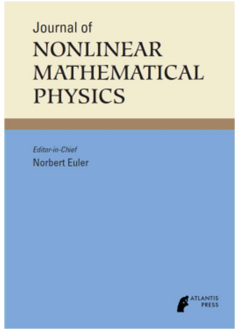

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SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS AND FIRST INTEGRALS OF THE FORM $A(t,x)\dot{x} + B(t,x)$

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We characterize the equations in the class $\mathcal{A}$ of the second-order ordinary differential equations $\ddot{x} = M(t,x,\dot{x})$ which have first integrals of the form $A(t,x)\dot{x} + B(t,x)$. We give an intrinsic characterization of the equations in $\mathcal{A}$ and an algorithm to calculate explicitly such first integrals. Although $\mathcal{A}$ includes equations that lack Lie point symmetries, the equations in $\mathcal{A}$ do admit λ-symmetries of a certain form and can be characterized by the existence of such λ-symmetries. The equations in a well-defined subclass of $\mathcal{A}$ can completely be integrated by using two independent first integrals of the form $A(t,x)\dot{x} + B(t,x)$. The methods are applied to several relevant families of equations.

Keywords: Ordinary differential equations; first integrals; λ-symmetries; Sundman transformations.

1. Introduction

The search of new methods to solve ordinary differential equations (ODEs) plays a fundamental role in the treatment of physical models. In being faced with the problem of solving a given ODE one may try to transform it into another ODE with known solutions. Usually the considered transformed equations are linear equations and invertible point transformations are the most commonly used. The first linearization problem for ODEs was solved by Lie [11]. He showed that a second-order ODE

$$\ddot{x} = M(t,x,\dot{x}) \tag{1.1}$$

is linearizable by a (local) change of variables if and only if the equation is of the form

$$\ddot{x} + a_3(t,x)\dot{x}^3 + a_2(t,x)\dot{x}^2 + a_1(t,x)\dot{x} + a_0(t,x) = 0 \tag{1.2}$$

and the coefficients a_i, $0 \le i \le 3$, satisfy two conditions involving their partial derivatives [12,9,10]. If Eq. (1.1) is linearizable to equation $X_{TT} = 0$, then this last equation has two

independent first integrals of the form $\widetilde{A}(T,X)\dot{X}+\widetilde{B}(T,X)$, but in terms of variables (t,x) the original equation may lack first integrals of the form

$$A(t,x)\dot{x}+B(t,x). \tag{1.3}$$

In the literature there are plenty of examples of equations with first integrals of the form (1.3) but, as far as we know, no characterizations of these equations have been derived.

Throughout this paper we say that a second-order ODE (1.1) belongs to the class $\mathcal{A}$ if the equation has a first integral of the form (1.3), by using the same variables in (1.1) and (1.3).

In Sec. 2 we prove that equations in $\mathcal{A}$ must have the form (1.2) with $a_3=0$, i.e. the equations must be of the form

$$\ddot{x}+a_2(t,x)\dot{x}^2+a_1(t,x)\dot{x}+a_0(t,x)=0. \tag{1.4}$$

The relationships between the coefficients a_i, $1\leq i\leq 2$, and the functions A and B are also established. We also consider in this section an equation in $\mathcal{A}$ that lacks Lie point symmetries (see Eq. (2.6)). Another example is given in Sec. 4 (Eq. (4.12)).

In Sec. 3 we give an intrinsic characterization of the equations in the class $\mathcal{A}$. We also provide an algorithm to determine the first integrals of the form 1.3 in terms of the coefficients a_i, $1\leq i\leq 2$. The main result in Sec. 3 (Theorem 2) connects $\mathcal{A}$ with the class of equations that are linearizable by a generalized Sundman transformation of the form

$$X=F(t,x), \quad dT=G(t,x)dt. \tag{1.5}$$

This is a consequence of Theorem 2 and the results of Duarte *et al.* in [6]. The three equations appearing in the examples of [6] are also considered here to illustrate our algorithm to determine first integrals.

We have already mentioned that there are equations in $\mathcal{A}$ that lack Lie point symmetries. Recently, several relationships between first integrals and λ-symmetries have been derived ([14,15]). In Sec. 4 we characterize the equations in $\mathcal{A}$ in terms of the λ-symmetries of the equation.

If the vector field ∂_x is a λ-symmetry for two different functions $\lambda_1=-a_2\dot{x}+\beta_1(t,x)$ and $\lambda_2=-a_2\dot{x}+\beta_2(t,x)$ then in Sec. 5 we prove that the equation necessarily admits two independent first integrals of the form (1.3) and therefore the equation can completely be integrated.

This complete integrability is illustrated for a large family of equations (Eq. (5.7)) that includes several significant equations of mathematical physics. These equations have been studied by several authors in order to find first integrals by using different approaches. The algorithms presented in this paper allow us to unify the treatment of these equations in a systematic and general procedure. We also characterize the equations of the form (1.4) with $a_2=0$ which have two independent first integrals of the form (1.3) as the second-order linear equations.

Several aspects related to the linearization of the equations studied in this paper will be dealt with in a separate work.

2. A First Characterization of the Equations in $\mathcal{A}$

We consider a second-order ordinary differential equation

$$\ddot{x} = M(t,x,\dot{x}). \tag{2.1}$$

If $w = A(t,x)\dot{x} + B(t,x)$ is a first integral of (2.1) and $Z = \partial_t + \dot{x}\partial_x + M(t,x,\dot{x})\partial_{\dot{x}}$ is the linear operator associated to this equation then $Zw = 0$; i.e.

$$Zw = (A_t\dot{x} + B_t) + (A_x\dot{x} + B_x)\dot{x} + M(t,x,\dot{x})A = 0. \tag{2.2}$$

This proves that the equation is of the form

$$\ddot{x} + a_2(t,x)\dot{x}^2 + a_1(t,x)\dot{x} + a_0(t,x) = 0, \tag{2.3}$$

where the coefficients a_0, a_1, a_2 can be written in terms of A and B as

$$\begin{aligned} a_2(t,x) &= \frac{A_x}{A}, \\ a_1(t,x) &= \frac{B_x + A_t}{A}, \\ a_0(t,x) &= \frac{B_t}{A}. \end{aligned} \tag{2.4}$$

Conversely, we suppose that A and B are two functions verifying (2.4). If we define $w = A\dot{x} + B$ then w is a first integral of (2.3) since

$$D_t w = A \cdot \left(\ddot{x} + \frac{A_x}{A}\dot{x}^2 + \frac{B_x + A_t}{A}\dot{x} + \frac{B_t}{A}\right) = A \cdot (\ddot{x} + a_2\dot{x}^2 + a_1\dot{x} + a_0). \tag{2.5}$$

This proves the following theorem:

Theorem 1. *A function $w = A\dot{x} + B$ is a first integral of* (2.1) *if and only if Eq.* (2.1) *is of the form* (2.3) *and functions A and B satisfy* (2.4). *In this case A is an integrating factor of* (2.3).

Example 1. Ibragimov ([9]) considered the equation

$$\ddot{x} - \frac{\dot{x}^2}{x} - \frac{t^2+t}{x}\dot{x} + 2t + 1 = 0 \tag{2.6}$$

as an example of an equation that does not admit Lie point symmetries but has an integrating factor $\mu(t,x,\dot{x}) = 1/x$, that could be found by using a method based on variational derivatives. It can be checked that the corresponding equations (2.4) are satisfied by $A(t,x) = 1/x$ and $B(t,x) = (t^2+t)/x$. Therefore $w = \dot{x}/x + (t^2+t)/x$ is a first integral of (2.6).

3. Intrinsic Characterization of the Equations in $\mathcal{A}$ and Construction of First Integrals

Equations (2.4) allows us readily to obtain necessary conditions on a_0, a_1 and a_2 in order that (2.3) have a first integral of the form $A(t,x)\dot{x} + B(t,x)$, with $A \neq 0$. This can be

done by eliminating A, B and their derivatives from (2.4) by using the compatibility conditions

$$A_{xt} = A_{tx}, \quad B_{xt} = B_{tx}, \quad C_{xt} = C_{tx}, \tag{3.1}$$

where $C = B_x$ is an auxiliar function. Condition $C_{xt} = C_{tx}$ can be written in the form

$$CS_1 + AS_2 = 0, \tag{3.2}$$

where

$$\begin{aligned} S_1(t,x) &= a_{1x} - 2a_{2t}, \\ S_2(t,x) &= (a_0a_2 + a_{0x})_x + (a_{2t} - a_{1x})_t + (a_{2t} - a_{1x})a_1. \end{aligned} \tag{3.3}$$

The analysis of (3.2) leads us to consider two cases:

Case I. If $S_1 = 0$ then necessarily $S_2 = 0$, since $A \neq 0$. In this case a necessary condition for the existence of two functions A and B satisfying (2.4) is $S_1 = S_2 = 0$.

Case II. If $S_1 \neq 0$ then we can write $S_2/S_1 = -C/A$, since $A \neq 0$. By derivating this expression, and by using (2.4), we get:

$$S_3(t,x) \equiv \left(\frac{S_2}{S_1}\right)_x - (a_{2t} - a_{1x}) = 0, \tag{3.4}$$

$$S_4'(t,x) \equiv \left(\frac{S_2}{S_1}\right)_x + \left(\frac{S_2}{S_1}\right)_t + \left(\frac{S_2}{S_1}\right)^2 + a_1\left(\frac{S_2}{S_1}\right) - (a_{2t} - a_{1x}) + (a_0a_2 + a_{0x}) = 0. \tag{3.5}$$

By using (3.4) in (3.5), this second equation can be written as

$$S_4(t,x) \equiv \left(\frac{S_2}{S_1}\right)_t + \left(\frac{S_2}{S_1}\right)^2 + a_1\left(\frac{S_2}{S_1}\right) + a_0a_2 + a_{0x} = 0. \tag{3.6}$$

Therefore, in this case, a necessary condition for the existence of two functions A and B which satisfy (2.4) is $S_3 = S_4 = 0$.

We now investigate if the former conditions are sufficient for the existence of two functions A and B that verify (2.4). We consider the same two cases as above.

Case I. We suppose that the coefficients of (2.3) are such that $S_1 = S_2 = 0$. Condition $S_1 = 0$ implies that $a_{2t} = \frac{1}{2}a_{1x}$ and therefore the function S_2 can be written as $S_2(t,x) = f_x(t,x)$, where

$$f(t,x) = a_0a_2 + a_{0x} - \frac{1}{2}a_{1t} - \frac{1}{4}a_1^2. \tag{3.7}$$

Condition $S_2 = f_x = 0$ implies that f does not depend on x and, in this case, we can write $f = f(t)$.

Let $P = P(t,x)$ be a function such that

$$P_t = \frac{1}{2}a_1, \quad P_x = a_2. \tag{3.8}$$

The existence of such function P can be ensured, since the compatibility condition $[a_2]_t = [\frac{1}{2}a_1]_x$ is equivalent to condition $S_1 = 0$.

Let $g = g(t)$ be a nonzero solution of the linear equation

$$g''(t) + f(t) \cdot g(t) = 0 \tag{3.9}$$

and let $Q = Q(t,x)$ be a function such that

$$Q_t = a_0 \cdot g \cdot e^P, \quad Q_x = \left(\frac{1}{2}a_1 - \frac{g'}{g}\right) g \cdot e^P. \tag{3.10}$$

There exists a function Q which satisfies (3.10) due to the compatibility condition

$$\left[\left(\frac{1}{2}a_1 - \frac{g'}{g}\right) g \cdot e^P\right]_t = \left[a_0 \cdot g \cdot e^P\right]_x \tag{3.11}$$

is equivalent to (3.9).

If we define

$$A = g \cdot e^P, \quad B = Q \tag{3.12}$$

then it can be verified that A and B satisfy (2.4).

Case II. We suppose that $S_1 \neq 0$ and that the coefficients of (2.3) are such that $S_3 = S_4 = 0$. Since $S_3 = 0$, we have $[a_2]_t = [a_1 + S_2/S_1]_x$ and there exists a function $P = P(t,x)$ such that

$$P_t = a_1 + \frac{S_2}{S_1}, \quad P_x = a_2. \tag{3.13}$$

Let $Q = Q(t,x)$ be such that

$$Q_t = a_0 \cdot e^P, \quad Q_x = -\left(\frac{S_2}{S_1}\right) \cdot e^P. \tag{3.14}$$

There exists a function Q due to the compatibility condition

$$\left[a_0 \cdot e^P\right]_x = \left[-\left(\frac{S_2}{S_1}\right) \cdot e^P\right]_t \tag{3.15}$$

is equivalent to condition $S_4 = 0$.

If we define

$$A = e^P, \quad B = Q \tag{3.16}$$

then the functions A and B satisfy (2.4).

Therefore, we have proved the following theorem:

Theorem 2. *We consider an equation of the form* (2.3) *and let* S_1 *and* S_2 *be the functions defined by* (3.3). *The following alternatives hold*:

(1) *If* $S_1 = 0$ *then the equation has a first integral of the form* (1.3) *if and only if* $S_2 = 0$. *In this case* A *and* B *can be given by* (3.12), *where* P *is a solution of* (3.8), g *is a nonzero solution of* (3.9) *and* Q *is a solution of* (3.10).

(2) *If* $S_1 \neq 0$ *then the equation has a first integral of the form* (1.3) *if and only if* $S_3 = 0$ *and* $S_4 = 0$, *where* S_3 *and* S_4 *are the functions defined by* (3.4) *and* (3.6). *In this case* A *and* B *can be given by* (3.16), *where* P *is a solution of* (3.13) *and* Q *is a solution of* (3.14).

Duarte *et al.* studied in [6] some necessary conditions for which equation (2.3) is linearizable by means of a generalized Sundman transformation of the form

$$X = F(t,x), \quad dT = G(t,x)dt. \tag{3.17}$$

Conditions (10) and (11) in [6] can be written as

$$\begin{gathered} \widetilde{S}_1 \equiv a_{1x} - 2a_{2t} = 0, \\ \widetilde{S}_2 \equiv 2a_{0xx} - 2a_{1tx} + 2a_0 a_{2x} - a_{1x}a_1 + 2a_{0x}a_2 + 2a_{2tt} = 0. \end{gathered} \tag{3.18}$$

Some errata appear in expressions (12) and (13) in [6]. The correct expressions are

$$\begin{gathered} \widetilde{S}_3 \equiv \widetilde{S}_2^2 - 2\widetilde{S}_{1t}\widetilde{S}_2 - 2\widetilde{S}_1^2 a_{1t} + 4\widetilde{S}_1^2 a_{0x} + 4\widetilde{S}_1^2 a_0 a_2 + 2\widetilde{S}_1\widetilde{S}_{2t} - \widetilde{S}_1^2 a_1^2 = 0, \\ \widetilde{S}_4 \equiv -\widetilde{S}_{1x}\widetilde{S}_2 + \widetilde{S}_1^2 a_{1x} - 2\widetilde{S}_1^2 a_{2t} + \widetilde{S}_1\widetilde{S}_{2x} = 0. \end{gathered} \tag{3.19}$$

It can be verified that conditions (3.18) and (3.19) are equivalent to the conditions given in Theorem 2, since $\widetilde{S}_1 = S_1$, $\widetilde{S}_2 = 2S_2 + a_1 S_1$, $\widetilde{S}_3 = 4S_1^2\, S_4$ and $\widetilde{S}_4 = 2S_1^2\, S_3$.

Other generalizations of Sundman transformations have been considered in the literature (e.g. in [3]). Note also that the generalized Sundman transformation (3.17) was used to define so-called Sundman symmetries [7, 8] of ODEs. In [8] a rich structure of Sundman symmetries was reported for the equations in $\mathcal{A}$, and also for a large class of third-order ODEs. The complete classification of all linearizable third-order ODEs which can be transformed in $X''' = 0$ under the generalized Sundman transformation (3.17) was reported in [7]. For this classification the Sundman symmetries played a fundamental role.

Example 2. The three following equations have been considered in [6] as examples of equations that can be linearized by means of nonlocal transformations of the form (3.17).

$$\ddot{x} - \frac{2\dot{x}^2}{x} + \frac{2x}{t^2} = 0, \tag{3.20}$$

$$\ddot{x} + \left(t - \frac{1}{x}\right)\dot{x}^2 + 2x\dot{x} + \frac{x^2}{t} - \frac{x}{t^2} = 0, \tag{3.21}$$

$$\ddot{x} - \left(\tan(x) + \frac{1}{x}\right)\dot{x}^2 + \left(\frac{1}{t} - \frac{\tan(x)}{xt}\right)\dot{x} - \frac{\tan(x)}{t^2} = 0. \tag{3.22}$$

By the algorithm described in Theorem 2, it is possible to determine first integrals of the form $A(t,x)\dot{x} + B(t,x)$ for each equation listed above. It can be shown that the coefficients of equations (3.20) and (3.21) verify $S_1 = S_2 = 0$ and the coefficients of Eq. (3.22) satisfy $S_1 \neq 0$ but $S_3 = S_4 = 0$.

- For Eq. (3.20) the corresponding system (3.8) is

$$P_t = 0, \quad P_x = -2/x. \tag{3.23}$$

A solution of this system is given by $P(t,x) = -2\ln(x)$. The corresponding equation (3.9) is

$$g''(t) - 2/t^2 g(t) = 0. \tag{3.24}$$

Two linearly independent solutions of this equation are given by $g_1(t) = t^2$, $g_2(t) = 1/t$. By considering $g_1(t)$ the corresponding system (3.14) is

$$Q_t = \frac{2}{x}, \quad Q_x = -\frac{2t}{x^2}. \tag{3.25}$$

A solution of this system is $Q_1(t,x) = 2t/x$. By (3.12) a first integral of (3.20) is given by

$$w^1 = \frac{t^2}{x^2}\dot{x} + \frac{2t}{x}. \tag{3.26}$$

Similarly, by considering $g_2(t) = 1/t$ a second first integral of (3.20) is given by

$$w^2 = \frac{1}{tx^2}\dot{x} - \frac{1}{t^2 x}. \tag{3.27}$$

- For Eq. (3.21), a similar procedure can be followed to obtain two independent first integrals:

$$w^1(t,x,\dot{x}) = \frac{e^{tx}(x+t\dot{x})}{tx}, \quad w^2(t,x,\dot{x}) = \frac{e^{tx}(x+t\dot{x})}{x} - \mathrm{Ei}(tx), \tag{3.28}$$

where $\mathrm{Ei}(z)$ denotes the exponential integral function, i.e. a primitive of e^z/z.

- For Eq. (3.22) we have $S_1 \neq 0$, $S_3 = S_4 = 0$ and $S_2/S_1 = \tan(x) - x/(tx)$. The corresponding system (3.13) is

$$P_t = 0, \quad P_x = -\tan(x) - \frac{1}{x}. \tag{3.29}$$

A solution of this system is given by $P(t,x) = \ln(\cos(x)/x)$. The corresponding system (3.14) is

$$Q_t = -\frac{\sin(x)}{t^2 x}, \quad Q_x = \frac{\cos(x)(x-\tan(x))}{tx^2}, \tag{3.30}$$

a solution of which is $Q(t,x) = \sin(x)/(tx)$. By (3.16),

$$w(t,x,\dot{x}) = \frac{\cos(x)}{x}\dot{x} + \frac{\sin(x)}{tx} \tag{3.31}$$

is a first integral of (3.22).

4. First Integrals of the Form $A(t,x)\dot{x}+B(t,x)$ and λ-Symmetries

We recall [13] that the vector field $v = \partial_x$ is a λ-symmetry of (2.1) if and only if λ is a solution of the equation

$$M_x + \lambda M_{\dot{x}} = \lambda_t + \dot{x}\lambda_x + M\lambda_{\dot{x}} + \lambda^2. \tag{4.1}$$

We suppose that the coefficients a_0, a_1, a_2 of (2.3) are such that either

- $S_1 = S_2 = 0$, or
- $S_1 \neq 0$ and $S_3 = S_4 = 0$.

We now prove that (4.1) has solutions of the form $\lambda = \alpha(t,x)\dot{x} + \beta(t,x)$. In this case the following system must be compatible

$$\alpha_x + \alpha^2 + a_2\alpha + a_{2x} = 0, \tag{4.2}$$

$$\beta_x + 2(a_2 + \alpha)\beta + a_{1x} + \alpha_t = 0, \tag{4.3}$$

$$\beta_t + \beta^2 + a_1\beta + a_{0x} - a_0\alpha = 0. \tag{4.4}$$

It is obvious that $\alpha(t,x) = -a_2(t,x)$ solves Eq. (4.2). For this α, Eqs. (4.3)–(4.4) are

$$\beta_x + a_{1x} - a_{2t} = 0, \tag{4.5}$$

$$\beta_t + \beta^2 + a_1\beta + a_{0x} + a_0a_2 = 0. \tag{4.6}$$

Case I. If $S_1 = S_2 = 0$ then we have seen that the function f defined by (3.7) is such that $f = f(t)$. If $h(t)$ is any solution of the Riccati equation

$$h'(t) + h^2(t) + f(t) = 0, \tag{4.7}$$

then $\beta(t,x) = h(t) - \frac{1}{2}a_1(t,x)$ satisfies (4.5) and (4.6), since $S_1 = 0$ and $S_2 = 0$, respectively. Therefore $\lambda = -a_2\dot{x} + \beta$ is, in this case, such that ∂_x is a λ-symmetry of (2.3).

Case II. If $S_1 \neq 0$ and $S_3 = S_4 = 0$ then (3.4) and (3.6) prove that $\beta = S_2/S_1$ is a solution of (4.5)–(4.6). Therefore ∂_x is a λ-symmetry of (2.3) for $\lambda = -a_2\dot{x} + S_2/S_1$.

Conversely, let us suppose that ∂_x is a λ-symmetry for some function λ of the form $\lambda = -a_2\dot{x} + \beta(t,x)$. Then β satisfies Eqs. (4.5)–(4.6).

If we define $\gamma(t,x) = \beta(t,x) + \frac{1}{2}a_1(t,x)$ then γ satisfies the following system

$$\gamma_x + \frac{1}{2}S_1 = 0, \tag{4.8}$$

$$\gamma_t + \gamma^2 + f = 0, \tag{4.9}$$

where S_1 is defined by (3.3) and f is given by (3.7). Since $\gamma_{xt} = \gamma_{tx}$, Eqs. (4.8)–(4.9) imply that

$$S_1\gamma = -\frac{1}{2}S_{1t} + f_x. \tag{4.10}$$

This equation leads us to consider two cases, the same we have considered above.

Case I. If $S_1 = 0$ then (4.8) implies that $\gamma_x = 0$, and therefore $\gamma = \gamma(t)$, and that S_2 can be written as $S_2 = f_x$, where $f(t,x)$ is defined by (3.7). Similarly, (4.9) or (4.10) imply that $f = f(t)$. Therefore $f = f(t)$ and $S_2 = f_x = 0$.

Case II. If $S_1 \neq 0$ then γ is uniquely defined by (4.10):

$$\gamma(t,x) = \frac{1}{S_1}\left(-\frac{1}{2}S_{1t} + f_x\right). \tag{4.11}$$

Therefore, $\beta(t,x)=\gamma(t,x)-\frac{1}{2}a_1(t,x)$ is uniquely defined by the coefficients a_0,a_1,a_2 of (2.3) and it can be checked that $\beta=S_2/S_1$. Since β satisfies Eqs. (4.5)–(4.6), it is necessary that S_2/S_1 satisfies (3.4) and (3.6); i.e. $S_3=S_4=0$.

This proves the following theorem:

Theorem 3. *We consider an equation of the form* (2.3) *and let S_1,S_2,S_3 and S_4 be the functions defined by* (3.3), (3.4) *and* (3.6).

The equation is such that either $S_1=S_2=0$ or $S_3=S_4=0$ if and only if ∂_x is a λ-symmetry of (2.3) *for some $\lambda=-a_2(t,x)\dot{x}+\beta(t,x)$.*

The following theorem sum up our former results:

Theorem 4. *The following conditions on an ODE of the form* (2.1) *are equivalent*:

(1) *Equation* (2.1) *admits a first integral of the form $A(t,x)\dot{x}+B(t,x)$.*
(2) *Equation* (2.1) *is of the form* (2.3) *and there exist two functions $A(t,x)$ and $B(t,x)$ that satisfy* (2.4).
(3) *Equation* (2.1) *is of the form* (2.3) *and its coefficient are such that either $S_1=S_2=0$ or $S_3=S_4=0$.*
(4) *Equation* (2.1) *is of the form* (2.3) *and ∂_x is a λ-symmetry for some function λ of the form $\lambda=-a_2(t,x)\dot{x}+\beta(t,x)$.*

Example 3. We consider the equation

$$\ddot{x}-\frac{2}{x}\dot{x}^2+\left(\frac{4}{t}-e^{t/x}t\right)\dot{x}-tx^2-\frac{2x}{t^2}+e^{t/x}\left(\frac{3x^2}{t}+x\right)=0. \tag{4.12}$$

This equation does not have Lie point symmetries and hence it cannot be integrated by Lie's method ([2]). This equation is of the form (2.3) and its coefficients satisfy $S_1\neq 0$, and $S_3=S_4=0$. Hence statement 3 of Theorem 4 is satisfied. Consequently:

(1) Equation (4.12) has a first integral of the form $w(t,x,\dot{x})=A(t,x)\dot{x}+B(t,x)$, where A and B satisfy (2.4) and can be calculated by the algorithm of Sec. 2 (Case II):

$$A(t,x)=\frac{t^3}{x^2},\qquad B(t,x)=e^{t/x}t^3-\frac{t^2}{x}-\frac{t^5}{5}. \tag{4.13}$$

(2) Equation (4.12) admits ∂_x as a λ-symmetry for $\lambda=-a_2\dot{x}+S_2/S_1$:

$$\lambda(t,x,\dot{x})=\frac{2}{x}\dot{x}-\frac{1}{t}+e^{t/x}t. \tag{4.14}$$

5. Complete Integrability in Case I ($S_1=S_2=0$)

We observe that in Case I Eq. (3.9) has two linearly independent solutions $g_1(t)$ and $g_2(t)$. We denote by $W(g_1,g_2)$ the Wronskian of the functions g_1 and g_2. We can construct two functions Q^1 and Q^2 satisfying (3.10):

$$Q^i_t=a_0\cdot g_i\cdot e^P,\quad Q^i_x=\left(\frac{1}{2}a_1-\frac{g_i'}{g_i}\right)\cdot g_i\cdot e^P,\quad (i=1,2). \tag{5.1}$$

Two first integrals of Eq. (2.3) are given by

$$w^1 = g_1\, e^P \dot{x} + Q^1, \quad w^2 = g_2\, e^P \dot{x} + Q^2. \tag{5.2}$$

It can be verified that

$$\begin{vmatrix} w^1_x & w^1_{\dot{x}} \\ w^2_x & w^2_{\dot{x}} \end{vmatrix} = e^P \cdot W(g_1, g_2) \neq 0. \tag{5.3}$$

This proves that in Case I, w^1 and w^2 are two functionally independent first integrals of Eq. (2.3) that are of the form (1.3).

Conversely, let us suppose that Eq. (2.3) has two functionally independent first integrals of the form (1.3):

$$w^1(t, x, \dot{x}) = A^1(t, x)\dot{x} + B^1(t, x), \quad w^2(t, x, \dot{x}) = A^2(t, x)\dot{x} + B^2(t, x). \tag{5.4}$$

The functions A^i and B^i, $i = 1, 2$, must satisfy system (2.4):

$$\begin{aligned} a_2 &= \frac{A^1_x}{A^1} = \frac{A^2_x}{A^2}, \\ a_1 &= \frac{A^1_t + B^1_x}{A^1} = \frac{A^2_t + B^2_x}{A^2}, \\ a_0 &= \frac{B^1_t}{A^1} = \frac{B^2_t}{A^2}. \end{aligned} \tag{5.5}$$

The vector field ∂_x is a λ^i-symmetry for

$$\lambda^i = -\frac{w^i_x}{w^i_{\dot{x}}} = -a_2\dot{x} - \frac{B^i_x}{A^i}, \quad (i = 1, 2). \tag{5.6}$$

Since w^1 and w^2 are functionally independent, necessarily $\beta^1 = -B^1_x/A^1 \neq -B^2_x/A^2 = \beta^2$ and therefore the system (4.5)–(4.6) has two different solutions. This cannot happen in Case II since in this case the function β such that ∂_x is a λ-symmetry for $\lambda = -a_2\dot{x} + \beta$ is uniquely determined by $\beta = S_2/S_1$. Therefore $S_1 = S_2 = 0$ and we have proved the following result.

Theorem 5. *The following conditions on an equation of the form* (2.3) *are equivalent*:

(1) *The equation admits two functionally independent first integrals of the form* (1.3).
(2) $S_1 = S_2 = 0$.
(3) *The vector field* ∂_x *is a* λ^1*-symmetry and a* λ^2*-symmetry for some functions* $\lambda^1 = -a_2\dot{x} + \beta^1$ *and* $\lambda^2 = -a_2\dot{x} + \beta^2$ *with* $\beta^1 \neq \beta^2$.

5.1. *Some examples*

We now apply Theorem 5 to two families of second-order equations.

1. An equation of the form

$$\ddot{x} + a_2(x)\dot{x}^2 + a_1(t)\dot{x} = 0 \tag{5.7}$$

satisfies any of the conditions given in Theorem 5. By (3.7), $f(t) = -\frac{1}{4}a_1(t)^2 - \frac{1}{2}a_1'(t)$. Two linearly independent solutions $g_1(t)$ and $g_2(t)$ of the corresponding equation (3.9) are

given by

$$g_1(t)=\exp\left(\frac{1}{2}\int a_1(t)dt\right),\quad g_2(t)=g_1(t)\int\frac{dt}{g_1^2(t)}. \tag{5.8}$$

We define

$$\begin{aligned} h_1(t)&=\exp\left(-\int a_1(t)dt\right),\quad H_1(t)=\int h_1(t)dt,\\ h_2(x)&=\exp\left(\int a_2(x)dx\right),\quad H_2(x)=\int h_2(x)dx. \end{aligned} \tag{5.9}$$

A solution $P(t,x)$ of the corresponding system (3.8) can be written as $P(t,x)=\ln(g_1(t)\cdot h_2(x))$. Two particular solutions of systems (5.1) are given by $Q^1(t,x)=0$ and $Q^2(t,x)=-H_2(x)$. Two functionally independent first integrals of Eq. (5.7) are given by (5.10):

$$w^1(t,x,\dot{x})=\frac{h_2(x)}{h_1(t)}\dot{x},\quad w^2(t,x,\dot{x})=H_1(t)\cdot\frac{h_2(x)}{h_1(t)}\dot{x}-H_2(x). \tag{5.10}$$

By Theorem 1, $\mu_1(t,x)=h_2(x)/h_1(t)$ and $\mu_2(t,x)=H_1(t)\cdot\mu_1(t,x)$ are integrating factors of (5.7).

The general solution of Eq. (5.7) could be found by eliminating $\dot{x}$ from $w^1=C_1$ and $w^2=C_2$, $C_1,C_2\in\mathbb{R}$:

$$C_1\,H_1(t)+H_2(x)=C_2,\quad C_1,C_2\in\mathbb{R}. \tag{5.11}$$

The vector field ∂_x is a λ^1-symmetry and a λ^2-symmetry for

$$\lambda^1(t,x,\dot{x})=-a_2(x)\dot{x},\quad \lambda^2(t,x,\dot{x})=-a_2(x)\dot{x}+\frac{h_1(t)}{H_1(t)}. \tag{5.12}$$

As a consequence, we have proved the following corollary:

Corollary 1. *Any equation of the form* (5.7) *admits two functionally independent first integrals of the form* (1.3) *that are given by* (5.10), *where* h_1,h_2,H_1 *and* H_2 *are defined by* (5.9). *For Eq.* (5.7), *the vector field* ∂_x *is a* λ^1*-symmetry and a* λ^2*-symmetry for* λ^1,λ^2 *given by* (5.12).

We now consider three equations of the form (5.7) that have previously been used in the literature to illustrate several integration strategies. These equations can be solved by using Corollary 1:

(1) The equation

$$x\ddot{x}=3\dot{x}^2+\frac{x}{t}\dot{x} \tag{5.13}$$

was originally derived by Buchdahl [1] in the theory of general relativity. Duarte *et al.* [6] deduced a first integral by applying the extended Prelle–Singer method and Chandrasekar *et al.* derived a second one in [4].

By (5.10)

$$w^1(t,x,\dot{x})=\frac{\dot{x}}{tx^3},\quad w^2(t,x,\dot{x})=\frac{t}{x^3}\dot{x}+\frac{1}{x^2} \tag{5.14}$$

are two independent first integrals.

(2) The equation

$$\ddot{x} + \frac{\dot{x}^2}{x} + 3\frac{\dot{x}}{t} = 0 \tag{5.15}$$

has been considered in [9] to illustrate a method, based on variational derivatives, to find two integrating factors and therefore the general solution of the equation. The method in [9] requires to solve a system of two coupled second-order partial differential equations. By using Corollary 1, two functionally independent first integrals are given by (5.10):

$$w^1(t, x, \dot{x}) = t^3 x\dot{x}, \quad w^2(t, x, \dot{x}) = tx\dot{x} + x^2. \tag{5.16}$$

Two integrating factors of Eq. (5.15) can readily be found by Theorem 1:

$$\mu_1(t, x) = t^3 x, \quad \mu_2(t, x) = tx. \tag{5.17}$$

(3) Equation

$$tx\ddot{x} + (2tx + t)\dot{x}^2 + x\dot{x} = 0 \tag{5.18}$$

was proposed in [5] to show that the extended Prelle–Singer method can be applied to find a non-rational first integral $w = x + \frac{1}{2}\ln(tx\dot{x})$. Since (5.18) has the form (5.7) and a complete system of first integrals is given by the functions w^1, w^2 defined by (5.10), w must be a function of w^1 and w^2 (that are rational first integrals). In fact, it can be checked that $w = \ln(w^1)/2$, where $w^1 = te^{2x}x\dot{x}$. A second independent first integral is given by $w^2 = e^{2x}(t\,\ln(t)\,x\dot{x} + (1 - 2x)/4)$.

2. Theorem 5 can be used to obtain a characterization of second-order linear ODEs. Suppose that an equation of the form

$$\ddot{x} + a_1(t, x)\dot{x} + a_0(t, x) = 0 \tag{5.19}$$

admits two independent first integrals of the form (1.3). By Theorem 2, the coefficients a_0 and a_1 of (5.19) must satisfy

$$S_1 = a_{1_x} = 0 \quad \text{and} \quad S_2 = -a_1 a_{1_x} + a_{0_{xx}} - a_{1_{tx}} = 0. \tag{5.20}$$

This implies that $a_1 = a_{11}(t)$ and $a_{0xx} = 0$. Therefore $a_0(t, x) = a_{01}(t)x + a_{02}(t)$ for some functions $a_{01}(t)$ and $a_{02}(t)$ and (5.19) has the form

$$\ddot{x} + a_{11}(t)\dot{x} + a_{01}(t)x + a_{02}(t) = 0. \tag{5.21}$$

Conversely, it is obvious that any equation of the form (5.21) admits two independent first integrals of the form (1.3). We have achieved the following characterization of the second-order linear ODEs in terms of first integrals:

Corollary 2. *An equation of the form* (5.19) *has two independent first integrals of the form* (1.3) *if and only if it is a linear equation, i.e. it has the form* (5.21).

It must be observed that under the conditions of Corollary 2

$$f(t) = \frac{1}{4}(-a_1(t,x)^2 + 4a_{0x}(t,x) - 2a_{1t}(t,x)) = -\frac{a_{11}(t)}{4} - \frac{a'_{11}(t)}{2} + a_{01}(t). \tag{5.22}$$

Therefore, for equations of the form (2.3), the function f defined by (3.7) generalizes the usual invariant that appears in the study of equivalence transformations of second-order linear ODEs ([9]).

6. Conclusions

We have characterized the second-order ODEs that admit first integrals of the form $A(t,x)\dot{x} + B(t,x)$ through an easy-to-check criterion expressed in terms of functions S_1, S_2, S_3 and S_4 given by (3.3) and (3.4)–(3.6).

This criterion and a systematic procedure to calculate such first integrals have been derived in Theorem 2. The considered equations can also be characterized as the equations of the form (1.4) that admit the vector field ∂_x as a λ-symmetry for some $\lambda = -a_2\dot{x} + \beta(t,x)$.

The class of equations such that $S_1 = S_2 = 0$ is composed of the equations that admits two independent first integrals of the form (1.3). The determination of these first integrals requires the solution of a second-order linear ODE (3.9). For these equations there are infinitely many functions λ for which the vector field ∂_x is a λ-symmetry. They are of the form $\lambda = -a_2\dot{x} - a_1/2 + h$, where h denotes a particular solution of the Riccati equation (4.7).

If $S_1 \neq 0$ and $S_3 = S_4 = 0$, the equations have an unique (up to multipliers) first integral of the form (1.3), that can readily be obtained by quadratures. These equations can also be characterized as the equations of the form (1.4) that admit the vector field ∂_x as a λ-symmetry for $\lambda = -a_2\dot{x} + \beta$, where β is uniquely defined by $\beta = S_2/S_1$.

The equations classified in this paper are interestingly related to the equations that can be linearized by generalized Sundman transformations of the form (1.5). This relationship and other aspects of the problem of linearization are studied in detail in a forthcoming paper.

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References

[1] H. A. Buchdahl, A relativistic fluid spheres resembling the Emden polytrope of index 5, *Ap. J.* **140** (1964) 1512–1516.

[2] D. Catalano and P. Morando, Local and nonlocal solvable structures in the reduction of ODEs, *J. Phys. A: Math. Theor.* **42** (2009) 035210.

[3] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, A unification in the theory of linearization of second-order nonlinear ordinary differential equations, *J. Phys. A: Math. Gen.* **39** (2006) L69–L76.

[4] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, On the complete integrability and linearization of certain second-order nonlinear ordinary differential equations, *Proc. R. Soc. A* **461** (2005) 2451–2476.

[5] L. G. S. Duarte, S. E. S. Duarte, L. A. C. P. da Mota and J. E. F. Skea, Solving second-order ordinary differential equations by extending the Prelle–Singer method, *J. Phys. A: Math. Gen.* **34** (2001) 3015–3024.

[6] L. G. S. Duarte, I. C. Moreira and F. C. Santos, Linearization under non-point transformations, *J. Phys. A: Math. Gen.* **27** (1994) L739–L743.

[7] N. Euler and M. Euler, Sundman symmetries of nonlinear second-order and third-order ordinary differential equations, *J. Nonlinear Math. Phys.* **11** (2004) 399–421.

[8] N. Euler, T. Wolf, P. G. L. Leach and M. Euler, Linearisable third-order ordinary differential equations and generalised Sundman transformations: the case $X''' = 0$, *Acta Appl. Math.* **76** (2003) 89–115.

[9] N. H. Ibragimov, A practical course in differential equations and mathematical modelling: Classical and new methods, nonlinear mathematical models, symmetry and invariance principles, ALGA Publications, Karlskrona (2006).

[10] N. H. Ibragimov and F. Magri, Geometric proof of Lie's linearization theorem, *Nonlinear Dynamics* **36** (2004) 41–46.

[11] S. Lie, Klassifikation und integration von gewhnlichen differentialgleichungen zwischen x, y die eine gruppe von transformationen gestatten. III, *Arch. Mat. Naturvidenskab* **8** (1883) 371–458.

[12] S. V. Meleshko, On linearizaton of third-order ordinary differential equations, *J. Phys. A: Math. Gen.* **39** (2006) 15135–15145.

[13] C. Muriel and J. L. Romero, New methods of reduction for ordinary differential equations, *IMA J. Appl. Math.* **66** (2001) 111–125.

[14] C. Muriel and J. L. Romero, Integrating factors and λ-symmetries, *J. Nonlinear Math. Phys.* **15** (2008) 290–299.

[15] C. Muriel and J. L. Romero, First integrals, integrating factors and λ-symmetries of second-order differential equations, *J. Phys. A: Math. Theor.* **42** (2009) 365207.