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## MULTIPOTENTIALISATIONS AND ITERATING-SOLUTION FORMULAE: THE KRICHEVER–NOVIKOV EQUATION

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We derive solution-formulae for the Krichever–Novikov equation by a systematic multipotentialisation of the equation. The formulae are achieved due to the connections of the Krichever–Novikov equations to certain symmetry-integrable 3rd-order evolution equations which admit autopotentialisations.

*Keywords:* Integrable evolution equations in (1+1) dimensions; Auto-Bäcklund transformations; solution generators; potentialisation of evolution equations; conservation laws; Krichever–Novikov equation.

### 1. Introduction

Our main object of interest is the Krichever–Novikov equation

$$u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x} + \frac{P(u)}{u_x}, \quad P^{(5)}(u) = 0. \quad (1.1)$$

Equation (1.1), with  $P(u) = 4u^3 + a_1u + a_2$ ,  $a_j \in \mathfrak{R}$ , was first introduced in [8] by Krichever and Novikov in connection with a study of finite-gap solutions of the Kadomtsev–Petviashvili equation (or KP equation). In the sense of conservation laws and integrability this equation was investigated by Svinolupov and Sokolov in [10] and by Svinolupov, Sokolov and Yamilov in [11]. It is interesting to note that (1.1) admits a 4th-order integro-differential recursion operator ([9,4]) and not a 2nd-order recursion operator as most other 3rd-order symmetry-integrable evolution equations (which implies that (1.1) admits a 2nd-order non-local recursion operator which generates local Lie–Bäcklund symmetries).

In the current paper we report a detailed study of the systematic (multi-) potentialisations of (1.1). This leads to iterating-solution formulae for special cases of the polynomial  $P$  in (1.1).

In addition to the Krichever–Novikov equation, (1.1), the following symmetry-integrable evolution equations are considered in this paper:

- A linearisable equation (see e.g. [5]):

$$u_t = u_{xxx} - \frac{3}{4} \frac{u_{xx}^2}{u_x};$$

- The Schwarzian–Korteweg–de Vries equation (see e.g. [3]):

$$u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x};$$

- An extended Schwarzian–Korteweg–de Vries equation (see e.g. [5]):

$$u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x} + \frac{\lambda_1}{u_x} + \lambda_2 u_x^3.$$

The paper is organised as follows: In Sec. 2 we define the concept of *multipotentialisation* and *autopotentialisation* of evolution equations and give the connection to *iterating-solution formulae* of evolution equations. We study two examples, namely a simple linearisable 3rd-order evolution equation and the Schwarzian–KdV equation. In Sec. 3 we report the main results of this paper, namely we propose three iterating-solution formulae for three special cases of the Krichever–Novikov equation (1.1). Throughout the paper diagrams show the connections between the equations which are achieved by the given potentialisations. In Sec. 4 we draw some conclusions.

## 2. Autopotentialisations and Iterating-Solution Formulae

Consider an  $n$ th-order evolution equation of the general form

$$u_t = F(x, u, u_x, u_{xx}, u_{3x}, \dots, u_{nx}) \quad (2.1)$$

which admits a conservation law

$$(D_t \Phi^t + D_x \Phi^x)|_{u_t=F} = 0, \quad (2.2)$$

where  $\Phi_1^t$  and  $\Phi_1^x$  are a conserved current and a flux for (2.1), respectively. We recall that  $\Phi^t$  must satisfy the relation ([7,1])

$$\Lambda = \hat{E}[u] \Phi^t, \quad (2.3)$$

where  $\Lambda$  denotes the corresponding integrating factor of (2.1) such that

$$\hat{E}[u](\Lambda u_t - \Lambda F(x, u, u_x, u_{xx}, \dots, u_{nx})) = 0. \quad (2.4)$$

Here  $\hat{E}[u]$  is the Euler operator

$$\hat{E}[u] := \frac{\partial}{\partial u} - D_t \circ \frac{\partial}{\partial u_t} - D_x \circ \frac{\partial}{\partial u_x} + D_x^2 \circ \frac{\partial}{\partial u_{xx}} - D_x^3 \circ \frac{\partial}{\partial u_{3x}} + \dots \quad (2.5)$$

Under the assumption

$$\Phi^t = \Phi^t(x, u, u_x, u_{xx}, u_{3x}), \quad (2.6)$$

the flux,  $\Phi^x$ , is calculated by the formula [6]

$$\begin{aligned} \Phi^x = & -D_x^{-1}(\Lambda F) - \frac{\partial \Phi^t}{\partial u_x} F - \frac{\partial \Phi^t}{\partial u_{xx}} D_x F - \frac{\partial \Phi^t}{\partial u_{3x}} D_x^2 F \\ & + F D_x \left( \frac{\partial \Phi^t}{\partial u_{xx}} \right) - F D_x^2 \left( \frac{\partial \Phi^t}{\partial u_{3x}} \right) + (D_x F) D_x \left( \frac{\partial \Phi^t}{\partial u_{3x}} \right). \end{aligned} \quad (2.7)$$

Assume now that the evolution equation (2.1) admits a conserved current,  $\Phi_1^t$ , and flux,  $\Phi_1^x$ . Following [2] and [6] a *first potential variable*  $v$  is then defined by the auxiliary system:

$$v_x = \Phi_1^t(x, u, u_x, \dots) \quad (2.8a)$$

$$v_t = -\Phi_1^x(x, u, u_x, \dots). \quad (2.8b)$$

System (2.8a)–(2.8b) is known as the *first auxiliary system* of (2.1). Assume further that (2.8b) can be expressed in terms of the first potential variable  $v$ , i.e., (2.8b) becomes by (2.8a) the *first potential equation* of the general form

$$v_t = G(x, v_x, v_{xx}, \dots, v_{nx}) \quad (2.9)$$

which may again admit a conserved current,  $\Phi_2^t$ , and flux,  $\Phi_2^x$ . A further potential  $w$  is then introduced for (2.9) and named the *second potential* for (2.1) by the *second auxiliary system*

$$w_x = \Phi_2^t(x, v, v_x, \dots) \quad (2.10a)$$

$$w_t = -\Phi_2^x(x, v, v_x, \dots). \quad (2.10b)$$

The corresponding potential equation for (2.9) is then obtained from (2.10a) and (2.10b), which we assume to have the general form

$$w_t = H(x, w_x, w_{xx}, \dots, w_{nx}). \quad (2.11)$$

We name (2.11) the *second potential equation* for (2.1). In the same manner we can introduce a third potential equation and higher potential equations with corresponding potential variables. We introduce the following:

**Definition.** An evolution equation, (2.1), which admits at least a second potential equation, is said to be a multipotentialisable evolution equation. Furthermore an evolution equation in potential form,

$$u_t = F(u_x, u_{xx}, \dots, u_{nx}) \quad (2.12)$$

which admits at least one potentialisation into the same potential equation, i.e., it potentialises into

$$v_t = F(v_x, v_{xx}, \dots, v_{nx}) \quad (2.13)$$

by

$$v_x = \Phi^t(x, u, u_x, \dots) \quad (2.14a)$$

$$v_t = -\Phi^x(x, u, u_x, \dots), \quad (2.14b)$$

is said to be autopotentialisable. The change of variables, (2.14a), defines an auto-Bäcklund transformation for (2.12) and can be present in the form of an iterating-solution formula

$$(u_{j+1})_x = \Phi^t(x, u_j, (u_j)_x, \dots), \quad j = 0, 1, 2, \dots, \quad (2.15)$$

where each  $u_j$  is a solution of the potential equation

$$u_t = F(u_x, u_{xx}, \dots, u_{nx}) \quad (2.16)$$

and  $u_0$  is known as the seed-solution.

In the current paper we show that it is possible to formulate iterating-solution formulae for those multi-potentialisable evolution equations which contain at least one autopotentialisable equation in its chain of potentialisations, by the use of the auto-Bäcklund transformations that appears from the autopotentialisable equation. We exploit this observation in Sec. 3 and formulate several iterating-solution formulae for three special Krichever–Novikov equations.

In the current section we firstly discuss three examples of autopotentialisable evolution equations which are to our interest. In particular we study

$$u_t = u_{xxx} - \frac{3}{4} \frac{u_{xx}^2}{u_x} \quad (2.17a)$$

$$u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x}. \quad (2.17b)$$

The first equation, (2.17a), is a linearisable equation and follows by potentialising the linear equation

$$u_t = u_{xxx}. \quad (2.18)$$

Equations (2.17b) is the well-known are the Schwarzian–KdV and plays a fundamental role in our study of the Krichever–Novikov equation (1.1), in Sec. 3.

The chain of potentialisations of (2.17a) and (2.17b) are presented in Diagrams 1 and 2, respectively.

Diagram 1 gives the chain of equations which follow from the multipotentialisation of the linear third-order equation, (2.18). The conserved currents,  $\Phi^t$ , and flux,  $\Phi^x$ , corresponding to the linear equation, (2.18), and the equation, (2.17a), used in Diagram 1 to define the potential variables,  $v$ ,  $w$  and  $q$ , are

$$\Phi_v^t[u] = u^2, \quad \Phi_v^x[u] = -2uu_{xx} + u_x^2 \quad (2.19a)$$

$$\Phi_w^t[v] = \frac{v_{xx}^2}{v_x}, \quad \Phi_w^x[u] = - \left( \frac{2v_{xx}v_{4x}}{v_x} - \frac{2u_{xx}^2 u_{xxx}}{u_x^2} + \frac{5}{4} \frac{u_{xx}^4}{u_x^3} - \frac{u_{xxx}^2}{u_x} \right) \quad (2.19b)$$

$$\Phi_q^t[w] = \sqrt{w_x}, \quad \Phi_q^x[w] = \frac{w_{xx}^2}{4w_x^{3/2}} - \frac{w_{xxx}}{2\sqrt{w_x}}, \quad (2.19c)$$

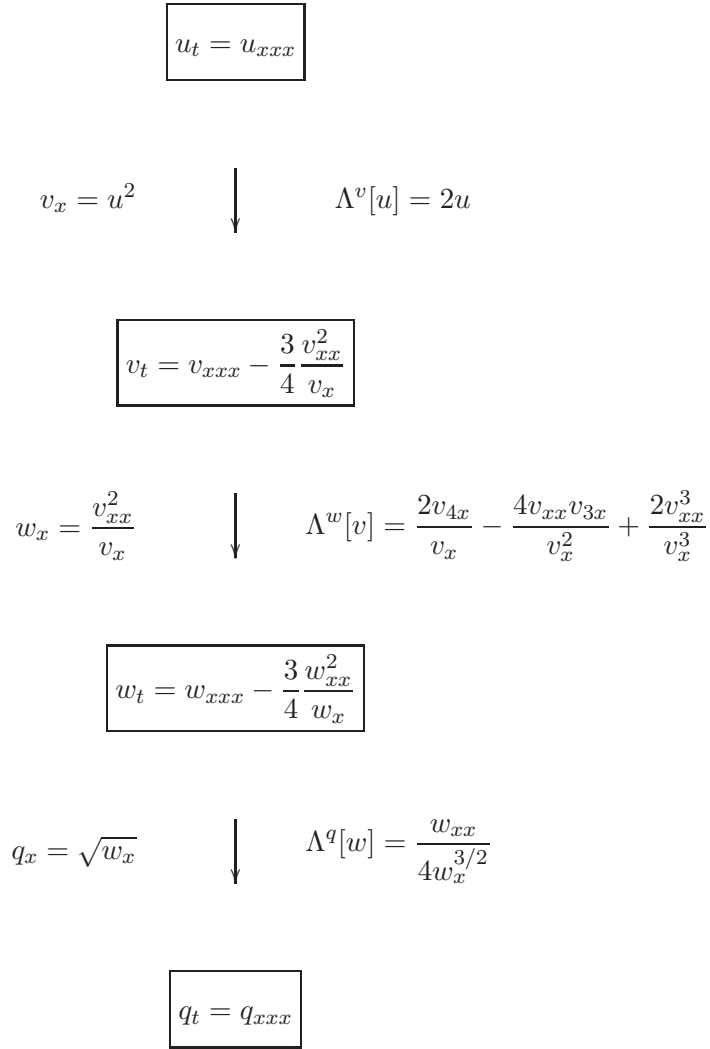


Diagram 1. Multipotentialisation of the linear equation  $u_t = u_{xxx}$ .

respectively. The corresponding integrating factors,  $\Lambda^v[u]$ ,  $\Lambda^w[v]$  and  $\Lambda^q[w]$ , are shown in Diagram 1. Following the results in Diagram 1, we have:

**Proposition 2.1.** *An iterating-solution formula for (2.17a), viz.*

$$u_t = u_{xxx} - \frac{3}{4} \frac{u_{xx}^2}{u_x},$$

takes the form

$$(u_{j+1})_x = \frac{(u_j)_{xx}^2}{(u_j)_x}, \quad j = 0, 1, 2, \dots, \quad (2.20)$$

where every  $u_j$  is a solution of (2.17a).

Diagram 2 gives the autopotentialisation of the Schwarzian–KdV equation (2.17b). The conserved currents,  $\Phi^t$ , and flux,  $\Phi^x$ , corresponding to the autopotentialisation as shown in

$$\begin{array}{ccc}
\boxed{u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x}} & & \\
v_x = u^2 u_x^{-1} \quad \downarrow & \Lambda^v[u] = -2u \left( \frac{u^2 u_{xx}}{u_x^3} - 2 \frac{u}{u_x} \right) & \\
\boxed{v_t = v_{xxx} - \frac{3}{2} \frac{v_{xx}^2}{v_x}} & & \\
w_x = v_x^{-1} \quad \downarrow & \Lambda^w[v] = -\frac{2v_{xx}}{v_x^3} & \\
\boxed{w_t = w_{xxx} - \frac{3}{2} \frac{w_{xx}^2}{w_x}} & & 
\end{array}$$

Diagram 2. Multipotentialisation of the Schwarzian–KdV equation  $u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x}$ .

Diagram 2 to define the potential variables,  $v$  and  $w$ , are

$$\Phi_v^t[u] = \frac{u^2}{u_x}, \quad \Phi_v^x[u] = \frac{u^2 u_{xxx}}{u_x^2} - \frac{1}{2} \frac{u^2 u_{xx}^2}{u_x^3} - \frac{4u u_{xx}}{u_x} + 4u_x \quad (2.21a)$$

$$\Phi_w^t[v] = \frac{1}{v_x}, \quad \Phi_w^x[v] = \frac{v_{xxx}}{v_x^2} - \frac{1}{2} \frac{v_{xx}^2}{v_x^3}, \quad (2.21b)$$

respectively. The corresponding integrating factors,  $\Lambda^v[u]$  and  $\Lambda^w[v]$ , are given in Diagram 2. Following the results in Diagram 2, we have:

**Proposition 2.2.** *Two iterating-solution formulae for (2.17b), viz.*

$$u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x},$$

take the form

$$(u_{j+1})_x = \frac{(u_j)^2}{(u_j)_x}, \quad j = 0, 1, 2, \dots, \quad (2.22)$$

and

$$(u_{j+1})_x = \frac{1}{(u_j)_x}, \quad j = 0, 1, 2, \dots, \quad (2.23)$$

where every  $u_j$  is a solution of (2.17b).

**Example.** As an example we use the iterating-solution formula (2.22) given in Proposition 2.2, *viz.*

$$(u_{j+1})_x = \frac{(u_j)^2}{(u_j)_x}, \quad j = 0, 1, 2, \dots,$$

for the Schwarzian–KdV equation (2.17b), *viz.*

$$u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x},$$

to generate two solutions,  $u_1$  and  $u_2$ . We start with the seed solution

$$u_0 = x.$$

Then

$$u_1 = \int x^2 \partial x + h_1(t) = \frac{1}{3} x^3 + h_1(t)$$

with

$$\frac{dh_1}{dt} + 4 = 0 \iff h_1(t) = -4t + a_1, \quad a_1 \in \mathfrak{R}.$$

Consequently

$$u_1 = \frac{1}{3} x^3 - 4t + a_1$$

is a solution of (2.17b). We now generate a second solution  $u_2$ :

$$\begin{aligned} u_2 &= \int \left( \frac{(u_1)^2}{(u_1)_x} \right) \partial x + h_2(t) \\ &= \frac{x^5}{45} + \frac{(a_1 - 4t)x^2}{3} + \frac{(a_1 - 4t)^2}{x} + h_2(t), \end{aligned}$$

which must satisfy (2.17b) and results in the condition

$$\frac{dh_2}{dt} = 0 \iff h_2(t) = a_2, \quad a_2 \in \mathfrak{R}.$$

Hence

$$u_2 = \frac{x^5}{45} + \frac{(a_1 - 4t)x^2}{3} + \frac{(a_1 - 4t)^2}{x} + a_2$$

is a solution of (2.17b). The iteration can obviously be continued to  $u_3, u_4, \dots$

### 3. Iterating Formulae for the Krichever–Novikov Equation

We study the following special Krichever–Novikov equation

$$u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x} + \frac{P(u)}{u_x}, \quad P(u) = k_2(u^2 + k_1u + k_0)^2 \quad (3.1)$$

with  $k_0, k_1, k_2 \neq 0$  being arbitrary constants. It is important to point out that (1.1) does not admit any zero-order integrating factor and, moreover, (3.1) admits a 2nd-order integrating



factor,  $\Lambda(u, u_x, u_{xx})$ , if and only if  $P$  is constrained by the following condition:

$$4P'''P^2 + 3(P')^3 - 6PP'P'' = 0. \quad (3.2)$$

The general solution of (3.2) is

$$P(u) = k_2(u^2 + k_1u + k_0)^2, \quad (3.3)$$

which is the reason that we are considering the special case, (3.1), rather than the general Krichever–Novikov equation (1.1) with  $P^{(5)} = 0$ .

For the multipotentialisation of (3.1) three different cases, depending on the values of  $k_0, k_1$  and  $k_2$ , have to be considered, namely  $4k_0 - k_1^2 = 0$ ,  $4k_0 - k_1^2 > 0$  and  $4k_0 - k_1^2 < 0$ .

**Case I.**  $4k_0 - k_1^2 = 0$ . The special Krichever–Novikov equation (3.1), takes the form

$$u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x} + \frac{(c_1u + c_0)^4}{u_x}. \quad (3.4)$$

Applying a systematic multipotentialisation of (3.4) we obtain the following:

**Proposition 3.1.** *Nonstationary solutions,  $u_j$ , of the special Krichever–Novikov equation (3.4), viz.*

$$u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x} + \frac{(c_1u + c_0)^4}{u_x},$$

*can be generated by nonstationary solutions,  $w_j$ , of the Schwarzian–KdV equation (2.17b), viz.*

$$w_t = w_{xxx} - \frac{3}{2} \frac{w_{xx}^2}{w_x}, \quad (3.5)$$

*by the relation*

$$u_j = \frac{1}{c_1} \left[ g_j(t) - \sqrt{\frac{2}{3}} c_1 \int \left( (w_j)_x h_j - (w_j)_x \int (w_j)_x^{-1} \partial x \right) \partial x \right]^{-1} - \frac{c_0}{c_1}, \quad (3.6)$$

*where  $w_j$  satisfy (3.5) and are iterated by either*

$$(w_{j+1})_x = \frac{(w_j)_x^2}{(w_j)_x}, \quad j = 1, 2, \dots \quad (3.7)$$

*or by*

$$(w_{j+1})_x = \frac{1}{(w_j)_x}, \quad j = 1, 2, \dots \quad (3.8)$$

*Here  $g_j(t)$  is determined by (3.4) for each solution,  $u_j$ , and  $h_j$  is an arbitrary constant for each  $u_j$ .*

The proof of Proposition 3.1 follows by composing  $\Phi_v^t$  and  $\Phi_w^t$ , as shown in Diagram 3.

**Example.** We apply Proposition 3.1 and start with the seed-solution

$$w_1 = \exp\left(x - \frac{t}{2}\right) + a_1,$$

( $a_1$  is an arbitrary constant) of the Schwarzian–KdV equation (2.17b). By the iterating-solution formula (3.6), we obtain the following solution for (3.4):

$$u_1 = \frac{1}{c_1} \left( g_1(t) - \frac{\sqrt{6}}{3} (h_1 e^{x-t/2} + x) \right)^{-1} - \frac{c_0}{c_1}, \quad (3.9)$$

where

$$g_1(t) = -\frac{\sqrt{6}}{2} c_1 t + b_1, \quad (3.10)$$

where  $b_1$  is an arbitrary constant. By the iteration-formula (3.7), we obtain the next solution for the Schwarzian–KdV equation as

$$w_2 = e^{x-t/2} - a_1^2 e^{-x+t/2} + 2a_1 x + 3a_1 t + a_2, \quad (3.11)$$

where  $a_2$  is another arbitrary constant. The iterating-solution formula (3.6), leads to the following solution for (3.4):

$$\begin{aligned} u_2 = & -[e^{x-t/2}(3 - 3c_0 g_2(t) + 2c_0 \sqrt{6} c_1 h_2 a_1 x + c_0 \sqrt{6} c_1 x) + c_0 c_1 h_2 \sqrt{6} e^{2x-t} \\ & - \sqrt{6} c_0 c_1 a_1 (h_2 a_1 + 1)] [c_1 e^{x-t/2} (-3g_2(t) + 2\sqrt{6} c_1 a_1 h_2 x + \sqrt{6} c_1 x) \\ & + \sqrt{6} c_1^2 h_2 e^{2x-t} - \sqrt{6} c_1^2 a_1 (a_1 h_2 + 1)]^{-1}, \end{aligned} \quad (3.12)$$

where

$$g_2(t) = -\sqrt{6} c_1 \left( a_1 h_2 + \frac{1}{2} \right) t + b_2 \quad (3.13)$$

and  $b_2$  is another arbitrary constant. This iteration can be continued to  $u_3, u_4$  etc.

**Case II.**  $4k_0 - k_1^2 > 0$ . The special Krichever–Novikov equation (3.1), takes the form

$$u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x} + \frac{k_2(u^2 + k_1 u + k_0)^2}{u_x}, \quad 4k_0 - k_1^2 > 0. \quad (3.14)$$

Applying a systematic multipotentialisation of (3.14) we obtain the following:

**Proposition 3.2.** *Nonstationary solutions,  $u_j$ , of the special Krichever–Novikov equation (3.14), viz.*

$$u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x} + \frac{k_2(u^2 + k_1 u + k_0)^2}{u_x}, \quad \text{where } 4k_0 - k_1^2 > 0,$$

*can be generated from nonstationary solutions,  $v_j$ , of the extended Schwarzian–KdV equation*

$$v_t = v_{xxx} - \frac{3}{2} \frac{v_{xx}^2}{v_x} - \frac{3}{2} \frac{\alpha^2 k_2 (4k_0 - k_1^2)}{v_x} - \frac{v_x^3}{3\alpha^2}, \quad 4k_0 - k_1^2 > 0 \quad (3.15)$$

by the relation

$$u_j = \frac{1}{2} \sqrt{4k_0 - k_1^2} \tan \left[ \frac{\alpha \sqrt{k_2}}{2} \sqrt{4k_0 - k_1^2} \left( \int \frac{1}{(v_j)_x} \partial x + h_j \right) \right] - \frac{k_1}{2}, \quad (3.16)$$

where  $v_j$  satisfy (3.15) and are iterated as follows:

$$(v_{j+1})_x = -\frac{\beta}{2} \frac{1}{(v_j)_x}, \quad \beta^2 = -6\alpha^2 k_2 (4k_0 - k_1^2), \quad 4k_0 - k_1^2 > 0, \quad j = 1, 2, \dots \quad (3.17)$$

Here  $h_j$  are arbitrary constants.

**Case III.**  $4k_0 - k_1^2 < 0$ . The special Krichever–Novikov equation (3.1), takes the form

$$u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x} + \frac{k_2(u^2 + k_1 u + k_0)^2}{u_x}, \quad 4k_0 - k_1^2 < 0. \quad (3.18)$$

Applying a systematic multipotentialisation of (3.18) we obtain the following:

**Proposition 3.3.** *Nonstationary solutions,  $u_j$ , of the special Krichever–Novikov equation (3.18), viz.*

$$u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x} + \frac{k_2(u^2 + k_1 u + k_0)^2}{u_x}, \quad \text{where } 4k_0 - k_1^2 < 0,$$

can be generated from nonstationary solutions,  $v_j$ , of the extended Schwarzian–KdV equation

$$v_t = v_{xxx} - \frac{3}{2} \frac{v_{xx}^2}{v_x} - \frac{3}{2} \frac{\alpha^2 k_2 (4k_0 - k_1^2)}{v_x} - \frac{v_x^3}{3\alpha^2}, \quad 4k_0 - k_1^2 < 0 \quad (3.19)$$

by the relation

$$u_j = \frac{(k_1 - \sqrt{k_1^2 - 4k_0}) h_j \exp[-\alpha \sqrt{k_2} \sqrt{k_1^2 - 4k_0} \int (v_j)_x^{-1} \partial x] - k_1 - \sqrt{k_1^2 - 4k_0}}{2 - 2h_j \exp[-\alpha \sqrt{k_2} \sqrt{k_1^2 - 4k_0} \int (v_j)_x^{-1} \partial x]}, \quad (3.20)$$

where  $v_j$  satisfy (3.19) and are iterated as follows:

$$(v_{j+1})_x = -\frac{\beta}{2} \frac{1}{(v_j)_x}, \quad \beta^2 = -6\alpha^2 k_2 (4k_0 - k_1^2), \quad 4k_0 - k_1^2 < 0, \quad j = 1, 2, \dots \quad (3.21)$$

Here  $h_j$  are arbitrary constants.

The proof of Propositions 3.2 and Proposition 3.3 consists of a systematic multipotentialisation of the Krichever–Novikov equation (3.1), followed by the composition of the potential variables. The potentialisations are shown in Diagram 4 below.

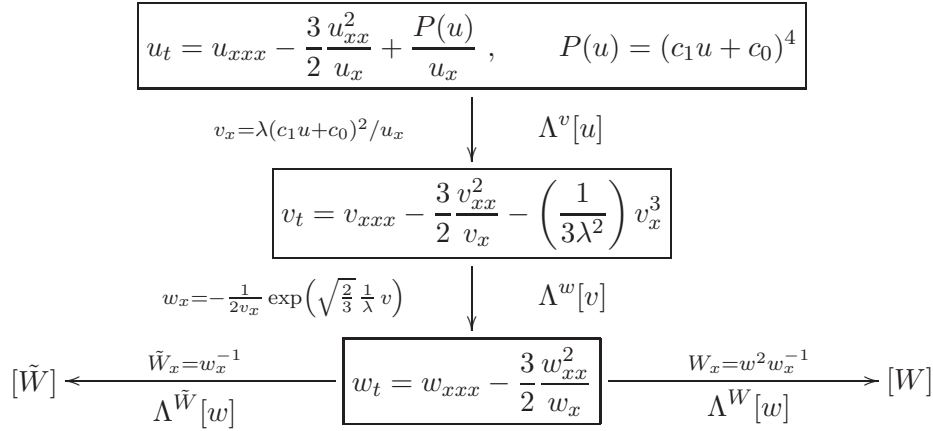


Diagram 3. Multipotentialisations of the special Krichever–Novikov equation (3.4).

Discussion of Diagram 3: The iterating-solution formula, (3.6), follows from the composition of the first two potential variables,  $v$  and  $w$ , shown in Diagram 3, i.e. the first two steps in the diagram. The symbols,  $[W]$  and  $[\tilde{W}]$ , refer to the autopotentialisation of the Schwarzian–KdV equation, that is,  $[W]$  represents the Schwarzian–KdV equation in terms of  $W$  and  $[\tilde{W}]$  the Schwarzian–KdV equation in terms of  $\tilde{W}$ .

**Example.** We consider Proposition 3.3 with  $k_0 = 0$ ,  $k_1 = 1$ ,  $k_2 = 1/9$  and  $\alpha = 1$ . A seed solution for (3.19) takes the form

$$v_1 = \frac{\sqrt{6}}{4} \ln \left[ \frac{1}{a_1} \tan \left( \frac{6^{1/4}}{3} \left( x - \frac{\sqrt{6}}{9} t + a_2 \right) \right)^2 + 1 \right], \quad (3.22)$$

where  $a_1$  and  $a_2$  are arbitrary constants. By the formula (3.20) a solution of (3.18) takes the form

$$u_1 = \frac{\tan \left( \frac{6^{1/4}}{27} (9x - \sqrt{6}t + 9a_2) \right)}{h_1 \left[ \tan \left( \frac{6^{1/4}}{27} (9x - \sqrt{6}t + 9a_2) \right)^2 + 1 \right]^{1/2} - \tan \left( \frac{6^{1/4}}{27} (9x - \sqrt{6}t + 9a_2) \right)}. \quad (3.23)$$

The solution  $v_1$  of (3.19) can now be iterated for more solutions  $u_2, u_3$  etc.

The conserved currents,  $\Phi^t$ , and flux,  $\Phi^x$ , used in Diagram 3 to defined the potential variables  $v, w, W$ , and  $\tilde{W}$ , are

$$\begin{aligned}
 \Phi_v^t[u] &= \frac{\lambda P^{1/2}}{u_x}, \quad P = (c_1 u + c_0)^4 \\
 \Phi_v^x[u] &= \lambda \left( P^{1/2} \frac{u_{xxx}}{u_x^2} - P^{1/2} \frac{u_{2x}^2}{2u_x^3} - P' P^{-1/2} \frac{u_{xx}}{u_x} - \frac{1}{2} (P')^2 P^{-3/2} u_x \right. \\
 &\quad \left. + P'' P^{-1/2} u_x + \frac{1}{3} \frac{P^{3/2}}{u_x^3} \right)
 \end{aligned}$$

$$u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x} + \frac{P(u)}{u_x}, \quad P(u) = k_2(u^2 + k_1 u + k_0)^2$$

$$v_x = \alpha P^{1/2} u_x^{-1} \quad \downarrow \quad \Lambda^v[u]$$

$$v_t = v_{xxx} - \frac{3}{2} \frac{v_{xx}^2}{v_x} - \left( \frac{3\alpha^2 k_2 (4k_0 - k_1^2)}{2} \right) \frac{1}{v_x} - \left( \frac{1}{3\alpha^2} \right) v_x^3$$

$$w_x = -\frac{\beta}{2} \frac{1}{v_x} \quad \downarrow \quad \Lambda^w[v], \quad \beta^2 = -6\alpha^2 k_2 (4k_0 - k_1^2)$$

$$w_t = w_{xxx} - \frac{3}{2} \frac{w_{xx}^2}{w_x} - \left( \frac{3\alpha^2 k_2 (4k_0 - k_1^2)}{2} \right) \frac{1}{w_x} - \left( \frac{1}{3\alpha^2} \right) w_x^3$$

where

$$\Phi_v^x[u] = \alpha \left( \frac{\sqrt{P} u_{xxx}}{u_x^2} - \frac{\sqrt{P} u_{xx}^2}{2 u_x^3} - \frac{P'}{\sqrt{P}} \frac{u_{xx}}{u_x} - \frac{(P')^2}{2} \frac{u_x}{P^{3/2}} + \frac{P'' u_x}{\sqrt{P}} + \frac{P^{3/2}}{3u_x^3} \right)$$

$$\Phi_w^x[v] = -\frac{\beta}{2} \frac{v_{xxx}}{v_x^2} + \frac{\beta}{4} \frac{v_{xx}^2}{v_x^3} + \frac{1}{4} \beta \alpha^2 k_2 (4k_0 - k_1^2) \frac{1}{v_x^3} + \frac{\beta}{2\alpha^2} v_x$$

$$\Lambda^v[u] = \alpha \left( \sqrt{P} \frac{u_{xx}}{u_x^3} - \frac{1}{2} \frac{P'}{\sqrt{P}} \frac{1}{u_x} \right), \quad P(u) = k_2(u^2 + k_1 u + k_0)^2$$

$$\Lambda^w[v] = \beta \frac{v_{xx}}{v_x^2}.$$

Diagram 4. Multipotentialisations of the special Krichever–Novikov equations (3.14) and (3.18).

$$\Phi_w^t[v] = -\frac{1}{2v_x} \exp \left\{ \frac{1}{\lambda} \sqrt{\frac{2}{3}} v \right\}$$

$$\Phi_w^x[v] = -\exp \left\{ \frac{1}{\lambda} \sqrt{\frac{2}{3}} v \right\} \left( \frac{1}{2} \frac{v_{3x}}{v_x^2} - \frac{1}{4} \frac{v_{2x}^2}{v_x^3} - \sqrt{\frac{2}{3}} \frac{1}{\lambda} \frac{v_{2x}}{v_x} + \frac{v_x}{6\lambda^2} \right),$$

respectively. Note that we do not give here the current and flux for the autopotentialisable case as this was already discussed and given in Sec. 2 (see Diagram 2).

The integrating factors,  $\Lambda$ , as shown in Diagram 3 are

$$\Lambda^v[u] = \lambda \left( \sqrt{P} \frac{u_{xx}}{u_x^3} - \frac{1}{2} \frac{P'}{\sqrt{P}} \frac{1}{u_x} \right), \quad P(u) = (c_1 u + c_0)^4$$

$$\Lambda^w[v] = e^{\sqrt{2/3}v/\lambda} \left( \frac{v_{xx}}{v_x^3} - \sqrt{\frac{2}{3}} \frac{1}{\lambda v_x} \right).$$

Diagram 4 shows the multipotentialisations of (3.14) and (3.18), which lead to the iterating-solution formulae (3.16) and (3.20), respectively, for the constants  $k_0$  and  $k_1$  as given in Propositions 2.2 and 2.3. Furthermore the autopotentialisation regarding the potential variable,  $w$ , leads to

**Proposition 3.4.** *An iterating-solution formula for*

$$u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x} - \frac{3}{2} \frac{\alpha^2 k_2 (4k_0 - k_1^2)}{u_x} - \frac{u_x^3}{3\alpha^2} \quad (3.24)$$

takes the form

$$(u_{j+1})_x = \frac{\beta}{2} \frac{1}{(u_j)_x}, \quad j = 0, 1, 2, \dots, \quad (3.25)$$

where  $\beta^2 = -6\alpha^2 k_2 (4k_0 - k_1^2)$  with  $4k_0 - k_1^2 > 0$  or  $4k_0 - k_1^2 < 0$  and every  $u_j$  is a solution of (3.24).

#### 4. Concluding Remarks

We have shown that a systematic potentialisation of a given evolution equation can lead to auto-Bäcklund transformations and hence to iterating-solution formulae for the equation. We have obtained several iterating-solution formulae for special cases of the Krichever–Novikov equation. We should, however, point out that we were not able to find such formulae for the full Krichever–Novikov equation, (1.1), and, in particular, the case of the original Krichever–Novikov equation, where  $P(u) = 4u^3 + a_1 u + a_2$ , is not included in the cases described by Propositions 2.1, 2.2 and 2.3. It might be possible to extend the current results for the Krichever–Novikov equation to include these important cases, but this seems not possible with the presented method of potentialisation.

We believe that the method presented in this paper can be applied to a large class of evolution equations to gain interesting connections between evolution equations as well as iterating-solution formulae, similar to those obtained here for some special Krichever–Novikov equations.

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