\Omega_3 \int_{\Gamma^+} \frac{8[-\frac{3}{4}h'(0)](\xi_n - i) + ih'(0)}{(\xi_n - i)^3 (\xi_n + i)^2} d\xi_n dx' \\ &= \frac{9}{2} \pi h'(0) \Omega_3 dx'. \end{aligned} \tag{3.50}$$

Similar to (3.44), we have

$$\begin{aligned} \text{tr}[\pi_{\xi_n}^+ \left[\lambda_2 \frac{c(\xi)l(v)(x_0)c(\xi)}{(1 + \xi_n^2)^2} \right] \times \partial_{\xi_n} \sigma_{-1}(D_1^{-1})(x_0)]_{|\xi'|=1} \\ = \frac{1}{2(1 + \xi_n^2)^2} \text{tr}[\lambda_2 c(\xi')l(v)(x_0)] + \frac{i}{2(1 + \xi_n^2)^2} \text{tr}[\lambda_2 c(dx_n)l(v)(x_0)]. \end{aligned} \tag{3.51}$$

By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, we have the equalities:

$$\text{tr}[c(dx_n)l(v)] = 8\langle v, dx_n \rangle; \quad \text{tr}[c(\xi')l(v)] = 8\langle v, \xi' \rangle; \tag{3.52}$$

By (3.51) and (3.52), we have

$$-i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \left[\frac{\lambda_2 c(\xi)l(v)c(\xi)}{(1 + \xi_n^2)^2} \right] \times \partial_{\xi_n} \sigma_{-1}(D_1^{-1})](x_0) d\xi_n \sigma(\xi') dx' = 2\lambda_2 \pi \langle v, dx_n \rangle \Omega_3 dx'. \tag{3.53}$$

By (3.44), (3.50) and (3.53), we have

$$\text{case 2) } = \left[\frac{9}{2} \pi h'(0) + 2\lambda_2 \pi \langle v, dx_n \rangle \right] \Omega_3 dx'. \tag{3.54}$$

case 3) $r = -1, l = -2, k = j = |\alpha| = 0$.

By (3.11), we get

$$\text{case 3) } = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \sigma_{-1}(D_2^{-1}) \times \partial_{\xi_n} \sigma_{-2}(D_1^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.55}$$

By (2.16) and Lemma 3.7, we have

$$\pi_{\xi_n}^+ \sigma_{-1}(D_2^{-1})_{|\xi'|=1} = \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)}. \tag{3.56}$$

By (3.36), (3.37) and (3.38), we have

$$\begin{aligned} \partial_{\xi_n} \sigma_{-2}(D_1^{-1})(x_0)_{|\xi'|=1} &= \partial_{\xi_n} \left\{ \frac{c(\xi)[A(x_0) + B(x_0) + (\lambda_1 l(v)(x_0))]c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n)[\partial_{x_n}[c(\xi')](x_0)|\xi|^2 - c(\xi)h'(0)] \right\} \\ &= \partial_{\xi_n} \left\{ \frac{c(\xi)A(x_0)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n)[\partial_{x_n}[c(\xi')](x_0)|\xi|^2 - c(\xi)h'(0)] \right\} + \partial_{\xi_n} \frac{c(\xi)B(x_0)c(\xi)}{|\xi|^4} \\ &\quad + \lambda_1 \partial_{\xi_n} \frac{c(\xi)l(v)(x_0)c(\xi)}{|\xi|^4}. \end{aligned} \tag{3.57}$$

By direct calculation, we have

$$\partial_{\xi_n} \frac{c(\xi)A(x_0)c(\xi)}{|\xi|^4} = \frac{c(dx_n)A(x_0)c(\xi)}{|\xi|^4} + \frac{c(\xi)A(x_0)c(dx_n)}{|\xi|^4} - \frac{4\xi_n c(\xi)A(x_0)c(\xi)}{|\xi|^6}; \tag{3.58}$$

$$\partial_{\xi_n} \frac{c(\xi)l(v)(x_0)c(\xi)}{|\xi|^4} = \frac{c(dx_n)l(v)(x_0)c(\xi)}{|\xi|^4} + \frac{c(\xi)l(v)(x_0)c(dx_n)}{|\xi|^4} - \frac{4\xi_n c(\xi)l(v)(x_0)c(\xi)}{|\xi|^6}. \tag{3.59}$$

Write

$$P_3 = \frac{c(\xi)B(x_0)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6}c(dx_n)[\partial_{x_n}[c(\xi')](x_0)|\xi|^2 - c(\xi)h'(0)],$$

then

$$\begin{aligned} \partial_{\xi_n}(P_3) &= \frac{1}{(1 + \xi_n^2)^3} \left[(2\xi_n - 2\xi_n^3)c(dx_n)Bc(dx_n) + (1 - 3\xi_n^2)c(dx_n)Bc(\xi') \right. \\ &\quad + (1 - 3\xi_n^2)c(\xi')Bc(dx_n) - 4\xi_n c(\xi')Bc(\xi') + (3\xi_n^2 - 1)\partial_{x_n}c(\xi') \\ &\quad \left. - 4\xi_n c(\xi')c(dx_n)\partial_{x_n}c(\xi') + 2h'(0)c(\xi') + 2h'(0)\xi_n c(dx_n) \right] \\ &\quad + 6\xi_n h'(0) \frac{c(\xi)c(dx_n)c(\xi)}{(1 + \xi_n^2)^4}. \end{aligned} \tag{3.60}$$

By (3.56) and (3.58), we have

$$\begin{aligned} &\text{tr}[\pi_{\xi_n}^+ \sigma_{-1}(D_2^{-1}) \times \partial_{\xi_n} \frac{c(\xi)Ac(\xi)}{|\xi|^4}](x_0)|_{|\xi'|=1} \\ &= \frac{-1}{(\xi - i)(\xi + i)^3} \text{tr}[c(\xi')A(x_0)] + \frac{i}{(\xi - i)(\xi + i)^3} \text{tr}[c(dx_n)A(x_0)]. \end{aligned} \tag{3.61}$$

By (3.42), we have

$$\text{tr}[\pi_{\xi_n}^+ \sigma_{-1}(D_2^{-1}) \times \partial_{\xi_n} \frac{c(\xi)Ac(\xi)}{|\xi|^4}](x_0)|_{|\xi'|=1} = \frac{-1}{(\xi - i)(\xi + i)^3} \text{tr}[c(\xi')A(x_0)]. \tag{3.62}$$

We note that $i < n$, $\int_{|\xi'|=1} \{\xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2d+1}}\} \sigma(\xi') = 0$, so $\text{tr}[c(\xi')A(x_0)]$ has no contribution for computing case 3).

By (3.56) and (3.60), we have

$$\text{tr}[\pi_{\xi_n}^+ \sigma_{-1}(D_2^{-1}) \times \partial_{\xi_n}(P_3)](x_0)|_{|\xi'|=1} = \frac{12h'(0)(i\xi_n^2 + \xi_n - 2i)}{(\xi - i)^3(\xi + i)^3} + \frac{48h'(0)i\xi_n}{(\xi - i)^3(\xi + i)^4}, \tag{3.63}$$

then

$$-i\Omega_3 \int_{\Gamma_+} \left[\frac{12h'(0)(i\xi_n^2 + \xi_n - 2i)}{(\xi_n - i)^3(\xi_n + i)^3} + \frac{48h'(0)i\xi_n}{(\xi_n - i)^3(\xi_n + i)^4} \right] d\xi_n dx' = -\frac{9}{2} \pi h'(0) \Omega_3 dx'. \tag{3.64}$$

By (3.56) and (3.59), we have

$$\begin{aligned} &\text{tr}[\pi_{\xi_n}^+ \sigma_{-1}(D_2^{-1}) \times \partial_{\xi_n} \frac{c(\xi)(\lambda_1 l(v))c(\xi)}{|\xi|^4}](x_0)|_{|\xi'|=1} \\ &= \frac{-1}{(\xi - i)(\xi + i)^3} \text{tr}[c(\xi')(\lambda_1 l(v))(x_0)] + \frac{i}{(\xi - i)(\xi + i)^3} \text{tr}[c(dx_n)(\lambda_1 l(v))(x_0)]. \end{aligned} \tag{3.65}$$

By (3.52), we have

$$\begin{aligned} &-i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \sigma_{-1}(D_2^{-1}) \times \partial_{\xi_n} \frac{c(\xi)(\lambda_1 l(v))c(\xi)}{|\xi|^4}](x_0) d\xi_n \sigma(\xi') dx' \\ &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{i}{(\xi - i)(\xi + i)^3} \left[\text{tr}[c(dx_n)(\lambda_1 l(v))(x_0)] + i \text{tr}[c(\xi')(\lambda_1 l(v))(x_0)] \right] d\xi_n \sigma(\xi') dx' \\ &= -\frac{\pi}{4} \left[\text{tr}[c(dx_n)(\lambda_1 l(v))] + i \text{tr}[c(\xi')(\lambda_1 l(v))(x_0)] \right] \Omega_3 dx' \\ &= -2\pi \lambda_1 \langle v, dx_n \rangle \Omega_3 dx'. \end{aligned} \tag{3.66}$$

So we have

$$\text{case 3) = } \left[-\frac{9}{2} \pi h'(0) - 2\lambda_1 \pi \langle v, dx_n \rangle \right] \Omega_3 dx'. \tag{3.67}$$

Since Φ_1 is the sum of the cases 1), 2) and 3), $\Phi_1 = 2(\lambda_2 - \lambda_1)\pi \langle v, dx_n \rangle \Omega_3 dx'$.

Theorem 3.8. Let M be a 4-dimensional oriented compact manifold with the boundary ∂M and the metric g^M as above, D_i ($i = 1, 2$) be statistical de Rham Hodge Operators on \widehat{M} , then

$$\begin{aligned} \widetilde{\text{Wres}}[\pi^+ D_2^{-1} \circ \pi^+ D_1^{-1}] &= 32\pi^2 \int_M \left(-\frac{4}{3}s - 4(\lambda_1^2 + \lambda_2^2)|v|^2 \right) d\text{Vol}_M \\ &\quad + \int_{\partial M} 2(\lambda_2 - \lambda_1)\pi \langle v, dx_n \rangle \Omega_3 dx', \end{aligned} \tag{3.68}$$

where s is the scalar curvature.

Let $D = d + \delta + l(v)$, $D^* = d + \delta + \varepsilon(v^*)$.

Corollary 3.9. For 4-dimensional oriented compact manifold M with the boundary ∂M , when $\lambda_1 = \lambda_2 = 1$, we get

$$\widetilde{\text{Wres}}[\pi^+ D^{-1} \circ \pi^+ D^{-1}] = 32\pi^2 \int_M \left(-\frac{4}{3}s - 8|v|^2 \right) d\text{Vol}_M,$$

where s is the scalar curvature.

On the other hand, we also prove the Kastler-Kalau-Walze type theorem for 4-dimensional manifolds with boundary associated with $(D_i^*)^2$ ($i = 1, 2$). By (3.3) and (3.4), we will compute

$$\widetilde{\text{Wres}}[\pi^+(D_1^*)^{-1} \circ \pi^+(D_2^*)^{-1}] = \int_M \int_{|\xi|=1} \text{trace}_{\wedge^* T^* M} [\sigma_{-4}((D_2^* D_1^*)^{-1})] \sigma(\xi) dx + \int_{\partial M} \Phi_2, \tag{3.69}$$

where

$$\begin{aligned} \Phi_2 = & \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \times \text{trace}_{\wedge^* T^* M} [\partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k \sigma_r^+((D_1^*)^{-1})(x', 0, \xi', \xi_n) \\ & \times \partial_{x'}^\alpha \partial_{\xi_n}^{j+1} \partial_{\xi_n}^k \sigma_l((D_2^*)^{-1})(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx', \end{aligned} \tag{3.70}$$

and the sum is taken over $r + l - k - j - |\alpha| = -3$, $r \leq -1, l \leq -1$.

Locally we can use Theorem 2.2 (2.43) to compute the interior of $\widetilde{\text{Wres}}[\pi^+(D_1^*)^{-1} \circ \pi^+(D_2^*)^{-1}]$, we have

$$\int_M \int_{|\xi|=1} \text{trace}_{\wedge^* T^* M} [\sigma_{-4}((D_2^* D_1^*)^{-1})] \sigma(\xi) dx = 32\pi^2 \int_M \left(-\frac{4}{3}s - 4(\lambda_1^2 + \lambda_2^2)|v^*|^2 \right) d\text{Vol}_M. \tag{3.71}$$

So we only need to compute $\int_{\partial M} \Phi_2$. From the remark above, now we can compute Φ_2 (see formula (3.70) for the definition of Φ_2). We use tr as shorthand of trace. Since $n = 4$, then $\text{tr}_{\wedge^* T^* M} [id] = 16$, since the sum is taken over $r + l - k - j - |\alpha| = -3$, $r \leq -1, l \leq -1$, then we have the following five cases:

case a) I) $r = -1, l = -1, k = j = 0, |\alpha| = 1$.

By (3.70), we get

$$\text{case a) I)} = - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr} [\partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}((D_1^*)^{-1}) \times \partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-1}((D_2^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.72}$$

case a) II) $r = -1, l = -1, k = |\alpha| = 0, j = 1$.

By (3.70), we get

$$\text{case a) II)} = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} [\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}((D_1^*)^{-1}) \times \partial_{\xi_n}^2 \sigma_{-1}((D_2^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.73}$$

case a) III) $r = -1, l = -1, j = |\alpha| = 0, k = 1$

By (3.70), we get

$$\text{case a) III)} = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} [\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}((D_1^*)^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-1}((D_2^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.74}$$

By Lemma 3.7, we have $\sigma_{-1}(D_i^{-1}) = \sigma_{-1}((D_i^*)^{-1})$ ($i = 1, 2$). By (3.19)-(3.34), so **case a)** vanishes.

case b) $r = -2, l = -1, k = j = |\alpha| = 0$.

By (3.70), we get

$$\text{case b)} = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} [\pi_{\xi_n}^+ \sigma_{-2}((D_1^*)^{-1}) \times \partial_{\xi_n} \sigma_{-1}((D_2^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.75}$$

By Lemma 3.7, we have $\sigma_{-1}(D_i^{-1}) = \sigma_{-1}((D_i^*)^{-1})$ ($i = 1, 2$) and

$$\sigma_{-2}((D_1^*)^{-1})(x_0) = \frac{c(\xi)\sigma_0(D_1^*)(x_0)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n) \left[\partial_{x_n} [c(\xi')](x_0)|\xi|^2 - c(\xi)h'(0)|\xi|_{\partial M}^2 \right], \tag{3.76}$$

where $\sigma_0(D_1^*)(x_0) = A(x_0) + B(x_0) + \lambda_1 \varepsilon(v^*)$. Then

$$\begin{aligned} \pi_{\xi_n}^+ \sigma_{-2}((D_1^*)^{-1}(x_0))|_{|\xi|=1} &= \pi_{\xi_n}^+ \left[\frac{c(\xi)A(x_0)c(\xi)}{(1 + \xi_n^2)^2} \right] + \pi_{\xi_n}^+ \left[\frac{c(\xi)(\lambda_1 \varepsilon(v^*)(x_0))c(\xi)}{(1 + \xi_n^2)^2} \right] \\ &+ \pi_{\xi_n}^+ \left[\frac{c(\xi)B(x_0)c(\xi) + c(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0)}{(1 + \xi_n^2)^2} - h'(0) \frac{c(\xi)c(dx_n)c(\xi)}{(1 + \xi_n^2)^3} \right]. \end{aligned} \tag{3.77}$$

By (3.40)-(3.50), we have

$$\begin{aligned} \text{case b)} &= \frac{9}{2} \pi h'(0) \Omega_3 dx' - i \int_{|\xi|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \left[\frac{c(\xi)(\lambda_1 \varepsilon(v^*)(x_0))c(\xi)}{(1 + \xi_n^2)^2} \right] \\ &\times \partial_{\xi_n} \sigma_{-1}((D_2^*)^{-1})(x_0) d\xi_n \sigma(\xi') dx']. \end{aligned} \tag{3.78}$$

Similar to (3.51), we have

$$\begin{aligned} &\text{tr}[\pi_{\xi_n}^+ \left[\frac{c(\xi)(\lambda_1 \varepsilon(v^*)(x_0))c(\xi)}{(1 + \xi_n^2)^2} \right] \times \partial_{\xi_n} \sigma_{-1}(D_2^*)^{-1}(x_0)]|_{|\xi|=1} \\ &= \frac{1}{2(1 + \xi_n^2)^2} \text{tr}[c(\xi') \lambda_1 \varepsilon(v^*)(x_0)] + \frac{i}{2(1 + \xi_n^2)^2} \text{tr}[c(dx_n) \lambda_1 \varepsilon(v^*)(x_0)]. \end{aligned} \tag{3.79}$$

By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, we have the equalities:

$$\text{tr}[c(dx_n) \varepsilon(v^*)] = -8 \langle v^*, \frac{\partial}{\partial x_n} \rangle; \quad \text{tr}[c(\xi') \varepsilon(v^*)] = -8 \langle v^*, g(\xi', \cdot) \rangle; \tag{3.80}$$

By (3.79) and (3.80), we have

$$-i \int_{|\xi|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \left[\frac{c(\xi) \lambda_1 \varepsilon(v^*)(x_0) c(\xi)}{(1 + \xi_n^2)^2} \right] \times \partial_{\xi_n} \sigma_{-1}(D_2^{-1})(x_0) d\xi_n \sigma(\xi') dx'] = -2\lambda_1 \pi \langle v^*, \frac{\partial}{\partial x_n} \rangle \Omega_3 dx'. \tag{3.81}$$

By (3.78) and (3.81), we have

$$\text{case b)} = [\frac{9}{2} \pi h'(0) - 2\lambda_1 \pi \langle v^*, \frac{\partial}{\partial x_n} \rangle] \Omega_3 dx'. \tag{3.82}$$

case c) $r = -1, l = -2, k = j = |\alpha| = 0$.

By (3.70), we get

$$\text{case c)} = -i \int_{|\xi|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \sigma_{-1}((D_1^*)^{-1}) \times \partial_{\xi_n} \sigma_{-2}((D_2^*)^{-1})(x_0) d\xi_n \sigma(\xi') dx']. \tag{3.83}$$

By Lemma 3.7, we have $\sigma_{-1}(D_i^{-1}) = \sigma_{-1}((D_i^*)^{-1})$ ($i = 1, 2$). Similar to (3.57), we have

$$\begin{aligned} \partial_{\xi_n} \sigma_{-2}((D_2^*)^{-1})(x_0)|_{|\xi|=1} &= \partial_{\xi_n} \left\{ \frac{c(\xi)[A(x_0) + B(x_0) + (\lambda_2 \varepsilon(v^*)(x_0))]c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n)[\partial_{x_n}[c(\xi')](x_0)|\xi|^2 - c(\xi)h'(0)] \right\} \\ &= \partial_{\xi_n} \left\{ \frac{c(\xi)[A(x_0)]c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n)[\partial_{x_n}[c(\xi')](x_0)|\xi|^2 - c(\xi)h'(0)] \right\} + \partial_{\xi_n} \frac{c(\xi)B(x_0)c(\xi)}{|\xi|^4} \\ &+ \partial_{\xi_n} \frac{c(\xi)(\lambda_2 \varepsilon(v^*)(x_0))c(\xi)}{|\xi|^4}. \end{aligned} \tag{3.84}$$

By (3.58)-(3.65), we have

$$\text{case c)} = -\frac{9}{2} \pi h'(0) - i \int_{|\xi|=1} \int_{-\infty}^{+\infty} \text{tr} \left[\pi_{\xi_n}^+ \sigma_{-1}((D_1^*)^{-1}) \times \partial_{\xi_n} \left(\frac{c(\xi)(\lambda_2 \varepsilon(v^*)(x_0))c(\xi)}{|\xi|^4} \right) \right] (x_0) d\xi_n \sigma(\xi') dx'. \tag{3.85}$$

Similar to (3.66), we have

$$\begin{aligned} &-i \int_{|\xi|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \sigma_{-1}((D_1^*)^{-1}) \times \partial_{\xi_n} \frac{c(\xi)(\lambda_2 \varepsilon(v^*)(x_0))c(\xi)}{|\xi|^4}](x_0) d\xi_n \sigma(\xi') dx' \\ &= -i \int_{|\xi|=1} \int_{-\infty}^{+\infty} \frac{i}{(\xi - i)(\xi + i)^3} \left[\text{tr}[c(dx_n)(\lambda_2 \varepsilon(v^*)(x_0))] + i \text{tr}[c(\xi')(\lambda_2 \varepsilon(v^*)(x_0))] \right] d\xi_n \sigma(\xi') dx' \\ &= -\frac{\pi}{4} \left[\text{tr}[c(dx_n)(\lambda_2 \varepsilon(v^*))] + i \text{tr}[c(\xi')(\lambda_2 \varepsilon(v^*))](x_0) \right] \Omega_3 dx' \\ &= 2\lambda_2 \pi \langle v^*, \frac{\partial}{\partial x_n} \rangle \Omega_3 dx'. \end{aligned} \tag{3.86}$$

So, we have

$$\text{case c)} = \left[-\frac{9}{2}\pi h'(0) + 2\lambda_2\pi \langle v^*, \frac{\partial}{\partial x_n} \rangle \right] \Omega_3 dx'. \tag{3.87}$$

Since Φ_2 is the sum of the cases a), b) and c), so $\Phi_2 = 2(\lambda_2 - \lambda_1)\pi \langle v^*, \frac{\partial}{\partial x_n} \rangle \Omega_3 dx'$.

Theorem 3.10. *Let M be a 4-dimensional oriented compact manifold with the boundary ∂M and the metric g^M as above, D_i^* ($i = 1, 2$) be statistical de Rham Hodge Operators on \widehat{M} , then*

$$\begin{aligned} \widetilde{\text{Wres}}[\pi^+(D_1^*)^{-1} \circ \pi^+(D_2^*)^{-1}] &= 32\pi^2 \int_M \left(-\frac{4}{3}s - 4(\lambda_1^2 + \lambda_2^2)|v^*|^2 \right) d\text{Vol}_M \\ &+ \int_{\partial M} 2(\lambda_2 - \lambda_1)\pi \langle v^*, \frac{\partial}{\partial x_n} \rangle \Omega_3 dx', \end{aligned} \tag{3.88}$$

where s is the scalar curvature.

Corollary 3.11. *For 4-dimensional oriented compact manifold M with the boundary ∂M , when $\lambda_1 = \lambda_2 = 1$, we get*

$$\widetilde{\text{Wres}}[\pi^+(D^*)^{-1} \circ \pi^+(D^*)^{-1}] = 32\pi^2 \int_M \left(-\frac{4}{3}s - 8|v^*|^2 \right) d\text{Vol}_M,$$

where s is the scalar curvature.

Next, we prove the Kastler-Kalau-Walze type theorem for 4-dimensional manifolds with boundary associated with $D_2^*D_1$. By (3.3) and (3.4), we will compute

$$\widetilde{\text{Wres}}[\pi^+D_1^{-1} \circ \pi^+(D_2^*)^{-1}] = \int_M \int_{|\xi|=1} \text{trace}_{\wedge^*T^*M}[\sigma_{-4}((D_2^*D_1)^{-1})]\sigma(\xi)dx + \int_{\partial M} \Phi_3, \tag{3.89}$$

where

$$\begin{aligned} \Phi_3 &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \times \text{trace}_{\wedge^*T^*M}[\partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k \sigma_r^+(D_1^{-1})(x', 0, \xi', \xi_n) \\ &\times \partial_{x'}^\alpha \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l((D_2^*)^{-1})(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx' \end{aligned} \tag{3.90}$$

and the sum is taken over $r+l-k-j-|\alpha| = -3, r \leq -1, l \leq -1$.

Locally we can use Theorem 2.2 (2.44) to compute the interior of $\widetilde{\text{Wres}}[\pi^+D_1^{-1} \circ \pi^+(D_2^*)^{-1}]$, we have

$$\begin{aligned} &\int_M \int_{|\xi|=1} \text{trace}_{\wedge^*T^*M}[\sigma_{-4}((D_2^*D_1)^{-1})]\sigma(\xi)dx \\ &= 32\pi^2 \int_M \left[-\frac{4}{3}s - 2(\lambda_1^2 + \lambda_2^2 - 4\lambda_1\lambda_2)|v|^2 + \frac{1}{2}\text{tr}[\lambda_2 \nabla_{e_j}^{TM}(\varepsilon(v^*))c(e_j) - \lambda_1 c(e_j) \nabla_{e_j}^{TM}(l(v))] \right] d\text{Vol}_M. \end{aligned} \tag{3.91}$$

So we only need to compute $\int_{\partial M} \Phi_3$. From the remark above, now we can compute Φ_3 (see formula (3.89) for the definition of Φ_3). We use tr as shorthand of trace. Since $n = 4$, then $\text{tr}_{\wedge^*T^*M}[\text{id}] = 16$, since the sum is taken over $r+l-k-j-|\alpha| = -3, r \leq -1, l \leq -1$, then we have the following five cases:

case a) I) $r = -1, l = -1, k = j = 0, |\alpha| = 1$.

By (3.89), we get

$$\text{case a) I)} = - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(D_1^{-1}) \times \partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-1}((D_2^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.92}$$

case a) II) $r = -1, l = -1, k = |\alpha| = 0, j = 1$.

By (3.89), we get

$$\text{case a) II)} = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}((D_1^{-1})) \times \partial_{\xi_n}^2 \sigma_{-1}((D_2^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.93}$$

case a) III) $r = -1, l = -1, j = |\alpha| = 0, k = 1$.

By (3.89), we get

$$\text{case a) III) } = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}((D_1^{-1})) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-1}((D_2^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.94}$$

By Lemma 3.7, we have $\sigma_{-1}(D_i^{-1}) = \sigma_{-1}((D_i^*)^{-1})$ ($i = 1, 2$). By (3.19)-(3.34), so case a) vanishes.

case b) $r = -2, l = -1, k = j = |\alpha| = 0$.

By (3.89), we get

$$\text{case b) } = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \sigma_{-2}(D_1^{-1}) \times \partial_{\xi_n} \sigma_{-1}((D_2^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.95}$$

By Lemma 3.7, we have $\sigma_{-1}(D_i^{-1}) = \sigma_{-1}((D_i^*)^{-1})$ ($i = 1, 2$). By (3.35)-(3.54), we have

$$\text{case b) } = \left[\frac{9}{2} \pi h'(0) + 2\lambda_1 \pi \langle dx_n, \nu \rangle \right] \Omega_3 dx'. \tag{3.96}$$

case c) $r = -1, l = -2, k = j = |\alpha| = 0$.

By (3.70), we get

$$\text{case c) } = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \sigma_{-1}(D_1^{-1}) \times \partial_{\xi_n} \sigma_{-2}((D_2^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.97}$$

By Lemma 3.7, we have $\sigma_{-1}(D_i^{-1}) = \sigma_{-1}((D_i^*)^{-1})$ ($i = 1, 2$). By (3.83)-(3.87), we have

$$\text{case c) } = \left[-\frac{9}{2} \pi h'(0) + 2\lambda_2 \pi \langle \nu^*, \frac{\partial}{\partial x_n} \rangle \right] \Omega_3 dx'. \tag{3.98}$$

Since Φ_3 is the sum of the cases a), b) and c), so $\Phi_3 = 2(\lambda_1 + \lambda_2) \pi \langle dx_n, \nu \rangle \Omega_3 dx'$.

Theorem 3.12. Let M be a 4-dimensional oriented compact manifold with the boundary ∂M and the metric g^M as above, D_i and D_i^* ($i = 1, 2$) be statistical de Rham Hodge Operators on \widehat{M} , then

$$\begin{aligned} \widetilde{\text{Wres}}[\pi^+ D_1^{-1} \circ \pi^+ (D_2^*)^{-1}] &= 32\pi^2 \int_M \left[-\frac{4}{3}s - 2(\lambda_1^2 + \lambda_2^2 - 4\lambda_1\lambda_2)|\nu|^2 + \frac{1}{2} \text{tr}[\lambda_2 \nabla_{e_j}^{TM}(\varepsilon(\nu^*))c(e_j) \right. \\ &\quad \left. - \lambda_1 c(e_j) \nabla_{e_j}^{TM}(l(\nu))] \right] d\text{Vol}_M + \int_{\partial M} 2(\lambda_1 + \lambda_2) \pi \langle dx_n, \nu \rangle \Omega_3 dx', \end{aligned} \tag{3.99}$$

where s is the scalar curvature.

Corollary 3.13. For a 4-dimensional oriented compact manifold M with the boundary ∂M , when $\lambda_1 = \lambda_2 = 1$, we get

$$\begin{aligned} \widetilde{\text{Wres}}[\pi^+ D^{-1} \circ \pi^+ (D^*)^{-1}] &= 32\pi^2 \int_M \left[-\frac{4}{3}s + 4|\nu|^2 + \frac{1}{2} \text{tr}[\nabla_{e_j}^{TM}(\varepsilon(\nu^*))c(e_j) - c(e_j) \nabla_{e_j}^{TM}(l(\nu))] \right] d\text{Vol}_M \\ &\quad + \int_{\partial M} 4\pi \langle dx_n, \nu \rangle \Omega_3 dx', \end{aligned}$$

where s is the scalar curvature.

4. A KASTLER-KALAU-WALZE TYPE THEOREM FOR 6-DIMENSIONAL MANIFOLDS WITH BOUNDARY

In this section, we prove the Kastler-Kalau-Walze type theorems for 6-dimensional manifolds with boundary. An application of (2.1.4) in [17] shows that

$$\widetilde{\text{Wres}}[\pi^+ D_1^{-1} \circ \pi^+ (D_2^* D_1 D_2^*)^{-1}] = \int_M \int_{|\xi|=1} \text{trace}_{\wedge^* T^* M} [\sigma_{-4}((D_2^* D_1)^{-2})] \sigma(\xi) dx + \int_{\partial M} \Psi, \tag{4.1}$$

where

$$\begin{aligned} \Psi &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \times \text{trace}_{\wedge^* T^* M} [\partial_{x_n}^j \partial_{\xi_n}^\alpha \partial_{\xi_n}^k \sigma_r^+(D_1^{-1})(x', 0, \xi', \xi_n) \\ &\quad \times \partial_{x_n}^\alpha \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l((D_2^* D_1 D_2^*)^{-1})(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx' \end{aligned} \tag{4.2}$$

and the sum is taken over $r + l - k - j - |\alpha| - 1 = -6, r \leq -1, l \leq -3$.

Locally we can use [Theorem 2.2 \(2.44\)](#) to compute the interior term of (4.1), we have

$$\int_M \int_{|\xi|=1} \text{trace}_{\wedge^* T^* M} [\sigma_{-4}((D_2^* D_1)^{-2})] \sigma(\xi) dx = 128\pi^3 \int_M \left[-\frac{16}{3}s - 8(\lambda_1^2 + \lambda_2^2 - 8\lambda_1\lambda_2)|v|^2 + \frac{1}{2} \text{tr}[\lambda_2 \nabla_{e_j}^{TM}(\varepsilon(v^*))c(e_j) - \lambda_1 c(e_j) \nabla_{e_j}^{TM}(l(v))] \right] d\text{Vol}_M. \tag{4.3}$$

So we only need to compute $\int_{\partial M} \Psi$. Let us now turn to compute the specification of $D_2^* D_1 D_2^*$.

$$\begin{aligned} D_2^* D_1 D_2^* &= \sum_{i=1}^n c(e_i) \langle e_i, dx_l \rangle (-g^{ij} \partial_l \partial_i \partial_j) + \sum_{i=1}^n c(e_i) \langle e_i, dx_l \rangle \left\{ -(\partial_l g^{ij}) \partial_i \partial_j - g^{ij} \left(4(\sigma_i + a_i) \partial_j - 2\Gamma_{ij}^k \partial_k \right) \partial_l \right\} \\ &+ \sum_{i=1}^n c(e_i) \langle e_i, dx_l \rangle \left\{ -2(\partial_l g^{ij})(\sigma_i + a_i) \partial_j + g^{ij} (\partial_l \Gamma_{ij}^k) \partial_k - 2g^{ij} [(\partial_l \sigma_i) + (\partial_l a_i)] \partial_j + (\partial_l g^{ij}) \Gamma_{ij}^k \partial_k \right\} \\ &+ \sum_{j,k} \left[\partial_l \left(c(e_j) \lambda_2 \varepsilon(v^*) + \lambda_1 l(v) c(e_j) \right) \right] \times \langle e_j, dx^k \rangle \partial_k + \sum_{j,k} \left(c(e_j) \lambda_2 \varepsilon(v^*) + \lambda_1 l(v) c(e_j) \right) \left[\partial_l \langle e_j, dx^k \rangle \right] \partial_k \left\{ \right. \\ &+ \left. [(\sigma_i + a_i) + \lambda_2 \varepsilon(v^*)] (-g^{ij} \partial_i \partial_j) + \sum_{i=1}^n c(e_i) \langle e_i, dx_l \rangle \left\{ 2 \sum_{j,k} \left(c(e_j) \lambda_2 \varepsilon(v^*) + \lambda_1 l(v) c(e_j) \right) \times \langle \tilde{e}_i, dx_k \rangle \right\} \partial_l \partial_k \right. \\ &+ \left. [(\sigma_i + a_i) + \lambda_2 \varepsilon(v^*)] \left\{ -\sum_{i,j} g^{ij} [2\sigma_i \partial_j + 2a_i \partial_j - \Gamma_{ij}^k \partial_k + (\partial_i \sigma_j) + \frac{1}{4}s + (\partial_i a_j) \right. \right. \\ &+ \left. \left. \sigma_i \sigma_j + \sigma_i a_j + a_i \sigma_j + a_i a_j - \Gamma_{ij}^k \sigma_k - \Gamma_{ij}^k a_k \right] + \sum_{i,j} g^{ij} \left(c(e_j) \lambda_2 \varepsilon(v^*) + \lambda_1 l(v) c(e_j) \right) \partial_j \right. \\ &+ \left. \sum_{i,j} g^{ij} \left[\lambda_1 l(v) c(\partial_i) \sigma_i + \lambda_1 l(v) c(\partial_i) a_i + c(\partial_i) \lambda_2 \partial_i (\varepsilon(v^*)) + c(\partial_i) \sigma_i \lambda_2 \varepsilon(v^*) + c(\partial_i) a_i \lambda_2 \varepsilon(v^*) \right] \right. \\ &\left. + \lambda_1 \lambda_2 l(v) \varepsilon(v^*) - \frac{1}{8} \sum_{ijkl} R_{ijkl} \bar{c}(e_i) \bar{c}(e_j) c(e_k) c(e_l) \right\}. \tag{4.4} \end{aligned}$$

Then, we obtain

Lemma 4.1. *The following identities hold:*

$$\begin{aligned} \sigma_2(D_2^* D_1 D_2^*) &= \sum_{i,j,l} c(dx_l) \partial_l (g^{ij}) \xi_i \xi_j + c(\xi) (4\sigma^k + 4a^k - 2\Gamma^k) \xi_k - 2[\lambda_1 c(\xi) l(v) c(\xi) \\ &- |\xi|^2 \lambda_2 \varepsilon(v^*)] + \frac{1}{4} |\xi|^2 \sum_{s,t,l} \omega_{s,t}(e_l) [c(e_l) \bar{c}(e_s) \bar{c}(e_t) - c(e_l) c(e_s) c(e_t)] + |\xi|^2 \lambda_2 \varepsilon(v^*); \\ \sigma_3(D_2^* D_1 D_2^*) &= ic(\xi) |\xi|^2. \tag{4.5} \end{aligned}$$

Write

$$\sigma(D_2^* D_1 D_2^*) = p_3 + p_2 + p_1 + p_0; \quad \sigma((D_2^* D_1 D_2^*)^{-1}) = \sum_{j=3}^{\infty} q_{-j}. \tag{4.6}$$

By the composition formula of pseudodifferential operators, we have

$$\begin{aligned} 1 &= \sigma((D_2^* D_1 D_2^*) \circ (D_2^* D_1 D_2^*)^{-1}) \\ &= \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} [\sigma(D_2^* D_1 D_2^*)] D_x^{\alpha} [(D_2^* D_1 D_2^*)^{-1}] \\ &= (p_3 + p_2 + p_1 + p_0)(q_{-3} + q_{-4} + q_{-5} + \dots) \\ &\quad + \sum_j (\partial_{\xi_j} p_3 + \partial_{\xi_j} p_2 + \partial_{\xi_j} p_1 + \partial_{\xi_j} p_0) (D_{x_j} q_{-3} + D_{x_j} q_{-4} + D_{x_j} q_{-5} + \dots) \\ &= p_3 q_{-3} + (p_3 q_{-4} + p_2 q_{-3} + \sum_j \partial_{\xi_j} p_3 D_{x_j} q_{-3}) + \dots, \tag{4.7} \end{aligned}$$

by (4.7), we have

$$q_{-3} = p_3^{-1}; \quad q_{-4} = -p_3^{-1} [p_2 p_3^{-1} + \sum_j \partial_{\xi_j} p_3 D_{x_j} (p_3^{-1})]. \tag{4.8}$$

By Lemma 4.1, we have some symbols of operators.

Lemma 4.2. *The following identities hold:*

$$\begin{aligned} \sigma_{-3}((D_2^*D_1D_2^*)^{-1}) &= \frac{ic(\xi)}{|\xi|^4}; \\ \sigma_{-4}((D_2^*D_1D_2^*)^{-1}) &= \frac{c(\xi)\sigma_2(D_2^*D_1D_2^*)c(\xi)}{|\xi|^8} \\ &\quad + \frac{ic(\xi)}{|\xi|^8} \left(|\xi|^4 c(dx_n) \partial_{x_n} c(\xi') - 2h'(0)c(dx_n)c(\xi) + 2\xi_n c(\xi) \partial_{x_n} c(\xi') + 4\xi_n h'(0) \right). \end{aligned} \tag{4.9}$$

In the normal coordinate, $g^{ij}(x_0) = \delta^i_j$ and $\partial_{x_j}(g^{\alpha\beta})(x_0) = 0$, if $j < n$; $\partial_{x_j}(g^{\alpha\beta})(x_0) = h'(0)\delta^\alpha_\beta$, if $j = n$. So by Lemma A.2 in [15], we have $\Gamma^n(x_0) = \frac{5}{2}h'(0)$ and $\Gamma^k(x_0) = 0$ for $k < n$. By the definition of δ^k and Lemma 2.3 in [15], we have $\delta^n(x_0) = 0$ and $\delta^k = \frac{1}{4}h'(0)c(e_k)c(e_n)$ for $k < n$. By Lemma 4.2, we obtain

$$\begin{aligned} \sigma_{-4}((D_2^*D_1D_2^*)^{-1})(x_0)|_{|\xi'|=1} &= \frac{c(\xi)\sigma_2((D_2^*D_1D_2^*)^{-1})(x_0)|_{|\xi'|=1}c(\xi)}{|\xi|^8} - \frac{c(\xi)}{|\xi|^4} \sum_j \partial_{\xi_j} (c(\xi)|\xi|^2) D_{x_j} \left(\frac{ic(\xi)}{|\xi|^4} \right) \\ &= \frac{1}{|\xi|^8} c(\xi) \left(\frac{1}{2} h'(0) c(\xi) \sum_{k < n} \xi_k c(e_k) c(e_n) - \frac{1}{2} h'(0) c(\xi) \sum_{k < n} \xi_k \bar{c}(e_k) \bar{c}(e_n) - \frac{5}{2} h'(0) \xi_n c(\xi) \right. \\ &\quad \left. - \frac{1}{4} h'(0) |\xi|^2 c(dx_n) - 2[c(\xi)\lambda_1 l(\nu)c(\xi) - |\xi|^2 \lambda_2 \varepsilon(\nu^*)] + |\xi|^2 \lambda_2 \varepsilon(\nu^*) \right) c(\xi) \\ &\quad + \frac{ic(\xi)}{|\xi|^8} \left(|\xi|^4 c(dx_n) \partial_{x_n} c(\xi') - 2h'(0)c(dx_n)c(\xi) + 2\xi_n c(\xi) \partial_{x_n} c(\xi') + 4\xi_n h'(0) \right). \end{aligned} \tag{4.10}$$

From the remark above, now we can compute Ψ (see formula (4.2) for the definition of Ψ). We use tr as shorthand of trace. Since $n = 6$, $\text{tr}_{\wedge^* T^* M}[\text{id}] = 64$. Since the sum is taken over $r + \ell - k - j - |\alpha| - 1 = -6$, $r \leq -1$, $\ell \leq -3$, we have the $\int_{\partial M} \Psi$ is the sum of the following five cases:

case (a) (I) $r = -1$, $l = -3$, $j = k = 0$, $|\alpha| = 1$.

By (4.2), we get

$$\text{case (a) (I)} = - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr} \left[\partial_{\xi'}^{\alpha} \pi_{\xi_n}^+ \sigma_{-1}(D_1^{-1}) \times \partial_{x_n}^{\alpha} \partial_{\xi_n} \sigma_{-3}((D_2^*D_1D_2^*)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \tag{4.11}$$

By Lemma 4.2, for $i < n$, we have

$$\partial_{x_i} \sigma_{-3}((D_2^*D_1D_2^*)^{-1})(x_0) = \partial_{x_i} \left[\frac{ic(\xi)}{|\xi|^4} \right] (x_0) = i \partial_{x_i} [c(\xi)] |\xi|^{-4}(x_0) - 2ic(\xi) \partial_{x_i} [|\xi|^2] |\xi|^{-6}(x_0) = 0. \tag{4.12}$$

so case (a) (I) vanishes.

case (a) (II) $r = -1$, $l = -3$, $|\alpha| = k = 0$, $j = 1$.

By (4.2), we have

$$\text{case (a) (II)} = - \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} \left[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(D_1^{-1}) \times \partial_{\xi_n}^2 \sigma_{-3}((D_2^*D_1D_2^*)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \tag{4.13}$$

By Lemma 4.2 and direct calculations, we have

$$\partial_{\xi_n}^2 \sigma_{-3}((D_2^*D_1D_2^*)^{-1}) = i \left[\frac{(20\xi_n^2 - 4)c(\xi') + 12(\xi_n^3 - \xi_n)c(dx_n)}{(1 + \xi_n^2)^4} \right]. \tag{4.14}$$

Since $n = 6$, $\text{tr}[-\text{id}] = -64$. By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, then

$$\begin{aligned} \text{tr}[c(\xi')c(dx_n)] &= 0; \text{tr}[c(dx_n)^2] = -64; \text{tr}[c(\xi')^2](x_0)|_{|\xi'|=1} = -64; \\ \text{tr}[\partial_{x_n}[c(\xi')]c(dx_n)] &= 0; \text{tr}[\partial_{x_n}c(\xi')c(\xi')](x_0)|_{|\xi'|=1} = -32h'(0). \end{aligned} \tag{4.15}$$

By (3.31), (4.14) and (4.15), we get

$$\text{tr} \left[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(D_1^{-1}) \times \partial_{\xi_n}^2 \sigma_{-3}((D_2^*D_1D_2^*)^{-1}) \right] (x_0) = 64h'(0) \frac{-1 - 3\xi_n i + 5\xi_n^2 + 3i\xi_n^3}{(\xi_n - i)^6(\xi_n + i)^4}. \tag{4.16}$$

Then we obtain

$$\begin{aligned} \text{case (a) (II)} &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} h'(0) \frac{-8 - 24\xi_n i + 40\xi_n^2 + 24i\xi_n^3}{(\xi_n - i)^6 (\xi_n + i)^4} d\xi_n \sigma(\xi') dx' \\ &= -\frac{15}{2} \pi h'(0) \Omega_4 dx', \end{aligned} \tag{4.17}$$

where Ω_4 is the canonical volume of S^4 .

case (a) (III) $r = -1, l = -3, |\alpha| = j = 0, k = 1$.

By (4.2), we have

$$\text{case (a) (III)} = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} \left[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1} (D_1^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-3} ((D_2^* D_1 D_2^*)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \tag{4.18}$$

By Lemma 4.2 and direct calculations, we have

$$\partial_{\xi_n} \partial_{x_n} \sigma_{-3} ((D_2^* D_1 D_2^*)^{-1}) = -\frac{4i\xi_n \partial_{x_n} c(\xi') (x_0)}{(1 + \xi_n^2)^3} + i \frac{12h'(0)\xi_n c(\xi')}{(1 + \xi_n^2)^4} - i \frac{(2 - 10\xi_n^2)h'(0)c(dx_n)}{(1 + \xi_n^2)^4}. \tag{4.19}$$

Combining (3.31) and (4.19), we have

$$\text{tr} \left[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1} (D_1^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-3} ((D_2^* D_1 D_2^*)^{-1}) \right] (x_0) |_{|\xi'|=1} = 8h'(0) \frac{8i - 32\xi_n - 8i\xi_n^2}{(\xi_n - i)^5 (\xi + i)^4}. \tag{4.20}$$

Then

$$\begin{aligned} \text{case (a) III)} &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} 8h'(0) \frac{8i - 32\xi_n - 8i\xi_n^2}{(\xi_n - i)^5 (\xi + i)^4} d\xi_n \sigma(\xi') dx' \\ &= \frac{25}{2} \pi h'(0) \Omega_4 dx'. \end{aligned} \tag{4.21}$$

case (b) $r = -1, l = -4, |\alpha| = j = k = 0$.

By (4.2), we have

$$\begin{aligned} \text{case (b)} &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} \left[\pi_{\xi_n}^+ \sigma_{-1} (D_1^{-1}) \times \partial_{\xi_n} \sigma_{-4} ((D_2^* D_1 D_2^*)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\ &= i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} [\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1} (D_1^{-1}) \times \sigma_{-4} ((D_2^* D_1 D_2^*)^{-1})] (x_0) d\xi_n \sigma(\xi') dx'. \end{aligned} \tag{4.22}$$

By (3.31) and (4.23), we have

$$\begin{aligned} &\text{tr} [\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1} (D_1^{-1}) \times \sigma_{-4} (D_2^* D_1 D_2^*)^{-1}] (x_0) |_{|\xi'|=1} \\ &= \frac{1}{2(\xi_n - i)^2 (1 + \xi_n^2)^4} \left(\frac{3}{4}i + 2 + (3 + 4i)\xi_n + (-6 + 2i)\xi_n^2 + 3\xi_n^3 + \frac{9i}{4}\xi_n^4 \right) h'(0) \text{tr}[id] \\ &\quad + \frac{1}{2(\xi_n - i)^2 (1 + \xi_n^2)^4} (-1 - 3i\xi_n - 2\xi_n^2 - 4i\xi_n^3 - \xi_n^4 - i\xi_n^5) \text{tr}[c(\xi') \partial_{x_n} c(\xi')] \\ &\quad - \frac{1}{2(\xi_n - i)^2 (1 + \xi_n^2)^4} \left(\frac{1}{2}i + \frac{1}{2}\xi_n + \frac{1}{2}\xi_n^2 + \frac{1}{2}\xi_n^3 \right) \text{tr}[c(\xi') \bar{c}(\xi') c(dx_n) \bar{c}(dx_n)] \\ &\quad + \text{tr} \left[\pi_{\xi_n}^+ \sigma_{-1} (D_1^{-1}) \times \partial_{\xi_n} \left(\frac{3c(\xi) \lambda_2 \varepsilon(v^*) c(\xi)}{|\xi|^6} - \frac{2\lambda_1 l(v)}{|\xi|^4} \right) \right] (x_0) |_{|\xi'|=1} \end{aligned} \tag{4.23}$$

By direct calculations, we have

$$\begin{aligned} &\text{tr} \left[\pi_{\xi_n}^+ \sigma_{-1} (D_1^{-1}) \times \partial_{\xi_n} \left(\frac{3c(\xi) \lambda_2 \varepsilon(v^*) c(\xi)}{|\xi|^6} - \frac{2\lambda_1 l(v)}{|\xi|^4} \right) \right] (x_0) |_{|\xi'|=1} \\ &= \frac{3(4i\xi_n + 2)i}{2(\xi_n + i)(1 + \xi_n^2)^3} \text{tr}[\lambda_2 \varepsilon(v^*) c(\xi')] + \frac{3(4i\xi_n + 2)}{2(\xi_n + i)(1 + \xi_n^2)^3} \text{tr}[\lambda_2 \varepsilon(v^*) c(dx_n)] \\ &\quad + \frac{4\xi_n}{2(\xi_n - i)(1 + \xi_n^2)^3} \text{tr}[\lambda_1 l(v) c(\xi')] + \frac{4\xi_n i}{2(\xi_n - i)(1 + \xi_n^2)^3} \text{tr}[\lambda_1 l(v) c(dx_n)]. \end{aligned} \tag{4.24}$$

By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, then we have the following equalities:

$$\begin{aligned} \text{tr}[c(dx_n)l(v)] &= 32\langle dx_n, v \rangle; \quad \text{tr}[c(\xi')l(v)] = 32\langle \xi', v \rangle; \\ \text{tr}[c(dx_n)\varepsilon(v^*)] &= -32\langle v^*, \frac{\partial}{\partial x_n} \rangle; \quad \text{tr}[c(\xi')\varepsilon(v^*)] = -32\langle v^*, g(\xi', \cdot) \rangle; \\ \text{tr}[c(e_i)\bar{c}(e_i)c(e_n)\bar{c}(e_n)] &= 0 \quad (i < n). \end{aligned} \tag{4.25}$$

So

$$\text{tr}[c(\xi')\bar{c}(\xi')c(dx_n)\bar{c}(dx_n)] = \sum_{i < n, j < n} \text{tr}[\xi_i \xi_j c(e_i)\bar{c}(e_j)c(dx_n)\bar{c}(dx_n)] = 0. \tag{4.26}$$

By (4.25), then we have

$$\begin{aligned} \text{case (b)} &= ih'(0) \int_{|\xi'|=1} \int_{-\infty}^{+\infty} 64 \times \frac{\frac{3}{4}i + 2 + (3 + 4i)\xi_n + (-6 + 2i)\xi_n^2 + 3\xi_n^3 + \frac{9i}{4}\xi_n^4}{2(\xi_n - i)^5(\xi_n + i)^4} d\xi_n \sigma(\xi') dx' \\ &\quad + ih'(0) \int_{|\xi'|=1} \int_{-\infty}^{+\infty} 32 \times \frac{1 + 3i\xi_n + 2\xi_n^2 + 4i\xi_n^3 + \xi_n^4 + i\xi_n^5}{2(\xi_n - i)^2(1 + \xi_n^2)^4} d\xi_n \sigma(\xi') dx' \\ &\quad - i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} \left[\pi_{\xi_n}^+ \sigma_{-1}(D_1^{-1}) \times \partial_{\xi_n} \left(\frac{3c(\xi)\lambda_2 \varepsilon(v^*)c(\xi)}{|\xi|^6} - \frac{2\lambda_1 l(v)}{|\xi|^4} \right) \right] (x_0) \Big|_{|\xi'|=1} d\xi_n \sigma(\xi') dx' \\ &= \left(-\frac{41}{8}i - \frac{195}{8} \right) \pi h'(0) \Omega_4 dx' + (4\lambda_1 + 18\lambda_2) \pi \langle dx_n, v \rangle \Omega_4 dx'. \end{aligned} \tag{4.27}$$

case (c) $r = -2, l = -3, |\alpha| = j = k = 0$.

By (4.2), we have

$$\text{case (c)} = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} \left[\pi_{\xi_n}^+ \sigma_{-2}(D_1^{-1}) \times \partial_{\xi_n} \sigma_{-3}((D_2^* D_1 D_2^*)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \tag{4.28}$$

By (3.36) and (3.37), we have

$$\begin{aligned} \pi_{\xi_n}^+ \sigma_{-2}(D_1^{-1}(x_0)) \Big|_{|\xi'|=1} &= \pi_{\xi_n}^+ \left[\frac{c(\xi)A(x_0)c(\xi)}{(1 + \xi_n^2)^2} \right] + \pi_{\xi_n}^+ \left[\frac{c(\xi)(\lambda_1 l(v)(x_0))c(\xi)}{(1 + \xi_n^2)^2} \right] \\ &\quad + \pi_{\xi_n}^+ \left[\frac{c(\xi)B(x_0)c(\xi) + c(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0)}{(1 + \xi_n^2)^2} - h'(0) \frac{c(\xi)c(dx_n)c(\xi)}{(1 + \xi_n^2)^3} \right]. \end{aligned} \tag{4.29}$$

By (3.40), we have

$$\pi_{\xi_n}^+ \left[\frac{c(\xi)A(x_0)c(\xi)}{(1 + \xi_n^2)^2} \right] = -\frac{c(\xi')A(x_0)c(\xi')(2 + i\xi_n)}{4(\xi_n - i)^2} + \frac{ic(\xi')A(x_0)c(dx_n)}{4(\xi_n - i)^2} + \frac{ic(dx_n)A(x_0)c(\xi')}{4(\xi_n - i)^2} + \frac{-i\xi_n c(dx_n)A(x_0)c(dx_n)}{4(\xi_n - i)^2}. \tag{4.30}$$

By (3.45)-(3.47), we have

$$\pi_{\xi_n}^+ \left[\frac{c(\xi)B(x_0)c(\xi) + c(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0)}{(1 + \xi_n^2)^2} \right] - h'(0) \pi_{\xi_n}^+ \left[\frac{c(\xi)c(dx_n)c(\xi)}{(1 + \xi_n^2)^3} \right] := P_1 - P_2, \tag{4.31}$$

where

$$\begin{aligned} P_1 &= \frac{-1}{4(\xi_n - i)^2} [(2 + i\xi_n)c(\xi')b_0^2(x_0)c(\xi') + i\xi_n c(dx_n)b_0^2(x_0)c(dx_n) \\ &\quad + (2 + i\xi_n)c(\xi')c(dx_n)\partial_{x_n}c(\xi') + ic(dx_n)b_0^2(x_0)c(\xi') + ic(\xi')b_0^2(x_0)c(dx_n) - i\partial_{x_n}c(\xi')] \end{aligned} \tag{4.32}$$

and

$$P_2 = \frac{h'(0)}{2} \left[\frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3} [ic(\xi') - c(dx_n)] \right]. \tag{4.33}$$

Similar to (4.30), we have

$$\begin{aligned} \pi_{\xi_n}^+ \left[\frac{c(\xi)(l(v)(x_0))c(\xi)}{(1 + \xi_n^2)^2} \right] &= -\frac{c(\xi')(l(v)(x_0))c(\xi')(2 + i\xi_n)}{4(\xi_n - i)^2} + \frac{ic(\xi')(l(v)(x_0))c(dx_n)}{4(\xi_n - i)^2} + \frac{ic(dx_n)(l(v)(x_0))c(\xi')}{4(\xi_n - i)^2} \\ &\quad + \frac{-i\xi_n c(dx_n)(l(v)(x_0))c(dx_n)}{4(\xi_n - i)^2}. \end{aligned} \tag{4.34}$$

On the other hand,

$$\partial_{\xi_n} \sigma_{-3}((D_2^* D_1 D_2^*)^{-1}) = \frac{-4i\xi_n c(\xi')}{(1 + \xi_n^2)^3} + \frac{i(1 - 3\xi_n^2)c(dx_n)}{(1 + \xi_n^2)^3}. \tag{4.35}$$

By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, then we have equalities:

$$\text{tr}[Ac(dx_n)] = 0; \quad \text{tr}[\bar{c}(\xi')\bar{c}(dx_n)] = 0. \tag{4.36}$$

Then we have

$$\text{tr} \left[\pi_{\xi_n}^+ \left(\frac{c(\xi)A(x_0)c(\xi)}{(1 + \xi_n^2)^2} \right) \times \partial_{\xi_n} \sigma_{-3}((D_2^*D_1D_2^*)^{-1})(x_0) \right] \Big|_{|\xi'|=1} = \frac{2 - 8i\xi_n - 6\xi_n^2}{4(\xi_n - i)^2(1 + \xi_n^2)^3} \text{tr}[A(x_0)c(\xi')], \tag{4.37}$$

We note that $i < n$, $\int_{|\xi'|=1} \{\xi_{i_1}\xi_{i_2} \cdots \xi_{i_{2d+1}}\} \sigma(\xi') = 0$, so $\text{tr}[A(x_0)c(\xi')]$ has no contribution for computing case c).

By (4.32) and (4.35), we have

$$\begin{aligned} & \text{tr}[P_1 \times \partial_{\xi_n} \sigma_{-3}((D_2^*D_1D_2^*)^{-1})(x_0)] \Big|_{|\xi'|=1} \\ &= \text{tr} \left\{ \frac{1}{4(\xi_n - i)^2} \left[\frac{5}{2} h'(0)c(dx_n) - \frac{5i}{2} h'(0)c(\xi') - (2 + i\xi_n)c(\xi')c(dx_n)\partial_{\xi_n}c(\xi') + i\partial_{\xi_n}c(\xi') \right] \times \frac{-4i\xi_n c(\xi') + (i - 3i\xi_n^2)c(dx_n)}{(1 + \xi_n^2)^3} \right\} \\ &= 8h'(0) \frac{3 + 12i\xi_n + 3\xi_n^2}{(\xi_n - i)^4(\xi_n + i)^3}. \end{aligned} \tag{4.38}$$

By (4.33) and (4.35), we have

$$\begin{aligned} & \text{tr}[P_2 \times \partial_{\xi_n} \sigma_{-3}((D_2^*D_1D_2^*)^{-1})(x_0)] \Big|_{|\xi'|=1} \\ &= \text{tr} \left\{ \frac{h'(0)}{2} \left[\frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3} [ic(\xi') - c(dx_n)] \right] \times \frac{-4i\xi_n c(\xi') + (i - 3i\xi_n^2)c(dx_n)}{(1 + \xi_n^2)^3} \right\} \\ &= 8h'(0) \frac{4i - 11\xi_n - 6i\xi_n^2 + 3\xi_n^3}{(\xi_n - i)^5(\xi_n + i)^3}. \end{aligned} \tag{4.39}$$

By (4.34) and (4.35), we have

$$\begin{aligned} & \text{tr} \left[\pi_{\xi_n}^+ \left(\frac{c(\xi)[\lambda_1 l(v)(x_0)]c(\xi)}{(1 + \xi_n^2)^2} \right) \times \partial_{\xi_n} \sigma_{-3}((D_2^*D_1D_2^*)^{-1})(x_0) \right] \Big|_{|\xi'|=1} \\ &= \frac{-2 - 8i\xi_n + 6\xi_n^2}{4(\xi_n - i)^2(1 + \xi_n^2)^3} \text{tr}[\lambda_1 l(v)(x_0)c(\xi')] + \frac{-2i + 8\xi_n + 6i\xi_n^2}{4(\xi_n - i)^2(1 + \xi_n^2)^3} \text{tr}[\lambda_1 c(dx_n)l(v)(x_0)]. \end{aligned} \tag{4.40}$$

By (4.25), we have

$$\begin{aligned} \text{case (c)} &= -ih'(0) \int_{|\xi'|=1} \int_{-\infty}^{+\infty} 8 \times \frac{-7i + 26\xi_n + 15i\xi_n^2}{(\xi_n - i)^5(\xi_n + i)^3} d\xi_n \sigma(\xi') dx' \\ &\quad - i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} \left[\pi_{\xi_n}^+ \left(\frac{c(\xi)[\lambda_1 l(v)(x_0)]c(\xi)}{(1 + \xi_n^2)^2} \right) \times \partial_{\xi_n} \sigma_{-3}((D_2^*D_1D_2^*)^{-1})(x_0) \right] \Big|_{|\xi'|=1} d\xi_n \sigma(\xi') dx' \\ &= -8ih'(0) \times \frac{2\pi i}{4!} \left[\frac{-7i + 26\xi_n + 15i\xi_n^2}{(\xi_n + i)^3} \right]^{(5)} \Big|_{\xi_n=i} \Omega_4 dx' + (9\pi i - 4\pi) (\lambda_1 \langle dx_n, v \rangle - i\lambda_1 \langle \xi', v \rangle) \Omega_4 dx' \\ &= \left[\frac{55}{2} \pi h'(0) + (9\pi i - 4\pi) \lambda_1 \langle dx_n, v \rangle \right] \Omega_4 dx'. \end{aligned} \tag{4.41}$$

Since Ψ is the sum of the cases a), b) and c),

$$\Psi = \frac{(65 - 41i)\pi h'(0)}{8} \Omega_4 dx' + (18\pi \lambda_2 + 9\lambda_1 \pi i) \langle dx_n, v \rangle \Omega_4 dx'.$$

Theorem 4.3. Let M be a 6-dimensional compact oriented manifold with the boundary ∂M and the metric g^M as above, D_i and D_i^* ($i = 1, 2$) be statistical de Rham Hodge Operators on \bar{M} , then

$$\begin{aligned} \widetilde{\text{Wres}}[\pi^+ D_1^{-1} \circ \pi^+ (D_2^* D_1 D_2^*)^{-1}] &= 128\pi^3 \int_M \left[-\frac{16}{3}s - 8(\lambda_1^2 + \lambda_2^2 - 8\lambda_1\lambda_2)|v|^2 + \frac{1}{2} \text{tr}[\lambda_2 \nabla_{e_j}^{TM}(\varepsilon(v^*))c(e_j) \right. \\ &\quad \left. - \lambda_1 c(e_j) \nabla_{e_j}^{TM}(l(v))] \right] d\text{Vol}_M + \int_{\partial M} \left[\frac{(65 - 41i)\pi h'(0)}{8} \Omega_4 dx' \right. \\ &\quad \left. + (18\pi \lambda_2 + 9\lambda_1 \pi i) \langle dx_n, v \rangle \right] \Omega_4 dx', \end{aligned} \tag{4.42}$$

where s is the scalar curvature.

Corollary 4.4. When $\lambda_1 = \lambda_2 = 1$, we get for a 6-dimensional oriented compact manifold M with the boundary ∂M

$$\begin{aligned} \widetilde{\text{Wres}}[\pi^+ D^{-1} \circ \pi^+ (D^* D D^*)^{-1}] &= 128\pi^3 \int_M \left[-\frac{16}{3}s + 48|v|^2 + \frac{1}{2} \text{tr}[\nabla_{e_j}^{TM}(\varepsilon(v^*))c(e_j) - c(e_j)\nabla_{e_j}^{TM}(l(v))] \right] d\text{Vol}_M \\ &+ \int_{\partial M} \left(\frac{(65 - 41i)\pi h'(0)}{8} \Omega_4 dx' + (18\pi + 9\pi i) \langle dx_n, v \rangle \Omega_4 dx' \right), \end{aligned}$$

where s is the scalar curvature.

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

AUTHORS' CONTRIBUTION

SW contributed in study conceptualization and writing (review and editing) the manuscript, data curation, formal analysis and writing (original draft). YW contributed in funding acquisition and project administration, supervised the project.

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