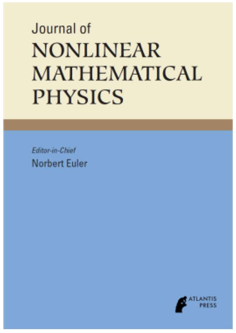

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INTEGRATION OF SYSTEMS OF FIRST-ORDER EQUATIONS ADMITTING NONLINEAR SUPERPOSITION

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Systems of two nonlinear ordinary differential equations of the first order admitting nonlinear superpositions are investigated using Lie's enumeration of groups on the plane. It is shown that the systems associated with two-dimensional Vessiot–Guldberg–Lie algebras can be integrated by quadrature upon introducing Lie's canonical variables. The knowledge of a symmetry group of a system in question is not needed in this approach. The systems associated with three-dimensional Vessiot–Guldberg–Lie algebras are classified into 13 standard forms 10 of which are integrable by quadratures and three are reduced to Riccati equations.

Keywords: Nonlinear superposition; Vessiot–Guldberg–Lie algebras; standard forms of Lie algebras; canonical variables.

1. Lie's Classification of L_2 and L_3

1.1. *Canonical variables for two-dimensional Lie algebras*

Consider linearly independent first-order linear partial differential operators

$$X_1 = \xi_1(x,y)\frac{\partial}{\partial x} + \eta_1(x,y)\frac{\partial}{\partial y}, \quad X_2 = \xi_2(x,y)\frac{\partial}{\partial x} + \eta_2(x,y)\frac{\partial}{\partial y} \tag{1.1}$$

with two variables x, y. The *commutator* $[X_1, X_2]$ of the operators (1.1) is a linear partial differential operator defined by the formula

$$[X_1, X_2] = X_1X_2 - X_2X_1,$$

or equivalently

$$[X_1, X_2] = (X_1(\xi_2) - X_2(\xi_1))\frac{\partial}{\partial x} + (X_1(\eta_2) - X_2(\eta_1))\frac{\partial}{\partial y}. \tag{1.2}$$

The linear space L_2 spanned by the operators (1.1) is a two-dimensional Lie algebra if

$$[X_1, X_2] \in L_2. \tag{1.3}$$

In order to formulate the result about canonical variables in two-dimensional Lie algebras it is convenient to use, along with the commutator $[X_1, X_2]$ of the operators (1.1), their *pseudoscalar product*

$$X_1 \vee X_2 = \xi_1\eta_2 - \eta_1\xi_2. \tag{1.4}$$

Recall that the operators (1.1) are said to be *linearly connected* if the equation

$$\lambda_1(x, y)X_1 + \lambda_2(x, y)X_2 = 0$$

holds identically in x, y with certain functions $\lambda_1(x, y)$, $\lambda_2(x, y)$, not both zero.

A geometrical significance of the pseudo-scalar product is clarified by the following statement: *the operators* (1.1) *are linearly connected if and only if their pseudo-scalar product* (1.4) *vanishes.*

Lie's method of integration of second-order ordinary differential equations by using their symmetries is based on the existence of so-called *canonical coordinates* in two-dimensional Lie algebras. These variables provide for every L_2 the simplest form of its basis and therefore reduce a differential equation admitting L_2 to an integrable form. The basic statement is formulated as follows (for the proof, see [1, Chapter 18, §1]; see also [2, Section 12.2.2]).

Theorem 1. *Any two-dimensional Lie algebra can be transformed by a proper choice of its basis and suitable variables*, t *and* u, *to one and only one of the four nonsimilar standard forms presented in Table* 1.

The variables, t and u, presented in Table 1 are called *canonical variables.* They are found for each type by solving the following systems of first-order linear partial differential equations:

$$\begin{aligned} &\textbf{Type I:}\ X_1(t) = 1,\ X_2(t) = 0; \quad X_1(u) = 0,\ X_2(u) = 1.\\ &\textbf{Type II:}\ X_1(t) = 0,\ X_2(t) = 0; \quad X_1(u) = 1,\ X_2(u) = t.\\ &\textbf{Type III:}\ X_1(t) = 0,\ X_2(t) = t; \quad X_1(u) = 1,\ X_2(u) = u.\\ &\textbf{Type IV:}\ X_1(t) = 0,\ X_2(t) = 0; \quad X_1(u) = 1,\ X_2(u) = u. \end{aligned} \tag{1.5}$$

1.2. *Three-dimensional Lie algebras*

Lie showed that the basis X_1, X_2, X_3 of any three-dimensional algebra of operators in two variables can be mapped by a complex change of variables to one of the following 13 *standard forms* (see Table 2) (see [1, Chapter 22]; see also [2, Section 7.3.8]).

Table 1. Structure and standard forms of L_2.

Type	Structure of L_2		Standard form of L_2
I	$[X_1, X_2] = 0,$	$X_1 \vee X_2 \neq 0$	$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial u}$
II	$[X_1, X_2] = 0,$	$X_1 \vee X_2 = 0$	$X_1 = \frac{\partial}{\partial u}, \quad X_2 = t\frac{\partial}{\partial u}$
III	$[X_1, X_2] = X_1,$	$X_1 \vee X_2 \neq 0$	$X_1 = \frac{\partial}{\partial u}, \quad X_2 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$
IV	$[X_1, X_2] = X_1,$	$X_1 \vee X_2 = 0$	$X_1 = \frac{\partial}{\partial u}, \quad X_2 = u\frac{\partial}{\partial u}$

Table 2. Standard forms of three-dimensional Lie algebras.

A. The first derived algebra has dimension three:

(1) $X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad X_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y},$

(2) $X_1 = \frac{\partial}{\partial x}, \quad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y},$

(3) $X_1 = \frac{\partial}{\partial y}, \quad X_2 = y\frac{\partial}{\partial y}, \quad X_3 = y^2\frac{\partial}{\partial y}.$

B. The first derived algebra has dimension two:

(4) $X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x} + cy\frac{\partial}{\partial y}$ $(c \neq 0, \neq 1),$

(5) $X_1 = \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial y}, \quad X_3 = (1-c)x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ $(c \neq 0, \neq 1),$

(6) $X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$

(7) $X_1 = \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial y}, \quad X_3 = y\frac{\partial}{\partial y},$

(8) $X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = (x+y)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$

(9) $X_1 = \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$

C. The first derived algebra has dimension one:

(10) $X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x},$

(11) $X_1 = \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$

(12) $X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial y}.$

D. The first derived algebra has dimension zero:

(13) $X_1 = \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial y}, \quad X_3 = p(x)\frac{\partial}{\partial y}.$

Recall that the derived algebra L_3' of the Lie algebra L_3 with a basis X_1, X_2, X_3 is the algebra spanned by the commutators $[X_1, X_2]$, $[X_1, X_3]$ and $[X_2, X_3]$. The higher derivatives are defined by induction, $L_3'' = (L_3')'$, etc. A Lie algebra is *solvable* if its derivative of a certain order vanishes. It is obvious that L_3 is solvable if $\dim L_3' \leq 2$ and not solvable if $\dim L_3' = 3$.

Remark 1. In (13) $p(x)$ is any given function. Lie uses the algebras (1.1) and (1.2) also in the following alternative forms:

$$(1') \; X_1 = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}, \quad X_3 = (x^2 - y)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y},$$

$$(2') \; X_1 = x\frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad X_3 = y\frac{\partial}{\partial x}.$$

Remark 2. It is also useful to utilize, instead of Lie's complex classification, the classification of three-dimensional Lie algebras over the reals [3] (see also [4, Chapter 8] written by N. H. Ibragimov and F. M. Mahomed).

2. Systems with Nonlinear Superposition

2.1. *Integration of systems associated with L_2*

A method for integrating systems of ordinary differential equations admitting nonlinear superpositions with two-dimensional associated Lie algebras L_2 was suggested in [2, Section 11.2.6]. The result is formulated as follows.

Theorem 2. *Consider a system of coupled nonlinear first-order ordinary differential equations of the form*

$$\begin{aligned}\frac{dx}{dt} &= T_1(t)\xi_1^1(x,y) + T_2(t)\xi_2^1(x,y),\\ \frac{dy}{dt} &= T_1(t)\xi_1^2(x,y) + T_2(t)\xi_2^2(x,y)\end{aligned} \tag{2.1}$$

admitting a nonlinear superposition principle. Let the operators

$$X_1 = \xi_1^1(x,y)\frac{\partial}{\partial x} + \xi_1^2(x,y)\frac{\partial}{\partial y}, \quad X_2 = \xi_2^1(x,y)\frac{\partial}{\partial x} + \xi_2^2(x,y)\frac{\partial}{\partial y} \tag{2.2}$$

associated with the system (2.1) *span a two-dimensional Lie algebra L_2, i.e.*

$$[X_1, X_2] = c_1X_1 + c_2X_2, \quad c_1, c_2 = \text{const.}$$

Then Eq. (2.1) *can be solved by quadratures upon introducing canonical variables.*

Proof. After a change of the variables x, y into new variables

$$\tilde{x} = \tilde{x}(x,y), \quad \tilde{y} = \tilde{y}(x,y) \tag{2.3}$$

without changing t, the operators (2.2) are transformed into the operators

$$X_1 = \tilde{\xi}_1^1(\tilde{x},\tilde{y})\frac{\partial}{\partial \tilde{x}} + \tilde{\xi}_1^2(\tilde{x},\tilde{y})\frac{\partial}{\partial \tilde{y}}, \quad X_2 = \tilde{\xi}_2^1(\tilde{x},\tilde{y})\frac{\partial}{\partial \tilde{x}} + \tilde{\xi}_2^2(\tilde{x},\tilde{y})\frac{\partial}{\partial \tilde{y}}, \tag{2.4}$$

where the vectors $(\tilde{\xi}_\alpha^1, \tilde{\xi}_\alpha^2)$, $\alpha = 1, 2$ are obtained from the vectors $(\xi_\alpha^1, \xi_\alpha^2)$ by the transformation law for contravariant vectors:

$$\begin{aligned}\tilde{\xi}_\alpha^1 &= \frac{\partial \tilde{x}(x,y)}{\partial x}\xi_\alpha^1 + \frac{\partial \tilde{x}(x,y)}{\partial y}\xi_\alpha^2,\\ \tilde{\xi}_\alpha^2 &= \frac{\partial \tilde{y}(x,y)}{\partial x}\xi_\alpha^1 + \frac{\partial \tilde{y}(x,y)}{\partial y}\xi_\alpha^2.\end{aligned}$$

The derivative of (x, y) with respect to t obeys the same transformation law:

$$\begin{aligned}\frac{d\tilde{x}}{dt} &= \frac{\partial \tilde{x}(x,y)}{\partial x}\frac{dx}{dt} + \frac{\partial \tilde{x}(x,y)}{\partial y}\frac{dy}{dt},\\ \frac{d\tilde{y}}{dt} &= \frac{\partial \tilde{y}(x,y)}{\partial x}\frac{dx}{dt} + \frac{\partial \tilde{y}(x,y)}{\partial y}\frac{dy}{dt}.\end{aligned}$$

Therefore Eq. (2.1) are written in the form

$$\frac{d\tilde{x}}{dt} = T_1(t)\tilde{\xi}_1^1(\tilde{x},\tilde{y}) + T_2(t)\tilde{\xi}_2^1(\tilde{x},\tilde{y}),$$
$$\frac{d\tilde{y}}{dt} = T_1(t)\tilde{\xi}_1^2(\tilde{x},\tilde{y}) + T_2(t)\tilde{\xi}_2^2(\tilde{x},\tilde{y}) \tag{2.5}$$

with the same coefficients $T_1(t), T_2(t)$ as those in the system (2.1).

To complete the proof we chose for $\tilde{x}, \tilde{y}$ canonical variables mapping the operators (2.2) to the standard forms from Table 1 and hence convert Eq. (2.1) to the simple integrable forms given in the following table, where $\tilde{x}, \tilde{y}$ are denoted again by x, y. □

Example 1. We apply the method to the following nonlinear system:

$$\frac{dx}{dt} = xy^2 - \frac{x}{2t}, \quad \frac{dy}{dt} = x^2y - \frac{y}{2t}. \tag{2.6}$$

In this case we have Eq. (2.1) with

$$T_1(t) = 1, \qquad \xi_1^1(x,y) = xy^2, \quad \xi_1^2(x,y) = x^2y,$$
$$T_2(t) = -\frac{1}{2t}, \quad \xi_2^1(x,y) = x, \qquad \xi_2^2(x,y) = y. \tag{2.7}$$

Hence the operators (2.2) are written:

$$X_1 = xy^2\frac{\partial}{\partial x} + x^2y\frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{2.8}$$

We have

$$[X_1,\ X_2] = -2X_1, \quad X_1 \vee X_2 \equiv \xi_1\eta_2 - \eta_1\xi_2 = xy(y^2 - x^2) \neq 0.$$

Hence the operators (2.8) span a two-dimensional Lie algebra of type III. Therefore we can transform the operators (2.8) and Eq. (2.6) to the form III from Table 3.

Let us find canonical variables $\tilde{x}, \tilde{y}$ for the first operator (2.8) by solving the equations

$$X_1(\tilde{x}) = 0, \quad X_1(\tilde{y}) = 1$$

Table 3. Standard forms of operators (2.2) and systems (2.1).

	Standard forms of operators (2.2)	Standard forms of Eq. (2.1)
I	$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}$	$\frac{dx}{dt} = T_1(t), \quad \frac{dy}{dt} = T_2(t)$
II	$X_1 = \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial y}$	$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = T_1(t) + T_2(t)x$
III	$X_1 = \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$	$\frac{dx}{dt} = T_2(t)x, \quad \frac{dy}{dt} = T_1(t) + T_2(t)y$
IV	$X_1 = \frac{\partial}{\partial y}, \quad X_2 = y\frac{\partial}{\partial y}$	$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = T_1(t) + T_2(t)y$

in accordance with Eq. (2.5) for Type III. These equations are written

$$xy^2\frac{\partial \tilde{x}}{\partial x} + x^2y\frac{\partial \tilde{x}}{\partial y} = 0, \quad xy^2\frac{\partial \tilde{y}}{\partial x} + x^2y\frac{\partial \tilde{y}}{\partial y} = 1.$$

The characteristic equation

$$\frac{dx}{y} - \frac{dy}{x} = 0$$

for the equation $X_1(\tilde{x}) = 0$ has the first integral $x^2 - y^2 = \text{const}$. Hence, $\tilde{x}$ is an arbitrary function of $x^2 - y^2$. One can take it in the simplest form $\tilde{x} = x^2 - y^2$.

We solve the equation $X_1(\tilde{y}) = 1$. Consider its characteristic system

$$\frac{dx}{xy^2} = \frac{dy}{x^2y} = d\tilde{y}.$$

Using the integral $x^2 - y^2 = a^2$ given by the first equation of this system we write the second equation $dx/(xy^2) = d\tilde{y}$ of the characteristic system in the form

$$d\tilde{y} = \frac{1}{a^2}\left[\frac{1}{2(x-a)} + \frac{1}{2(x+a)} - \frac{1}{x}\right] dx.$$

The resulting integral

$$\tilde{y} - \frac{1}{a^2}[\ln\sqrt{x^2 - a^2} - \ln|x|] = C$$

together with $x^2 - y^2 = a^2$ provide the solution to the equation $X_1(\tilde{y}) = 1$:

$$\tilde{y} = \frac{\ln|y| - \ln|x|}{x^2 - y^2} + F(x^2 - y^2).$$

Letting $F = 0$ and assuming that x, y are positive, we obtain the following variables:

$$\tilde{x} = x^2 - y^2, \quad \tilde{y} = \frac{\ln y - \ln x}{x^2 - y^2}. \tag{2.9}$$

One can verify that the variables (2.9) are the canonical variables required for our algebra L_2. Indeed, the operators (2.8) are written in the form of Type III of Table 3 (up to inessential constant factors in X_2):

$$X_1 = \frac{\partial}{\partial \tilde{y}}, \quad X_2 = 2\left(\tilde{x}\frac{\partial}{\partial \tilde{x}} - \tilde{y}\frac{\partial}{\partial \tilde{y}}\right).$$

These operators have the form (2.4) with

$$\tilde{\xi}_1^1 = 0, \quad \tilde{\xi}_1^2 = 1, \quad \tilde{\xi}_2^1 = 2\tilde{x}, \quad \tilde{\xi}_2^2 = -2\tilde{y}.$$

Substituting these expressions into (2.5) (or differentiating (2.9) with respect to t and using Eq. (2.6)) we see that Eq. (2.6) are written in the variables (2.9) as the following simple linear equations:

$$\frac{d\tilde{x}}{dt} = -\frac{\tilde{x}}{t}, \quad \frac{d\tilde{y}}{dt} = 1 + \frac{\tilde{y}}{t}. \tag{2.10}$$

Integration of Eq. (2.10) yields:

$$\tilde{x} = \frac{C_1}{t}, \quad \tilde{y} = C_2 t + t \ln t. \tag{2.11}$$

Now we solve Eq. (2.9) with respect to x and y:

$$x = \sqrt{\frac{\tilde{x}}{1 - \mathrm{e}^{2\tilde{x}\,\tilde{y}}}}, \quad y = \sqrt{\frac{\tilde{x}}{\mathrm{e}^{-2\tilde{x}\,\tilde{y}} - 1}},$$

substitute here the solutions (2.11) and finally arrive at the following general solution to the system of Eq. (2.6):

$$x = \sqrt{\frac{k}{t(1-\zeta^2)}}, \quad y = \zeta\sqrt{\frac{k}{t(1-\zeta^2)}}. \tag{2.12}$$

Here $\zeta = Ct^k$, where C and k are arbitrary constants.

2.2. *Integration of systems associated with L_3*

Using Lie's classification of three-dimensional algebras, we can extend Theorem 2 from Subsec. 2.1 as follows.

Theorem 3. *Consider a system of coupled nonlinear first-order ordinary differential equations of the form*

$$\begin{aligned} \frac{dx}{dt} &= T_1(t)\xi_1^1(x,y) + T_2(t)\xi_2^1(x,y) + T_3(t)\xi_3^1(x,y), \\ \frac{dy}{dt} &= T_1(t)\xi_1^2(x,y) + T_2(t)\xi_2^2(x,y) + T_3(t)\xi_3^1(x,y) \end{aligned} \tag{2.13}$$

admitting a nonlinear superposition principle. Let the operators

$$\begin{aligned} X_1 &= \xi_1^1(x,y)\frac{\partial}{\partial x} + \xi_1^2(x,y)\frac{\partial}{\partial y}, \\ X_2 &= \xi_2^1(x,y)\frac{\partial}{\partial x} + \xi_2^2(x,y)\frac{\partial}{\partial y}, \\ X_3 &= \xi_3^1(x,y)\frac{\partial}{\partial x} + \xi_3^2(x,y)\frac{\partial}{\partial y} \end{aligned} \tag{2.14}$$

associated with the system (2.13) *span a three-dimensional Lie algebra L_3. Then Eq.* (2.13) *can be solved by quadratures if the algebra L_3 is solvable and reduced to integration of Riccati equations if L_3 is not solvable.*

Proof. We transform the Lie algebra L_3 associated with the system (2.13) to an appropriate standard form given in Table 2 and map Eq. (2.13) to the following forms (see Table 4).

In Table 4 $\dot{x}$ and $\dot{y}$ are the derivatives of x and y with respect to t.

It is manifest that the systems of forms B, C and D can be solved by quadratures. It is also obvious that the systems in A require integration of Riccati equations and, in general, cannot be solved by quadratures. This completes the proof of the theorem. □

Table 4. Standard forms of Eq. (2.13).

A. dim $L'_3 = 3$:	
(1) $\dot{x} = T_1(t) + T_2(t)x + T_3(t)x^2,$	$\dot{y} = T_1(t) + T_2(t)y + T_3(t)y^2;$
(2) $\dot{x} = T_1(t) + 2T_2(t)x + T_3(t)x^2,$	$\dot{y} = T_2(t)y + T_3(t)xy;$
(3) $\dot{x} = 0,$	$\dot{y} = T_1(t) + T_2(t)y + T_3(t)y^2.$
B. dim $L'_3 = 2$:	
(4) $\dot{x} = T_1(t) + 2T_2(t)x + T_3(t)x^2,$	$\dot{y} = T_2(t)y + T_3(t)xy \ (c \neq 0, \neq 1);$
(5) $\dot{x} = (1-c)T_3(t)x \ (c \neq 0, \neq 1),$	$\dot{y} = T_1(t) + T_2(t)x + T_3(t)y;$
(6) $\dot{x} = T_1(t) + T_3(t)x,$	$\dot{y} = T_2(t) + T_3(t)y;$
(7) $\dot{x} = 0,$	$\dot{y} = T_1(t) + T_2(t)x + T_3(t)y;$
(8) $\dot{x} = T_1(t) + T_2(t)(x+y),$	$\dot{y} = T_2(t) + T_3(t)y;$
(9) $\dot{x} = T_3(t),$	$\dot{y} = T_1(t) + T_2(t)x + T_3(t)y.$
C. dim $L'_3 = 1$:	
(10) $\dot{x} = T_1(t) + T_3(t)x,$	$\dot{y} = T_2(t);$
(11) $\dot{x} = T_3(t)x,$	$\dot{y} = T_1(t) + T_2(t)x + T_3(t)y;$
(12) $\dot{x} = T_1(t),$	$\dot{y} = T_2(t) + T_3(t)x.$
D. dim $L'_3 = 0$:	
(13) $\dot{x} = 0,$	$\dot{y} = T_1(t) + T_2(t)x + T_3(t)p(x).$

Remark 3. The alternative forms mentioned in Remark 1 provide the following alternative standard forms of Eq. (2.13):

$$(1') \ \dot{x} = T_1(t) + T_2(t)x + T_3(t)(x^2-1), \quad \dot{y} = T_1(t)x + 2T_2(t)y + T_3(t)xy;$$
$$(2') \ \dot{x} = T_2(t)x + T_3(t)y, \qquad \dot{y} = T_1(t)x - T_2(t)y.$$

In certain particular cases the systems in A can be integrated either by quadratures or in terms of special functions. The simplest case is $T_3(t) = 0$. Then Eqs. (2.1)–(2.3) become easily solvable linear systems. Furthermore, if $T_1(t) = 0$, the Riccati equations in systems (2.1)–(2.3) can be linearized by a change of the dependent variables (see [5, Chapter 1]). Moreover it is demonstrated in [5] that the Riccati equations in systems (2.1), (2.3) can be linearized by a change of the dependent variables if

$$T_3(t) = k[T_2(t) - kT_1(t)], \quad k = \text{const}.$$

In the case of system (2.2) this condition is replaced by $T_3(t) = k[2T_2(t) - kT_1(t)]$.

It is well known that if $T_3(t) \neq 0$, one can transform the Riccati equations in question to the equivalent form with $T_3(t) = -1$ and $T_2(t) = 0$. Assuming that this transformation has been done, we consider, e.g., system (2.2),

$$\dot{x} + x^2 = T_1(t), \quad \dot{y} + xy = 0. \tag{2.15}$$

We set $x = (\ln|u|)'$ and rewrite the first equation of this system in the form of a linear second-order equation

$$u'' = T_1(t)u,$$

where u' is the derivative of u with respect to t. The above equation can be solved in terms of special functions if $T_1(t)$ is a linear function. Indeed, let $T_1(t) = \alpha t + \beta, \ \alpha \neq 0$. Then our equation

$$u'' = (\alpha t + \beta)u$$

becomes the *Airy equation*

$$\frac{d^2u}{d\tau^2} - \tau u = 0$$

upon introducing the new independent variable

$$\tau = \alpha^{-2/3}[\alpha t + \beta].$$

The general solution to the Airy equation is given by the linear combination

$$u = C_1 \mathrm{Ai}(\tau) + C_2 \mathrm{Bi}(\tau)$$

of the *Airy functions*

$$\begin{aligned} \mathrm{Ai}(\tau) &= \frac{1}{\pi}\int_0^\infty \cos\left(s\tau + \frac{1}{3}\,s^3\right) ds, \\ \mathrm{Bi}(\tau) &= \frac{1}{\pi}\int_0^\infty \left[\exp\left(s\tau - \frac{1}{3}\,s^3\right) + \sin\left(s\tau + \frac{1}{3}\,s^3\right)\right] ds. \end{aligned}$$

Assuming that $C_1 \neq 0$ and introducing the new constant $K_1 = C_2/C_1$ we obtain

$$x(t) = \frac{d}{dt} \ln |\mathrm{Ai}(\alpha^{-2/3}[\alpha t + \beta]) + K_1 \mathrm{Bi}(\alpha^{-2/3}[\alpha t + \beta])|. \tag{2.16}$$

Now we substitute (2.16) into the second equation of the system (2.15) and obtain upon integration:

$$y(t) = K_2\{\mathrm{Ai}(\alpha^{-2/3}[\alpha t + \beta]) + K_1 \mathrm{Bi}(\alpha^{-2/3}[\alpha t + \beta])\}^{-1}. \tag{2.17}$$

Thus the solution of the system (2.15) with $T_1(t) = \alpha t + \beta$ is given in terms of the special functions (2.16) and (2.17).

Example 2. Consider the nonlinear system

$$\begin{aligned} \frac{dx}{dt} &= -T_1(t) y \exp\left[\arctan\left(\frac{y}{x}\right)\right] + T_2(t)x + T_3(t)y, \\ \frac{dy}{dt} &= T(t) x \exp\left[\arctan\left(\frac{y}{x}\right)\right] + T_2(t)y - T_3(t)x \end{aligned} \tag{2.18}$$

with arbitrary coefficients $T_1(t), T_2(t), T_3(t)$. The operators (2.14) associated with Eq. (2.18) have the form

$$\begin{aligned} X_1 &= \exp\left[\arctan\left(\frac{y}{x}\right)\right]\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right), \\ X_2 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \\ X_3 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \end{aligned} \tag{2.19}$$

and span a three-dimensional Lie algebra L_3 with the following commutator relations:

$$[X_1,\ X_2] = 0, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = 0. \tag{2.20}$$

It follows that the derived algebra L_3' has dimension one and hence our algebra L_3 belongs to the category C of Table 2. Specifically comparison of the commutator relations (2.20) with the commutators of the standard operators (2.10), (2.11) or (2.12) from Table 2 shows that the operators (2.19) can be mapped by a change of variables (2.3) either to (2.10) or to (2.11). However, it is easy to show that they cannot be mapped to the form (2.11). Indeed the change of variables (2.3) converts (2.19) to the form (2.11),

$$X_1 = \frac{\partial}{\partial \tilde{y}}, \quad X_2 = \tilde{x}\frac{\partial}{\partial \tilde{y}}, \quad X_3 = \tilde{x}\frac{\partial}{\partial \tilde{x}} + \tilde{y}\frac{\partial}{\partial \tilde{y}},$$

if $\tilde{x}$ and $\tilde{y}$ solve the following over-determined system:

$$\begin{aligned} X_1(\tilde{x}) &= 0, \quad X_1(\tilde{y}) = 1,\\ X_2(\tilde{x}) &= 0, \quad X_2(\tilde{y}) = \tilde{x},\\ X_3(\tilde{x}) &= \tilde{x}, \quad X_3(\tilde{y}) = \tilde{y}, \end{aligned}$$

where X_1, X_2 and X_3 are the operators (2.19). These equations are not compatible. For example, the equations $X_1(\tilde{x}) = 0$ and $X_3(\tilde{x}) = \tilde{x}$ contradict each other because X_1 differs from X_3 by a non-vanishing factor only. For another reasoning see the general construction of similarity transformations given in [2, Section 7.3.7].

We find the change of variables (2.3) mapping (2.19) to the form (2.10), namely

$$X_1 = \frac{\partial}{\partial \tilde{x}}, \quad X_2 = \frac{\partial}{\partial \tilde{y}}, \quad X_3 = \tilde{x}\frac{\partial}{\partial \tilde{x}}. \tag{2.21}$$

Now $\tilde{x}$ and $\tilde{y}$ should solve the following over-determined systems:

$$\begin{aligned} X_1(\tilde{x}) &= 1, \quad X_1(\tilde{y}) = 0,\\ X_2(\tilde{x}) &= 0, \quad X_2(\tilde{y}) = 1,\\ X_3(\tilde{x}) &= \tilde{x}, \quad X_3(\tilde{y}) = 0. \end{aligned}$$

Substituting the expressions (2.19) for X_1, X_2, X_3 we write these equations in the form

$$\begin{aligned} x\frac{\partial \tilde{x}}{\partial y} - y\frac{\partial \tilde{x}}{\partial x} &= \mathrm{e}^{-\arctan(y/x)}, \quad x\frac{\partial \tilde{y}}{\partial y} - y\frac{\partial \tilde{y}}{\partial x} = 0,\\ x\frac{\partial \tilde{x}}{\partial x} + y\frac{\partial \tilde{x}}{\partial y} &= 0, \quad x\frac{\partial \tilde{y}}{\partial x} + y\frac{\partial \tilde{y}}{\partial y} = 1,\\ x\frac{\partial \tilde{x}}{\partial y} - y\frac{\partial \tilde{x}}{\partial x} &= \tilde{x}, \quad x\frac{\partial \tilde{y}}{\partial y} - y\frac{\partial \tilde{y}}{\partial x} = 0. \end{aligned} \tag{2.22}$$

Comparison of the first and third equations for $\tilde{x}$ yields $\tilde{x} = \mathrm{e}^{-\arctan(y/x)}$. One can readily verify that this function solves all three equations (2.22) for $\tilde{x}$. The equations for $\tilde{y}$ are easy to solve and yield $\tilde{y} = \ln\sqrt{x^2 + y^2}$. Thus the canonical variables mapping the operators (2.19) to the standard form (2.21) are given by

$$\tilde{x} = \exp\left[-\arctan\left(\frac{y}{x}\right)\right], \quad \tilde{y} = \ln\sqrt{x^2+y^2}. \tag{2.23}$$

In these variables Eq. (2.18) are written in the integrable form (2.10) from Table 4:

$$\frac{d\tilde{x}}{dt} = T_1(t) + T_3(t)\tilde{x}, \quad \frac{d\tilde{y}}{dt} = T_1(t). \tag{2.24}$$

The results presented in this paper have been included in [6].

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