Research Article

Similarity Measures and Multi-person TOPSIS Method Using m-polar Single-Valued Neutrosophic Sets

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ABSTRACT

In this paper, we give a new notion of the m-polar single-valued neutrosophic sets (m-PSVNSs) which is a hybrid of the single-valued neutrosophic sets (SVNSs) and the m-polar fuzzy sets (m-PFSs) and study several of the structure operations including subset, equal, union, intersection, and complement. Subsequently, we present the basic definitions, theorems, and examples on m-PSVNSs. Also, we define the certain distance between two m-PSVNSs and a novel similarity measure for m-PSVNSs based on distances. A multi criteria decision-making (MCDM) problem is animated for m-PSVNS data that takes into account the distances for the best alternative (solution) by an application of similarity measure for m-PSVNSs in brand recognition. Finally, we construct a new methodology to extend the TOPSIS to m-PSVNS and illustrate its applicability via a numerical example.

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1. INTRODUCTION

Zadeh [1] defined the concept of fuzzy set (FS) as the generalization of crisp set theory and highlighted the complications of some models (e.g., educational mathematical and crisp set theory, which are unable to handle the difficulty of the statistics containing doubts) with the approach of membership. Smarandache [2,3] defined the neutrosophic set (NS) and gave several operations on NS. Wang et al. [4] defined the single valued neutrosophic set (SVNS) by separately assigning the degree of truth membership, the degree of indeterminacy membership, and the degree of falsity membership. Zhang [5] proposed the concept of bipolar fuzzy sets (BFSs) and relations. Chen et al. [6] presented the concept of m-polar fuzzy set (m-PFS) as a simplified type of BFS. The notions of m-PF graph, characterization of m-PF graph, and metrics in m-PF graph were introduced in [7–10]. A multi-criteria decision-making (MCDM) method on m-PF soft rough sets, hybrid m-PF models with m-PF ELECTRE-I were presented in [11–13].

A similarity measure (SM) for the fuzzy system plays a very considerable role in handling problems that comprehend ambiguous information, but unable to deal with the unclarness and uncertainty of the problems having normal information. The idea of SM of two sets (i.e., fuzzy values and vague sets) were discussed in [14,15], but it had failed to hold in some problems. To resolve this problem, Hong and Kim [16] brought into light some modified measures. Several SM based on SVNSs were proposed in [17–20]. Akram and Waseem [21] implemented a m-PFS and m-PF soft set on SMs to medical diagnosis. Yong [22] gave the new approach of the selection of location for plantation under the terms of the linguistic environment through graded mean reparation based on fuzzy TOPSIS. Adeel et al. [23] animates m-PF through the extension of order preference of MCGDM-TOPSIS for the best alternative. Shih et al. [24] described the technique of order preference through similarity using MADM in a group decision environment based on group preference for TOPSIS. Saeed et al. [25] gave an application in SM on multipoar neutrosophic soft sets structure in decision-making for medical diagnosis. Kang et al. [26] presented an application in BCK/BCI-algebras on multipolar IFS with finite degree. Akram [27,28] defined the notions of single-valued neutrosophic graphs and m-polar fuzzy graphs. There are many published papers on TOPSIS in different fields (for instance, [29–33]).

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In normal practice, the problem of SVN information occurs (i.e., which cannot be elaborated well using the existing approaches), while an m-polar single-valued neutrosophic set (m-PSVNS) is used to resolve the uncertain and more variant data, specifically in the SVNS form with m-PFS. Our motivation is to be able to find a solution of many daily life problems enhanced through distance-based SM with the m-PSVNS. It increases the number of applications in various fields, including electronic optimization, industries, and forensic facial portrait. Moreover, a new approach for the best m-PSVNS alternatives based on distance SMs in MCDM is animated.

This article is structured as follows: In Section 2, a brief overview of some fundamental concepts is provided. In Section 3, the concept of m-PSVNS and its basic operations are defined. In Section 4, distance measure formulas are presented on m-PSVNS. Further, the m-PSVNS is used to investigate a problem involving distance-based SMs with an algorithm. In Section 5, the MCDM for m-PSVN data is described with an algorithm for the best solution (alternative). The article is concluded with a review and further work outlook.

2. PRELIMINARIES
We give a short survey of concepts of BFSs, m-PFSs, and SVNSs as indicated below.

2.1. BFSs and m-PFSs

**Definition 1.** (cf. [5]). Suppose that $Z$ (i.e., $Z = \{z_1, z_2, ..., z_r | r = 1, 2, ..., n\}$, $n \in N$ (natural Numbers)) be a set of elements. A BFS on $Z$ is a pair $(\alpha, \beta)$, where

$$
\alpha : Z \rightarrow [0, 1] \text{ and } \beta : Z \rightarrow [-1, 0]
$$

are two mappings. We note that BFS are an extension of FSs whose membership degree range is $[-1, 1]$.

**Definition 2.** (cf. [6]). An m-PFS on $Z$ (i.e., $Z = \{z_1, z_2, ..., z_r\}$) be a set of elements, is a mapping

$$
\alpha : Z \rightarrow [0, 1]^m.
$$

**Example 1.**
Suppose that 3-PFS on $Z$ (i.e., $Z = \{z_1, z_2, z_3\}$) is defined by

$$
\alpha(z_1) = (0.45, 0.42, 0.59), \ \alpha(z_2) = (0.51, 0.52, 0.7), \ \text{and} \ \alpha(z_3) = (0.4, 0.53, 0.9).
$$

2.2. SVNSs

**Definition 3.** (cf. [4]). A SVNS $\Phi = (\alpha, \beta, \gamma)$ on $Z$ (i.e., $Z = \{z_1, z_2, ..., z_r\}$) be a set of elements, is a mapping

$$(\alpha, \beta, \gamma) : Z \rightarrow [0, 1] \times [0, 1] \times [0, 1],$$

where $\alpha : Z \rightarrow [0, 1]$ (i.e., the degree of truth membership), $\beta : Z \rightarrow [0, 1]$ (i.e., the degree of indeterminacy membership), and $\gamma : Z \rightarrow [0, 1]$ (i.e., the degree of falsity membership) are FSs over $Z$ such that $0 \leq a_{\Phi}(z_r) + b_{\Phi}(z_r) + \gamma_{\Phi}(z_r) \leq 3$ ($\forall z_r \in Z$, $r = 1, 2, ..., n$) or

$$
\Phi = \left\{ \frac{(a_{\Phi}(z_r), b_{\Phi}(z_r), \gamma_{\Phi}(z_r))}{z_r} \right\} \quad z_r \in Z.
$$

**Definition 4.** (cf. [4]). Let

$$
\Phi = \left\{ \frac{(a_{\Phi}(z_r), b_{\Phi}(z_r), \gamma_{\Phi}(z_r))}{z_r} \right\} \quad z_r \in Z
$$

and

$$
\Psi = \left\{ \frac{(\xi_{\Psi}(z_r), \eta_{\Psi}(z_r), \epsilon_{\Psi}(z_r))}{z_r} \right\} \quad z_r \in Z
$$

are SVNSs. The following five operations are defined by
Definition 7. (Continued from Example 2). Suppose that $D = \{C \cup \{z_r \mid z_r \in Z\}\}$.

Definition 5. An m-PSVNS $\Phi$ on $Z$ (i.e., $Z = \{z_1, z_2, ..., z_n\}$ be a set of elements), is a mapping

$$\Phi : Z \rightarrow [0, 1]^m \times [0, 1]^m \times [0, 1]^m.$$ 

It can be written as

$$\Phi = \left\{ \left( (p_1 \circ \alpha_p(z_r), p_1 \circ \beta_p(z_r), p_1 \circ \gamma_p(z_r)), \; (p_2 \circ \alpha_p(z_r), p_2 \circ \beta_p(z_r), p_2 \circ \gamma_p(z_r)), \; ... , \; (p_k \circ \alpha_p(z_r), p_k \circ \beta_p(z_r), p_k \circ \gamma_p(z_r)) \right) \mid z_r \in Z \right\},$$

where $(p_k \circ \alpha_p, p_k \circ \beta_p, p_k \circ \gamma_p) : ([0, 1] \times [0, 1] \times [0, 1]) \rightarrow [0, 1] \times [0, 1] \times [0, 1]$ is the kth projection mapping, $p_k \circ \alpha_p$ is the kth truth membership value, $p_k \circ \beta_p$ is the kth indeterminacy membership value, and $p_k \circ \gamma_p$ is the kth falsity membership value of element in $\Phi = \{ (p_1 \circ \alpha_p(z_r), p_1 \circ \beta_p(z_r), p_1 \circ \gamma_p(z_r)), \; (p_2 \circ \alpha_p(z_r), p_2 \circ \beta_p(z_r), p_2 \circ \gamma_p(z_r)), \; ... , \; (p_k \circ \alpha_p(z_r), p_k \circ \beta_p(z_r), p_k \circ \gamma_p(z_r)) \}$ such that $0 \leq p_k \circ \alpha_p(z_r) + p_k \circ \beta_p(z_r) + p_k \circ \gamma_p(z_r) \leq 3$.

Example 2. Suppose that an m-PSVNS on three of elements $Z = \{z_1, z_2, z_3\}$ is defined by

$$\Phi = \left\{ \left( (0.6, 0.3, 0.7), (0.4, 0.2, 0.8) \right), \; (0.1, 0.7, 0.9), \; (0.4, 0.3, 0.8) \right\} \times \left\{ (0.1, 0.3, 0.7), (0.2, 0.4, 0.7) \right\} \times \left\{ z_1, z_2, z_3 \right\}.$$ 

Definition 6. Assume that $\Phi$ and $\Psi$ are m-PSVNS on a set of elements $Z$, where

$$\Phi = \left\{ \left( (p_1 \circ \alpha_p(z_r), p_1 \circ \beta_p(z_r), p_1 \circ \gamma_p(z_r)), \; (p_2 \circ \alpha_p(z_r), p_2 \circ \beta_p(z_r), p_2 \circ \gamma_p(z_r)), \; ... , \; (p_k \circ \alpha_p(z_r), p_k \circ \beta_p(z_r), p_k \circ \gamma_p(z_r)) \right) \mid z_r \in Z \right\},$$

and

$$\Psi = \left\{ \left( (p_1 \circ \zeta_p(z_r), p_1 \circ \eta_p(z_r), p_1 \circ \epsilon_p(z_r)), \; (p_2 \circ \zeta_p(z_r), p_2 \circ \eta_p(z_r), p_2 \circ \epsilon_p(z_r)), \; ... , \; (p_k \circ \zeta_p(z_r), p_k \circ \eta_p(z_r), p_k \circ \epsilon_p(z_r)) \right) \mid z_r \in Z \right\}. $$

Then, $\Phi \subseteq \Psi$ (i.e., $\Phi$ is an m-PSVN subset of $\Psi$) if for all $z_r \in Z$, $p_k \circ \alpha_p(z_r) \leq p_k \circ \zeta_p(z_r)$, $p_k \circ \beta_p(z_r) \geq p_k \circ \eta_p(z_r)$, and $p_k \circ \gamma_p(z_r) \geq p_k \circ \epsilon_p(z_r)$.

Example 3. (Continued from Example 2). Suppose that $\Psi$ be an m-PSVNS on three of elements $Z = \{z_1, z_2, z_3\}$ is defined by

$$\Psi = \left\{ \left( (0.7, 0.3, 0.4), (0.5, 0.1, 0.5) \right), \; (0.2, 0.6, 0.4), \; (0.5, 0.2, 0.4) \right\} \times \left\{ (0.6, 0.1, 0.6), (0.4, 0.4, 0.5) \right\} \times \left\{ z_1, z_2, z_3 \right\}.$$ 

Hence, $\Phi \subseteq \Psi$ ($\forall z_r \in Z, r = 1, 2, 3$).

Definition 7. Assume that $\Phi$ and $\Psi$ are m-PSVNS on a set of elements $Z$, where

$$\Phi = \left\{ \left( (p_1 \circ \alpha_p(z_r), p_1 \circ \beta_p(z_r), p_1 \circ \gamma_p(z_r)), \; (p_2 \circ \alpha_p(z_r), p_2 \circ \beta_p(z_r), p_2 \circ \gamma_p(z_r)), \; ... , \; (p_k \circ \alpha_p(z_r), p_k \circ \beta_p(z_r), p_k \circ \gamma_p(z_r)) \right) \mid z_r \in Z \right\}.$$
and
\[
\Psi = \left\{ \left( p_i \circ \zeta z_i, p_i \circ \eta z_i, p_i \circ \epsilon z_i \right) \mid z_i \in Z \right\}.
\]

Then, \( \Phi = \Psi \) (i.e., \( \Phi \) is an m-PSVN equal of \( \Psi \)) if \( \Phi \subseteq \Psi \) and \( \Phi \supseteq \Psi \).

**Definition 8.** Assume that \( \Phi \) be an m-PSVNS on a set of elements \( Z \), where
\[
\Phi = \left\{ \left( p_i \circ \alpha \alpha_i, p_i \circ \beta \beta_i, p_i \circ \gamma \gamma_i \right) \mid z_i \in Z \right\}.
\]
Then,

1. \( \Phi \) is called an m-PSVN null set (denoted by \( \hat{\Phi} \)), is defined by
   \[
   \hat{\Phi} = \left\{ \frac{(0, 1, 1)}{z_i} \mid z_i \in Z \right\}.
   \]
2. \( \Phi \) is called an m-PSVN absolute set (denoted by \( \hat{Z} \)), is defined by
   \[
   \hat{Z} = \left\{ \frac{(1, 0, 0)}{z_i} \mid z_i \in Z \right\}.
   \]

**Example 4.** (Continued from Example 2). Then, an 2-PSVNSs \( \hat{\Phi} \) and \( \hat{Z} \) are defined by
\[
\hat{\Phi} = \left\{ \frac{((0, 1, 1), (0, 1, 1))}{z_1}, \frac{((0, 1, 1), (0, 1, 1))}{z_2}, \frac{((0, 1, 1), (0, 1, 1))}{z_3} \right\}
\]
and
\[
\hat{Z} = \left\{ \frac{((1, 0, 0), (1, 0, 0))}{z_1}, \frac{((1, 0, 0), (1, 0, 0))}{z_2}, \frac{((1, 0, 0), (1, 0, 0))}{z_3} \right\}.
\]

**Definition 9.** Assume that \( \Phi \) and \( \Psi \) are m-PSVNSs on a set of elements \( Z \), where
\[
\Phi = \left\{ \left( p_i \circ \alpha \alpha_i, p_i \circ \beta \beta_i, p_i \circ \gamma \gamma_i \right) \mid z_i \in Z \right\},
\]
and
\[
\Psi = \left\{ \left( p_i \circ \zeta z_i, p_i \circ \eta z_i, p_i \circ \epsilon z_i \right) \mid z_i \in Z \right\}.
\]

1. The intersection \( \Phi \cap \Psi \), is defined as
   \[
   \Phi \cap \Psi = \left\{ \frac{p_i \circ \alpha \alpha_i \land p_i \circ \beta \beta_i \lor p_i \circ \gamma \gamma_i}{z_i} \mid z_i \in Z, k = 1, 2, \ldots, m \right\}.
   \]
2. The union \( \Phi \cup \Psi \), is defined as
   \[
   \Phi \cup \Psi = \left\{ \frac{p_i \circ \alpha \alpha_i \lor p_i \circ \beta \beta_i \land p_i \circ \gamma \gamma_i}{z_i} \mid z_i \in Z, k = 1, 2, \ldots, m \right\}.
   \]
Example 5. (Continued from Examples 2 and 3) Then, the intersection and union of an m-PSVNSs, respectively, given by

\[\Phi \cap \Psi = \left\{ \frac{(0.6,0.3,0.7),(0.4,0.2,0.8)}{z_1}, \frac{(0.1,0.7,0.9),(0.4,0.3,0.8)}{z_2}, \frac{(0.1,0.3,0.7),(0.2,0.4,0.7)}{z_3} \right\}\]

and

\[\Phi \cup \Psi = \left\{ \frac{(0.7,0.3,0.4),(0.5,0.1,0.5)}{z_1}, \frac{(0.2,0.6,0.4),(0.5,0.2,0.4)}{z_2}, \frac{(0.6,0.1,0.6),(0.4,0.4,0.5)}{z_3} \right\}\]

**Theorem 1.** (Identity law of m-PSVNSs) Assume that \(\Phi\) be m-PSVNS, \(\emptyset\) be m-PSVN null set, and \(\mathcal{Z}\) be m-PSVN absolute set. Then the followings hold:

1. \(\Phi \cup \emptyset = \Phi\),
2. \(\Phi \cap \mathcal{Z} = \Phi\).

**Proof.**

1. \(\Phi \cup \emptyset\)

\[
\frac{(p_k \circ \alpha_{\Phi}(z_r), p_k \circ \beta_{\Phi}(z_r), p_k \circ \gamma_{\Phi}(z_r))}{z_r} \quad \left| z_r \in Z, k = 1, 2, \ldots, m \right\} \cup \frac{(0,1,1)}{z_r} \quad \left| z_r \in Z \right\}
\]

\[
= \left\{ \frac{(p_k \circ \alpha_{\Phi}(z_r) \lor 0, p_k \circ \beta_{\Phi}(z_r) \land 1, p_k \circ \gamma_{\Phi}(z_r) \land 1)}{z_r} \left| z_r \in Z, k = 1, 2, \ldots, m \right\} \right\}
\]

\[
= \left\{ \frac{(p_k \circ \alpha_{\Phi}(z_r), p_k \circ \beta_{\Phi}(z_r), p_k \circ \gamma_{\Phi}(z_r))}{z_r} \left| z_r \in Z, k = 1, 2, \ldots, m \right\} \right\}
\]

= \(\Phi\).

2. \(\Phi \cap \mathcal{Z}\)

\[
= \left\{ \frac{(p_k \circ \alpha_{\Phi}(z_r), p_k \circ \beta_{\Phi}(z_r), p_k \circ \gamma_{\Phi}(z_r))}{z_r} \left| z_r \in Z, k = 1, 2, \ldots, m \right\} \cap \frac{(1,0,0)}{z_r} \quad \left| z_r \in Z \right\}
\]

\[
= \left\{ \frac{(p_k \circ \alpha_{\Phi}(z_r) \land 1, p_k \circ \beta_{\Phi}(z_r) \lor 0, p_k \circ \gamma_{\Phi}(z_r) \lor 0)}{z_r} \left| z_r \in Z, k = 1, 2, \ldots, m \right\} \right\}
\]

\[
= \left\{ \frac{(p_k \circ \alpha_{\Phi}(z_r), p_k \circ \beta_{\Phi}(z_r), p_k \circ \gamma_{\Phi}(z_r))}{z_r} \left| z_r \in Z, k = 1, 2, \ldots, m \right\} \right\}
\]

= \(\Phi\).

**Theorem 2.** (Domination law of m-PSVNSs) Assume that \(\Phi\) be m-PSVNS, \(\emptyset\) be m-PSVN null set, and \(\mathcal{Z}\) be m-PSVN absolute set. Then the followings hold:

1. \(\Phi \cup \mathcal{Z} = \mathcal{Z}\),
2. \(\Phi \cap \emptyset = \emptyset\).
Proof.

(1) \( \Phi \cup \hat{Z} \)

\[
\left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \left| z_r \in \mathbb{Z}, k = 1, 2, ..., m \right\} \cup \left\{ \frac{(1,0,0)}{z_r} \left| z_r \in \mathbb{Z} \right\} \right.
\]

\[
= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r) \lor p_k \circ \beta_\Phi(z_r) \land 0, p_k \circ \gamma_\Phi(z_r) \lor 0)}{z_r} \left| z_r \in \mathbb{Z}, k = 1, 2, ..., m \right\}
\]

\[
= \left\{ \frac{(1,0,0)}{z_r} \left| z_r \in \mathbb{Z} \right\} \right.
\]

\( = \hat{Z}. \)

(2) \( \Phi \cap \hat{\Phi} \)

\[
= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \left| z_r \in \mathbb{Z}, k = 1, 2, ..., m \right\} \cap \left\{ \frac{(0,1,1)}{z_r} \left| z_r \in \mathbb{Z} \right\} \right.
\]

\[
= \left\{ \frac{(0,1,1)}{z_r} \left| z_r \in \mathbb{Z} \right\} \right.
\]

\( = \emptyset. \)

Theorem 3. (Idempotent law of m-PSVNSs) Assume that \( \Phi \) be m-PSVNS. Then the followings hold:

(1) \( \Phi \cup \Phi = \Phi. \)

(2) \( \Phi \cap \Phi = \Phi. \)

Proof.

(1) \( \Phi \cup \Phi = \Phi \)

\[
= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \left| z_r \in \mathbb{Z}, k = 1, 2, ..., m \right\} \cup \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \left| z_r \in \mathbb{Z}, k = 1, 2, ..., m \right\}
\]

\[
= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r) \lor p_k \circ \beta_\Phi(z_r) \land p_k \circ \gamma_\Phi(z_r) \lor p_k \circ \gamma_\Phi(z_r))}{z_r} \left| z_r \in \mathbb{Z}, k = 1, 2, ..., m \right\}
\]

\[
= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \left| z_r \in \mathbb{Z}, k = 1, 2, ..., m \right\}
\]

\( = \Phi. \)

(2) \( \Phi \cap \Phi \)

\[
= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \left| z_r \in \mathbb{Z}, k = 1, 2, ..., m \right\} \cap \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \left| z_r \in \mathbb{Z}, k = 1, 2, ..., m \right\}
\]

\[
= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r) \land p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r) \lor p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r) \lor p_k \circ \gamma_\Phi(z_r))}{z_r} \left| z_r \in \mathbb{Z}, k = 1, 2, ..., m \right\}
\]

\[
= \left\{ \frac{(p_k \circ \alpha_\Phi(z_r), p_k \circ \beta_\Phi(z_r), p_k \circ \gamma_\Phi(z_r))}{z_r} \left| z_r \in \mathbb{Z}, k = 1, 2, ..., m \right\}
\]

\( = \Phi. \)
Theorem 4. (Commutative law of m-PSVNSs) Assume that $\Phi$ and $\Psi$ are m-PSVNSs. Then the followings hold:

1. $\Phi \cap \Psi = \Psi \cap \Phi$.
2. $\Phi \cup \Psi = \Psi \cup \Phi$.

Proof.

1. $\Phi \cap \Psi$

$$
= \left\{ \left( p_k \circ \alpha_k(z_r), p_k \circ \beta_k(z_r), p_k \circ \gamma_k(z_r) \right) \mid z_r \in Z, k = 1, 2, \ldots, m \right\} \cap \left\{ \left( p_l \circ \alpha_l(z_r), p_l \circ \eta_l(z_r), p_l \circ \epsilon_l(z_r) \right) \mid z_r \in Z, k = 1, 2, \ldots, m \right\}.
$$

2. $\Phi \cup \Psi$

$$
= \left\{ \left( p_k \circ \alpha_k(z_r), p_k \circ \beta_k(z_r), p_k \circ \gamma_k(z_r) \right) \mid z_r \in Z, k = 1, 2, \ldots, m \right\} \cup \left\{ \left( p_l \circ \alpha_l(z_r), p_l \circ \beta_l(z_r), p_l \circ \gamma_l(z_r) \right) \mid z_r \in Z, k = 1, 2, \ldots, m \right\}.
$$

Theorem 5. (Associative law of m-PSVNSs) Assume that $\Phi, \Psi,$ and $\Omega$ are m-PSVNSs. Then the followings hold:

1. $(\Phi \cap \Psi) \cap \Omega = \Phi \cap (\Psi \cap \Omega)$.
2. $(\Phi \cup \Psi) \cup \Omega = \Phi \cup (\Psi \cup \Omega)$.

Proof.

1. $(\Phi \cap \Psi) \cap \Omega$

$$
= \left\{ \left( p_k \circ \alpha_k(z_r), p_k \circ \beta_k(z_r), p_k \circ \gamma_k(z_r) \right) \mid z_r \in Z, k = 1, 2, \ldots, m \right\} \cap \left\{ \left( p_l \circ \alpha_l(z_r), p_l \circ \beta_l(z_r), p_l \circ \gamma_l(z_r) \right) \mid z_r \in Z, k = 1, 2, \ldots, m \right\}.
$$
2. \((\Phi \cup \Psi) \cup \Omega\) 

\[
\begin{align*}
(\Phi \cup \Psi) \cup \Omega &= \bigg\{\left(p_k \circ a_q(z), p_k \circ b_q(z), p_k \circ \eta_q(z), p_k \circ \gamma_q(z), \right) & \left| z_r \in Z, k = 1, 2, \ldots, m \right\} \\
&\cup \bigg\{\left(p_k \circ a_q(z), p_1 \circ \lambda_q(z), p_k \circ \delta_q(z) \right) & \left| z_r \in Z, k = 1, 2, \ldots, m \right\} \\
&\cup \bigg\{\left(p_k \circ a_q(z), p_k \circ \beta_q(z), p_k \circ \gamma_q(z) \right) & \left| z_r \in Z, k = 1, 2, \ldots, m \right\} \\
= (\Phi \cup \Psi) \cup (\Phi \cap \Omega).
\end{align*}
\]

**Theorem 6.** (Distributive law of m-PSVNSs) Assume that \(\Phi, \Psi, \) and \(\Omega\) are m-PSVNSs. Then the following hold:

1. \(\Phi \cap (\Psi \cup \Omega) = (\Phi \cap \Psi) \cup (\Phi \cap \Omega)\).
2. \(\Phi \cup (\Psi \cap \Omega) = (\Phi \cup \Psi) \cap (\Phi \cup \Omega)\).

**Proof.**

1. \(\Phi \cap (\Psi \cup \Omega)\)

\[
\begin{align*}
\Phi \cap (\Psi \cup \Omega) &= \bigg\{\left(p_k \circ a_q(z), p_k \circ b_q(z), p_k \circ \eta_q(z), p_k \circ \gamma_q(z) \right) & \left| z_r \in Z, k = 1, 2, \ldots, m \right\} \\
&\cap \bigg\{\left(p_k \circ a_q(z), p_1 \circ \lambda_q(z), p_k \circ \delta_q(z) \right) & \left| z_r \in Z, k = 1, 2, \ldots, m \right\} \\
&\cap \bigg\{\left(p_k \circ a_q(z), p_k \circ \beta_q(z), p_k \circ \gamma_q(z) \right) & \left| z_r \in Z, k = 1, 2, \ldots, m \right\} \\
= (\Phi \cap \Psi) \cup (\Phi \cap \Omega).
\end{align*}
\]

2. \(\Phi \cup (\Psi \cap \Omega)\)

\[
\begin{align*}
\Phi \cup (\Psi \cap \Omega) &= \bigg\{\left(p_k \circ a_q(z), p_k \circ b_q(z), p_k \circ \eta_q(z), p_k \circ \gamma_q(z) \right) & \left| z_r \in Z, k = 1, 2, \ldots, m \right\} \\
&\cup \bigg\{\left(p_k \circ a_q(z), p_1 \circ \lambda_q(z), p_k \circ \delta_q(z) \right) & \left| z_r \in Z, k = 1, 2, \ldots, m \right\} \\
&\cup \bigg\{\left(p_k \circ a_q(z), p_k \circ \beta_q(z), p_k \circ \gamma_q(z) \right) & \left| z_r \in Z, k = 1, 2, \ldots, m \right\} \\
= (\Phi \cup \Psi) \cap (\Phi \cup \Omega).
\end{align*}
\]

**Definition 10.** Assume that \(\Phi\) be m-PSVNS on a set of elements \(Z,\) where

\[
\Phi = \left\{\left((p_1 \circ a_q(z), p_1 \circ b_q(z), p_1 \circ \eta_q(z), p_1 \circ \gamma_q(z)), \right) \cdots (p_k \circ a_q(z), p_k \circ b_q(z), p_k \circ \gamma_q(z)) \right\}_{z_r \in Z}
\]

Then, the complement \(\Phi'\) of \(\Phi\), is defined by

\[
\Phi' = \left\{\left((p_1 \circ a_q(z), 1 - p_1 \circ b_q(z), p_1 \circ \eta_q(z)), \right) \cdots (p_k \circ a_q(z), 1 - p_k \circ b_q(z), p_k \circ \gamma_q(z)) \right\}_{z_r \in Z}
\]

**Example 6.** (Continued from Example 2). The complement \(\Phi'\) of \(\Phi\), is calculated by

\[
\Phi' = \left\{\left((0.7, 0.7, 0.6), (0.8, 0.8, 0.4)), (0.9, 0.3, 0.1), (0.8, 0.7, 0.4)), (0.7, 0.7, 0.1), (0.7, 0.6, 0.2)\right\}_{z_r \in Z}
\]
Remark 1. By the following Example 7, we will show the equality of $\Phi^c \cup \Phi^c = \hat{X}$ and $\Phi^c \cap \Phi^c = \emptyset$ does not hold.

Example 7. (Continued from Examples 2 and 6). Then $\Phi^c \cup \Phi^c$ and $\Phi^c \cap \Phi^c$ are calculated by

$$
\Phi^c \cup \Phi^c = \left\{ \frac{(0.7,0.3,0.6),(0.8,0.2,0.4)}{z_1}, \frac{(0.9,0.3,0.1),(0.8,0.3,0.4)}{z_2}, \frac{(0.7,0.3,0.1),(0.7,0.4,0.2)}{z_3} \right\}
$$

and

$$
\Phi^c \cap \Phi^c = \left\{ \frac{(0.6,0.7,0.7),(0.4,0.8,0.8)}{z_1}, \frac{(0.7,0.7,0.9),(0.4,0.7,0.8)}{z_2}, \frac{(0.1,0.7,0.7),(0.2,0.6,0.7)}{z_3} \right\}.
$$

This show $\Phi^c \cup \Phi^c \neq \hat{X}$ and $\Phi^c \cap \Phi^c \neq \emptyset$.

Theorem 7. (Complementation and double complementation of m-PSVNSs) Assume that $\Phi$ be $m$-PSVNSs, $\emptyset$ be $m$-PSVN null set, and $\hat{Z}$ be $m$-PSVN absolute set. Then the followings hold:

1. $\hat{\Phi} = \hat{Z}$.
2. $\hat{Z} = \emptyset$.
3. $(\Phi^c)^c = \Phi^c$.

Proof.

1. $\hat{\Phi} = \left\{ \left\{ \frac{(0.1,0.1)}{z} \mid z \in Z \right\} \right\}^c = \left\{ \left\{ \frac{(0.1,0.0)}{z} \mid z \in Z \right\} \right\} = \hat{Z}$.
2. $\hat{Z} = \left\{ \left\{ \frac{(0.1,0.0)}{z} \mid z \in Z \right\} \right\}^c = \left\{ \left\{ \frac{(0.1,1.1)}{z} \mid z \in Z \right\} \right\} = \emptyset$.
3. $(\Phi^c)^c = \left\{ \left\{ \frac{p_i \circ r_i (z_r), 1 - p_k \circ r_k (z_r) \cdot p_l \circ r_l (z_r)}{z_r} \mid z_r \in Z, k, l = 1, 2, \ldots, m \right\} \right\}^c = \Phi^c$.

Remark 2. (De Morgan's laws of m-PSVNSs) Assume that $\Phi$, $\Psi$, and $\Omega$ are $m$-PSVNSs. Then the followings hold:

1. $(\Phi \cup \Psi)^c = \Phi^c \cap \Psi^c$.
2. $(\Phi \cap \Psi)^c = \Phi^c \cup \Psi^c$.

Proof.

1. $(\Phi \cup \Psi)^c$

$$
= \left\{ \left\{ \frac{p_i \circ r_i (z_r), p_j \circ r_j (z_r), p_k \circ r_k (z_r)}{z_r} \mid z_r \in Z, k, l = 1, 2, \ldots, m \right\} \cup \left\{ \frac{p_i \circ r_i (z_r), p_l \circ r_l (z_r), p_k \circ r_k (z_r)}{z_r} \mid z_r \in Z, k, l = 1, 2, \ldots, m \right\} \right\}^c
$$

$$
= \left\{ \left\{ \frac{p_i \circ r_i (z_r) \vee p_j \circ r_j (z_r), p_k \circ r_k (z_r) \vee p_l \circ r_l (z_r)}{z_r} \mid z_r \in Z, k, l = 1, 2, \ldots, m \right\} \cup \left\{ \frac{p_i \circ r_i (z_r) \vee p_l \circ r_l (z_r), p_k \circ r_k (z_r) \vee p_j \circ r_j (z_r)}{z_r} \mid z_r \in Z, k, l = 1, 2, \ldots, m \right\} \right\}^c
$$

$$
= \left\{ \left\{ 1 - p_i \circ r_i (z_r) \vee p_j \circ r_j (z_r), 1 - p_k \circ r_k (z_r) \vee p_l \circ r_l (z_r) \right\} \mid z_r \in Z, k, l = 1, 2, \ldots, m \right\}^c
$$

$$
= (\Phi^c \cap \Psi^c).
$$
2. \((\Phi \cap \Psi)^c\)

\[
= \left\{ \left( \frac{p_k \circ \mu_{a}(z_i)}{z_i}, \frac{p_k \circ \mu_{b}(z_i)}{z_i}, \frac{p_k \circ \gamma_{\Phi}(z_i)}{z_i} \right) \mid z_i \in Z, k = 1, 2, ..., m \right\} \cap \left\{ \left( \frac{p_k \circ \nu_{a}(z_i)}{z_i}, \frac{p_k \circ \nu_{b}(z_i)}{z_i}, \frac{p_k \circ \nu_{\Psi}(z_i)}{z_i} \right) \mid z_i \in Z, k = 1, 2, ..., m \right\}
\]

\[
= \left\{ \left( \frac{p_k \circ \mu_{a}(z_i) + p_k \circ \nu_{a}(z_i)}{z_i}, \frac{p_k \circ \mu_{b}(z_i) + p_k \circ \nu_{b}(z_i)}{z_i}, \frac{p_k \circ \gamma_{\Phi}(z_i) + p_k \circ \nu_{\Psi}(z_i)}{z_i} \right) \mid z_i \in Z, k = 1, 2, ..., m \right\}
\]

\[
= \left\{ \left( \frac{p_k \circ \mu_{a}(z_i) + p_k \circ \nu_{a}(z_i)}{z_i}, \frac{p_k \circ \mu_{b}(z_i) + p_k \circ \nu_{b}(z_i)}{z_i}, \frac{p_k \circ \gamma_{\Phi}(z_i) + p_k \circ \nu_{\Psi}(z_i)}{z_i} \right) \mid z_i \in Z, k = 1, 2, ..., m \right\}
\]

\[
= \Phi^c \cup \Psi^c.
\]

4. **DISTANCE MEASURE AND SM FOR m-PSVNSs**

We present a few new concepts of distances measure and SM for m-PSVNSs as indicated below.

**Definition 11.** Assume that \(\Phi\) and \(\Psi\) are m-PSVNSs on a set of elements \(Z\) (i.e., \(Z = \{z_1, z_2, ..., z_r \mid r = 1, 2, ..., n\}\)),

where

\[
\Phi = \left\{ \left( \frac{p_k \circ \mu_{a}(z_i), p_k \circ \mu_{b}(z_i), p_k \circ \gamma_{\Phi}(z_i)}{z_i} \right) \mid z_i \in Z \right\},
\]

and

\[
\Psi = \left\{ \left( \frac{p_k \circ \mu_{a}(z_i), p_k \circ \mu_{b}(z_i), p_k \circ \gamma_{\Psi}(z_i)}{z_i} \right) \mid z_i \in Z \right\}
\]

The distances measure between \(\Phi\) and \(\Psi\) are defined by

1. **Hamming distance:**

\[
d_1(\Phi, \Psi) = \frac{1}{m} \left\{ \sum_{k=1}^{m} \sum_{r=1}^{n} \left( \frac{p_k \circ \mu_{a}(z_i) + p_k \circ \mu_{b}(z_i) + p_k \circ \gamma_{\Phi}(z_i)}{2} - \frac{p_k \circ \mu_{a}(z_i) + p_k \circ \mu_{b}(z_i) + p_k \circ \gamma_{\Psi}(z_i)}{2} \right) \right\}
\]

2. **Normalized Hamming distance:**

\[
d_2(\Phi, \Psi) = \frac{1}{mn} \left\{ \sum_{k=1}^{m} \sum_{r=1}^{n} \left( \frac{p_k \circ \mu_{a}(z_i) + p_k \circ \mu_{b}(z_i) + p_k \circ \gamma_{\Phi}(z_i)}{2} - \frac{p_k \circ \mu_{a}(z_i) + p_k \circ \mu_{b}(z_i) + p_k \circ \gamma_{\Psi}(z_i)}{2} \right) \right\}
\]

3. **Euclidean distance:**

\[
d_3(\Phi, \Psi) = \frac{1}{m} \left\{ \sum_{k=1}^{m} \sum_{r=1}^{n} \left( \frac{p_k \circ \mu_{a}(z_i) + p_k \circ \mu_{b}(z_i) + p_k \circ \gamma_{\Phi}(z_i)}{2} - \frac{p_k \circ \mu_{a}(z_i) + p_k \circ \mu_{b}(z_i) + p_k \circ \gamma_{\Psi}(z_i)}{2} \right)^2 \right\}
\]

4. **Normalized Euclidean distance:**

\[
d_4(\Phi, \Psi) = \frac{1}{mn} \left\{ \sum_{k=1}^{m} \sum_{r=1}^{n} \left( \frac{p_k \circ \mu_{a}(z_i) + p_k \circ \mu_{b}(z_i) + p_k \circ \gamma_{\Phi}(z_i)}{2} - \frac{p_k \circ \mu_{a}(z_i) + p_k \circ \mu_{b}(z_i) + p_k \circ \gamma_{\Psi}(z_i)}{2} \right)^2 \right\}.
\]
Theorem 9. The distances between Im-PFSs Φ and Ψ satisfy the following four inequalities:

1. \( d_1(\Phi, \Psi) \leq n \),
2. \( d_2(\Phi, \Psi) \leq 1 \),
3. \( d_3(\Phi, \Psi) \leq \sqrt{n} \),
4. \( d_4(\Phi, \Psi) \leq 1 \).

Theorem 10. The distance functions \( d_1, d_2, d_3, \) and \( d_4 \) defined from \( m \) PSVNSs \( (\Phi, \Psi) \rightarrow \mathbb{R}^+ \), are metric distances.

Proof. Suppose that \( \Phi, \Psi, \) and \( \Omega \) are three m-PSVNSs on \( Z \). Then

1. \( d_1(\Phi, \Psi) \geq 0 \),
2. Let
   \[
   d_1(\Phi, \Psi) = 0 \iff \frac{1}{m} \left\{ \sum_{k=1}^{m} \sum_{r=1}^{n} \left| p_k \circ \alpha_k(z_r) + p_k \circ \beta_k(z_r) + p_k \circ \gamma_k(z_r) - \frac{1}{2} (p_k \circ \zeta_k(z_r) + p_k \circ \eta_k(z_r) + p_k \circ \epsilon_k(z_r)) \right| \right\} = 0
   \]
   \[
   \iff \frac{1}{2} (p_k \circ \alpha_k(z_r) + p_k \circ \beta_k(z_r) + p_k \circ \gamma_k(z_r) - p_k \circ \zeta_k(z_r) - p_k \circ \eta_k(z_r) - p_k \circ \epsilon_k(z_r)) = 0
   \]
   \[
   \iff p_k \circ \alpha_k(z_r) + p_k \circ \beta_k(z_r) + p_k \circ \gamma_k(z_r) = p_k \circ \zeta_k(z_r) + p_k \circ \eta_k(z_r) + p_k \circ \epsilon_k(z_r), \quad 1 \leq k \leq m, \quad 1 \leq r \leq n
   \]
   \[
   \iff \Phi = \Psi.
   \]
3. \( d_1(\Phi, \Psi) = d_1(\Psi, \Phi) \).
4. For any three m-PSVNSs \( \Phi, \Psi, \) and \( \Omega \),
   \[
   = \frac{1}{2} \left| p_k \circ \alpha_k(z_r) + p_k \circ \beta_k(z_r) + p_k \circ \gamma_k(z_r) - p_k \circ \zeta_k(z_r) - p_k \circ \eta_k(z_r) - p_k \circ \epsilon_k(z_r) \right|
   \]
   \[
   \leq \frac{1}{2} \left| p_k \circ \alpha_k(z_r) + p_k \circ \beta_k(z_r) + p_k \circ \gamma_k(z_r) - p_k \circ \zeta_k(z_r) - p_k \circ \eta_k(z_r) - p_k \circ \epsilon_k(z_r) \right|
   \]
   \[
   = \frac{1}{2} \left| p_k \circ \alpha_k(z_r) + p_k \circ \beta_k(z_r) + p_k \circ \gamma_k(z_r) - p_k \circ \zeta_k(z_r) - p_k \circ \eta_k(z_r) - p_k \circ \epsilon_k(z_r) \right|
   \]

Hence, \( d_1(\Phi, \Psi) \leq d_1(\Phi, \Omega) + d_1(\Omega, \Psi) \). ⊠

Definition 12. The SM of two m-PSVNSs \( \Phi \) and \( \Psi \), defined by

\[
SM(\Phi, \Psi) = \frac{1}{1 + d(\Phi, \Psi)}
\]  

where \( d(\Phi, \Psi) \) is any of the above four distances in Definition 11.

Definition 13. The two m-PSVNSs \( \Phi \) and \( \Psi \) are \( \sigma \) similar if and only if \( SM(\Phi, \Psi) \geq \sigma \), i.e.,

\[
\Phi \cong^\sigma \Psi \iff SM(\Phi, \Psi) \geq \sigma, \quad \sigma \in (0, 1)
\]  

\( \Phi \) and \( \Psi \) are significantly similar if \( SM(\Phi, \Psi) \geq \frac{1}{2} \).
Example 8. Suppose that an 2-PSVNSs on two of elements \( Z = \{z_1, z_2 \} \) are defined by
\[
\Phi = \left\{ \begin{array}{c}
(0.4, 0.3, 0.9), \langle 0.5, 0.2, 0.8 \rangle \\
(0.6, 0.3, 0.7), \langle 0.5, 0.5, 0.6 \rangle 
\end{array} \right\}
\]
and
\[
\Psi = \left\{ \begin{array}{c}
(0.6, 0.2, 0.8), \langle 0.1, 0.4, 0.7 \rangle \\
(0.3, 0.2, 0.6), \langle 0.7, 0.1, 0.8 \rangle 
\end{array} \right\}.
\]
Then, the Hamming distance is \( d_1(\Phi, \Psi) = 0.2 \) and the SM is \( SM(\Phi, \Psi) = 0.833 \). It shows \( \Phi \) is significantly similar to \( \Psi \).

Theorem 11. The SM of two m-PSVNSs \( \Phi \) and \( \Psi \) satisfy the following three properties:

(1) \( 0 \leq SM(\Phi, \Psi) \leq 1 \).
(2) \( SM(\Phi, \Psi) = SM(\Psi, \Phi) \).
(3) \( SM(\Phi, \Psi) = 1 \iff \Psi = \Phi \).

4.1. An Application of SM for m-PSVNSs in Brand Recognition

Based on the notion of Euclidean distance (i.e., Equation (3)) by SM, we will use m-PSVNSs information to solve brand recognition problem. In the following, we will build an Algorithm 1 to solve brand recognition problem (i.e., to apply m-PSVNS in brand recognition problem).

Algorithm 1

Determination of how an observer finds similarity to an unknown watch.

First step: Suppose that there are \( n \) watches represented by m-PSVNS \( T_e \) \((e = 1, 2, ..., n)\) in feature space \( Z \) (i.e., \( Z = \{z_1, z_2, ..., z_k\} \)).

Second step: Consider an m-PSVNS \( \psi \) is another unknown watch, which is to be recognized.

Third step: Convert the given m-PSVNS data into m-PFS by using the following formulas
\[
p_k \circ \alpha'_e (z_i) = \frac{p_k \circ \alpha_{\theta} (z_i) + p_k \circ \beta_{\theta} (z_i) + p_k \circ \gamma_{\theta} (z_i)}{3}
\]
and
\[
p_k \circ \alpha''_e (z_i) = \frac{p_k \circ \alpha'_{\theta} (z_i) + p_k \circ \beta'_{\theta} (z_i) + p_k \circ \gamma'_{\theta} (z_i)}{3}
\]

Fourth step: Calculate \( d_1, d_2, d_3, \) and \( d_4 \) distance between \( T_e \) and \( \psi \).

Fifth step: Calculate \( SM(T_e, \psi) \) between \( T_e \) and \( \psi \) by using the following formula
\[
SM(T_e, \psi) = \frac{1}{1 + d(T_e, \psi)}
\]

Sixth step: Evaluate the result by choosing the \( T_e \), which has the greatest SM with unknown \( \psi \).

Example 9. Suppose there are four brands of watches (i.e., \( T_1, T_2, T_3, \) and \( T_4 \)) and let \( Z = \{z_1, z_2, z_3, z_4\} \), where \( z_1 \) is “Material,” \( z_2 \) is “Glass kind,” \( z_3 \) is “Water Resistance,” and \( z_4 \) is “Beautiful Finishing” be feature space of watches. The data of 2-PSVNSs of four brands in the first step of the Algorithm 1 are given in Table 1:

Also, the 2-PSVNS of unknown watch in the second step of the Algorithm 1 is given by
\[
\psi = \left\{ \begin{array}{c}
(0.6, 0.3, 0.4), \langle 0.4, 0.2, 0.5 \rangle \\
(0.1, 0.7, 0.8), \langle 0.4, 0.3, 0.6 \rangle \\
(0.1, 0.3, 0.9), \langle 0.2, 0.4, 0.8 \rangle \\
(0.5, 0.1, 0.5), \langle 0.3, 0.2, 0.6 \rangle 
\end{array} \right\}.
\]
By the third step of the Algorithm 1 and by Equation (7), we convert 2-PSVNSs of four brands (i.e., \( T_1, T_2, T_3, \) and \( T_4 \)) into 2-PFSs as shown in the following Table 2:
By Equation (8) (i.e., the 2-PSVNS $\psi$ of unknown watch), we obtain 2-PFSs as follows:

$$\psi = \left\{ \frac{(0.43, 0.36)}{z_1}, \frac{(0.53, 0.43)}{z_2}, \frac{(0.43, 0.46)}{z_3}, \frac{(0.36, 0.36)}{z_4} \right\}.$$  

Then, by Equation (3) (i.e., Euclidean distance), we compute the Euclidean distance measure of $T_e$ ($e = 1, 2, 3, 4$) and $\psi$ in the fourth step of the Algorithm 1 as follows:

$$d_3(T_1, \psi) = 0.145, \quad d_3(T_2, \psi) = 0.039, \quad d_3(T_3, \psi) = 0.083, \quad d_3(T_4, \psi) = 0.058.$$  

By Equation (9) (i.e., SM), we compute the SM of $T_e$ ($e = 1, 2, 3, 4$) and $\psi$ in the fifth step of the Algorithm 1 as follows:

$$SM(T_1, \psi) = \frac{1}{1 + d_3(T_1, \psi)} = \frac{1}{1 + 0.145} = 0.873,$$

$$SM(T_2, \psi) = \frac{1}{1 + d_3(T_2, \psi)} = \frac{1}{1 + 0.039} = 0.962,$$

$$SM(T_3, \psi) = \frac{1}{1 + d_3(T_3, \psi)} = \frac{1}{1 + 0.083} = 0.923,$$

$$SM(T_4, \psi) = \frac{1}{1 + d_3(T_4, \psi)} = \frac{1}{1 + 0.058} = 0.945.$$  

Finally, although for every brand SM is greater than 0.5, SM of $T_2$ is the highest. Hence the unknown watch $\psi$ is closest in similarity to brand $T_2$.

### 5. A NOVEL METHODOLOGY TO EXTEND THE TOPSIS TO m-PSVNSs

We construct a new methodology to extend the TOPSIS to m-PSVNSs (i.e., this process is very applicable to deal with the group decision-making problem under m-PSVNS system).

Now, we propose an Algorithm 2 of the multi-decision maker multi-criteria decision-making of m-PSVNSs as follows:

**Algorithm 2**

Determination of the optimal decision based on m-PSVNSs.

**First step:** Assume that there is a group of $n$ persons decision-makers to evaluate the ratings of alternatives $Z_r$ ($r = 1, 2, ..., p$) concerning criteria $C_s$ ($s = 1, 2, ..., q$) in single valued neutrosophic value form where the information about criterion weights is known.
**Second step:** Create an m-PSVNS decision (i.e., a multi-criteria group decision-making problem) which can be represented in matrix form as

\[
\mathbf{M}_{\text{mPSVNS}} = \begin{pmatrix}
\theta_{11} & \theta_{12} & \cdots & \theta_{1q} \\
\theta_{21} & \theta_{22} & \cdots & \theta_{2q} \\
\vdots & \vdots & \ddots & \vdots \\
\theta_{p1} & \theta_{p2} & \cdots & \theta_{pq}
\end{pmatrix},
\]

where \( p \) (i.e., the number of alternatives), \( q \) (i.e., the number of criteria), and \( \theta_{rs} \forall r (r = 1, 2, \ldots, p), \forall s (s = 1, 2, \ldots, q) \) represents the ratings of \( r \)-alternatives concerning the \( s \)-criteria in the single valued neutrosophic value. The multiple data of \( r \) persons of decision-maker group for rating \( \theta_{rs} \) can be expressed as

\[
\theta_{rs} = (p_k \circ \alpha_{rs}, p_k \circ \beta_{rs}, p_k \circ \gamma_{rs})(k = 1, 2, \ldots, m).
\]

**Third step:** Convert the m-PSVNS value data of alternatives into m-PF \( n \)-single-value data by using the following formula

\[
\hat{f}_{rs} = (p_1 \circ \alpha_{rs}, p_2 \circ \alpha_{rs}, \ldots, p_m \circ \alpha_{rs})(k = 1, 2, \ldots, m)
\]

and we obtain rating of alternatives as

\[
\hat{f}_{rs} = (p_1 \circ \alpha_{rs}, p_2 \circ \alpha_{rs}, \ldots, p_m \circ \alpha_{rs})(k = 1, 2, \ldots, m)
\]

and the weights of criterion is expressed in information as

\[
\hat{w}_s = (p_1 \circ \omega_{ws}, p_2 \circ \omega_{ws}, \ldots, p_m \circ \omega_{ws})(k = 1, 2, \ldots, m; s = 1, 2, \ldots, q).
\]

**Fourth step:** By normalization (i.e., the normalization process is to preserve the property of m-PSVNS that the ranges of membership of elements is \([0, 1]\)) we get the normalized m-PSVNS to m-PF decision matrix (i.e., fuzzy numbers belong to \([0, 1]\)), denoted by \( \hat{E} \), is defined by

\[
\hat{E} = (\mathcal{E}_m)_{m \times q},
\]

where

\[
\mathcal{E}_m = (p_k \circ \mathcal{E}_{rs} = \frac{p_k \circ \alpha_{rs}}{\sqrt{\sum_{s=1}^{q} (p_k \circ \alpha_{rs})^2}})(k = 1, 2, \ldots, m; s = 1, 2, \ldots, q)
\]

**Fifth step:** Construct the weighted normalized m-PSVNS decision matrix, denoted by \( \hat{F} \), is defined by

\[
\hat{F} = (f_{rs})_{m \times q},
\]

such that

\[
f_{rs} = \mathcal{E}_{rs}(\cdot) \hat{w}_s = (p_k \circ \mathcal{E}_{rs}(\cdot) p_k \circ \hat{w}_s)(k = 1, 2, \ldots, m; s = 1, 2, \ldots, q)
\]

where the elements \( f_{rs} (r = 1, 2, \ldots; s = 1, 2, \ldots, q) \) are normalized the given \( m \) values to belong their ranges to from closed interval \([0, 1]\).

**Sixth step:** Determine the m-fuzzy positive ideal solution (denoted by \( A^+ \)) and m-fuzzy negative ideal solution (denoted by \( A^- \)), are defined by

\[
A^+ = (f_1^+, f_2^+, \ldots, f_q^+), \quad A^- = (f_1^-, f_2^-, \ldots, f_q^-),
\]

where

\[
f_k^+ = p_k \circ \hat{f}_k^+ = \begin{cases} (1, 1, \ldots, 1), & s \in B \\ (0, 0, \ldots, 0), & s \in C \end{cases}
\]

\[
f_k^- = p_k \circ \hat{f}_k^- = \begin{cases} (0, 0, \ldots, 0), & s \in B \\ (1, 1, \ldots, 1), & s \in C \end{cases}
\]

for all \( k = 1, 2, \ldots, m; s = 1, 2, \ldots, q \) and \( B \) denotes the benefit criteria and \( C \) denotes the cost criteria.

**Seventh step:** Compute the separation measure (i.e., Euclidean distance) of each alternative from \( A^+ \) and \( A^- \), respectively, as follows:

\[
\delta_k^+(f_{rs}, f_k^+) = \frac{1}{m} \left\{ \sum_{k=1}^{m} \sum_{s=1}^{q} (p_k \circ \hat{f}_{rs}^+ - p_k \circ \hat{f}_k^+) \right\}^2
\]

and

\[
\delta_k^-(f_{rs}, f_k^-) = \frac{1}{m} \left\{ \sum_{k=1}^{m} \sum_{s=1}^{q} (p_k \circ \hat{f}_{rs}^- - p_k \circ \hat{f}_k^-) \right\}^2
\]
Eighth step: Compute the closeness coefficient (denoted by \( C_y^i \)) of each alternative \( Z_i \) (\( r = 1, 2, ..., p \)) (i.e., as alternative \( Z_i \) is near to \( A^+ \) and far from \( A^- \) as closeness coefficient \( C_y^i \) go closer to 1), is defined by
\[
C_y^i = \frac{\delta^-_i}{\delta^-_i + \delta^+_i}
\]

(15)

Ninth step: According to the \( C_y^i \), we can give the rank to all alternatives and select the best one of them that nearer to 1.

In the following example, we will explain and apply the above nine steps of Algorithm 2.

Example 10. Let us consider an investment company, which wants to invest some money in the best alternative. The company forms a committee of three members for the selection of best alternative (i.e., \( Z_i \) (\( r = 1, 2, 3, 4 \)) among a panel of four possible alternatives to invest the money, where \( Z_1 \) is "a medicine company," \( Z_2 \) is "an electronics company," \( Z_3 \) is "a food company," and \( Z_4 \) is "a telecommunication company." The committee shall make a decision concerning the following four criteria (i.e., \( C_j \) (\( s = 1, 2, 3, 4 \))): \( C_j \) is "the environmental impact analysis," \( C_j \) is "the risk analysis," \( C_j \) is "the social-political impact analysis," and \( C_j \) is "the growth analysis." The data of 3-PSVN decision matrix (in the range \((0,10)\)) and 3-polar weights by the committee of three decision-makers \( p_k \) (\( k = 1, 2, 3 \)) evaluate all four possible alternatives (i.e., \( Z_1, Z_2, Z_3, Z_4 \)) in the first and the second steps of Algorithm 2 are given by

\[
\mathcal{M} = \begin{bmatrix}
Z_1 & (3, 4, 5, (2, 4, 5), (2, 3, 4)) & C_1 & C_2 & C_3 & C_4 \\
Z_2 & (2, 1, 5, (3, 2, 7), (1, 1, 3)) & (1, 1, 1) & (2, 5, 2, (3, 2, 6)) & (2, 4, 6, (4, 4, 1), (3, 4, 4)) \\
Z_3 & (1, 2, 2, (1, 3, 5), (2, 3, 6)) & (2, 4, 4, (2, 2, 2, 1, 2), 4) & (3, 4, 4, (2, 4, 4, (5, 4, 5))) & ((1, 3, 3), (2, 2, 4), (1, 3, 4)) \\
Z_4 & (1, 3, 4, (2, 2, 4, (4, 4, 4))] & ((2, 2, 2, (3, 2, 3), (3, 4, 6)) & (1, 5, 5, (3, 3, 6), (3, 6, 7)) & ((2, 2, 4), (1, 3, 3), (3, 4, 4)) \\
\end{bmatrix}
\]

By Equation (10) in the third step of Algorithm 2, we obtain the following 3-polar decision matrix as follows:

\[
\mathcal{M}' = \begin{bmatrix}
Z_1 & (6, 5, 5, 5, 5) & (2, 5, 6, 5, 5) & (2, 5, 4, 5, 5) & (6, 4, 5, 5) \\
Z_2 & (4, 6, 2, 5) & (2, 5, 7, 5, 5) & (8, 3, 5, 5) & (6, 3, 5, 5) \\
Z_3 & (2, 5, 4, 5, 5) & (5, 5, 5, 5) & (5, 5, 5, 5) & (3, 5, 4, 5) \\
Z_4 & (4, 4, 6) & (3, 4, 6, 5) & (5, 5, 6, 8) & (4, 3, 5, 5) \\
\end{bmatrix}
\]

By Equation (11) in the fourth step of Algorithm 2, we normalized the 3-polar decision matrix to get the normalized 3-PF decision matrix as follows:

\[
\mathcal{M}'' = \begin{bmatrix}
Z_1 & (0.69, 0.54, 0.47) & (0.36, 0.55, 0.53) & (0.21, 0.47, 0.41) & (0.59, 0.57, 0.55) \\
Z_2 & (0.46, 0.59, 0.26) & (0.36, 0.59, 0.43) & (0.69, 0.31, 0.41) & (0.59, 0.44, 0.45) \\
Z_3 & (0.29, 0.44, 0.57) & (0.73, 0.46, 0.34) & (0.48, 0.32, 0.53) & (0.34, 0.51, 0.40) \\
Z_4 & (0.46, 0.39, 0.62) & (0.43, 0.34, 0.63) & (0.48, 0.63, 0.60) & (0.39, 0.44, 0.55) \\
\end{bmatrix}
\]

By Equation (12) in the fifth step of the Algorithm 2, we obtain the weighted normalized 3-PSVNS decision matrix as follows:

\[
\mathcal{M}''' = \begin{bmatrix}
Z_1 & (0.24, 0.14, 0.17) & (0.15, 0.17, 0.13) & (0.05, 0.21, 0.06) & (0.11, 0.20, 0.14) \\
Z_2 & (0.16, 0.15, 0.09) & (0.15, 0.18, 0.10) & (0.17, 0.13, 0.06) & (0.11, 0.15, 0.10) \\
Z_3 & (0.10, 0.11, 0.21) & (0.30, 0.14, 0.08) & (0.12, 0.14, 0.07) & (0.06, 0.18, 0.09) \\
Z_4 & (0.16, 0.10, 0.23) & (0.12, 0.10, 0.15) & (0.12, 0.28, 0.09) & (0.07, 0.15, 0.13) \\
\end{bmatrix}
\]

As \( C_1 \) and \( C_2 \) (resp., \( C_3 \) and \( C_4 \)) are cost criteria (resp., are benefit criteria). Then the 3-fuzzy positive ideal solution \( A^+ \) and 3-fuzzy negative ideal solution \( A^- \) in sixth step of the Algorithm 2 are given as follows:

\[
A^+ = ((0, 0, 0), (0, 0, 0), (1, 1, 1), (1, 1, 1))
\]

\[
A^- = ((1, 1, 1), (1, 1, 1), (0, 0, 0), (0, 0, 0)).
\]
In the seventh step of the Algorithm 2, by computing separation measure \((S^*_r, S^-_r)\) between each attribute with \(A^*\) and \(A^-\) by using Equations (13) and (14), respectively, we get the following results as follows \((r = 1, 2, 3, 4)\):

\[
S^*_1(Z_1, A^*) = 1.25, \quad S^*_2(Z_2, A^*) = 1.26, \quad S^*_3(Z_3, A^*) = 1.28, \quad S^*_4(Z_4, A^*) = 1.23
\]

and

\[
S^-_1(Z_1, A^-) = 1.19, \quad S^-_2(Z_2, A^-) = 1.23, \quad S^-_3(Z_3, A^-) = 1.20, \quad S^-_4(Z_4, A^-) = 1.23.
\]

In the eighth step of the Algorithm 2, by computing closeness coefficient \(\overline{C_{fr}}\) of each alternative \(Z_r\) \((r = 1, 2, 3, 4)\) and by Equation (15), we get the following results as follows:

\[
\overline{C_{fr}}(Z_1) = 0.48, \quad \overline{C_{fr}}(Z_2) = 0.49, \quad \overline{C_{fr}}(Z_3) = 0.48, \quad \overline{C_{fr}}(Z_4) = 0.5.
\]

Finally, as \(\overline{C_{fr}}(Z_4) = 0.5\) is the highest among others, \(Z_4\) is the best ideal solution for this multi-person TOPSIS method. Consequently, the outcome of the decision-making committee of the company is to invest money in a medicine company to obtain the best return of investment.

6. CONCLUSION

We presented the concept of the m-PSVNS as a new m-PFS model. We studied the several structure operations of the m-PSVNS and also discussed the basic properties of the m-PSVNS. Then, we introduced novel concepts of distances measure and SM for m-PSVNSs. Further, we constructed an Algorithm 1 to solve brand recognition problem by applying the m-PSVNS. Moreover, a new approach for the best m-PSVNS alternatives based on distance SMs in MCDM is animated. Finally, a new methodology to extend the TOPSIS to m-PSVNSs is proposed and its applicability was illustrated through a numerical example. In future, an m-PSVNS will definitely open new ways to apply with or without restriction existing results of m-F soft set, m-F soft rough set. Also, by combining the m-PFS and the other sets (e.g., interval-valued FSs [34], picture FSs [35], and spherical FSs [36]) we can extend our work to obtain more novel m-PFS models.

CONFLICTS OF INTEREST

The authors declare no conflicts of interest.

AUTHORS’ CONTRIBUTIONS

Author 1, author 5 and author 6 sourced the funding; author 2 drafted the manuscript; author 3 revised the manuscript; author 4 supervised the work.

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