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NONSTANDARD SEPARABILITY ON THE MINKOWSKI PLANE

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We present examples of nonstandard separation of the natural Hamilton–Jacobi equation on the Minkowski plane \mathbb{M}^2 . By “nonstandard” we refer to the cases in which the form of the metric, when expressed in separating coordinates, does not have the usual Liouville structure. There are two possibilities: the “complex-Liouville” (or “harmonic”) case and the “linear/null” (or “Jordan block”) case. By means of explicit examples, we show that, in all cases, a suitable glueing of coordinate patches of the different structures allows us to separate natural systems with indefinite kinetic energy all over \mathbb{M}^2 .

Keywords: Integrable Hamiltonian systems; separability by quadrature.

1. Introduction

The study of separation of variables on the Minkowski plane \mathbb{M}^2 is very classical [1] and almost as old as the corresponding problem on the Euclidean plane \mathbb{E}^2 . A general approach based on conformal coordinate transformations to solve Killing tensor equations has been given in [2]. It has been applied [3–5] to get a complete classification of separating coordinate systems of the Hamilton–Jacobi equation with the corresponding separated potentials and second integral of motion.

Separability of free motion on the hyperbolic plane has already been investigated [7–9]. In particular, [9] describes a general theory of complex variable separation on pseudo-Riemannian manifolds. In [3, 4] the general picture of separability is extended to indefinite natural systems, showing how different kinds of separation structures are needed for different regions in configuration space. In particular, it is in general necessary to use different separating variables, even for the integration of a single orbit. Associating as usual the existence of a 2nd-rank Killing tensor to that of a system of separating coordinates [6], the

picture can be illustrated as follows: for $(1+1)$ -dimensional systems there are three possible types of conformal Killing tensors, and therefore, three distinct separability structures in contrast to the single standard (*Liouville*) type separation of the positive definite case [10]. One of the new separability structures is the *complex-Liouville/harmonic* type which is characterized by complex separation variables and the metric is an harmonic function. The other new type is the *linear/null* separation which occurs when the conformal Killing tensor has a null eigenvector so that it has the structure of a Jordan block and the metric depends only linearly on one of the separation variables.

In the general case the components of the conformal part of the Killing tensor are two arbitrary real functions. It can be proved [3] that for a natural system on \mathbb{M}^2 , these functions are two quadratic polynomials with equal leading-order coefficient. There are therefore five real constants determining nine different coordinate systems [7]. Six of these admit both Liouville and complex-Liouville separation and a subset of three of these admits also the null separation. In the particular case of a flat space, the additional structures do not in general provide new kinds of dynamical systems. Rather they in general coexist in determining the dynamics of a given system and are required to obtain separated solutions on the whole plane as we see below in four representative cases.

The layout of the paper is as follows: in Sec. 2 we recall separability on the Minkowski plane; in Sec. 3 we resume the classification of the coordinate systems and the corresponding separation structures; in Sec. 4 we provide a detailed analysis of four natural Hamiltonian systems that give an overview of the possible cases; Sec. 5 contains concluding remarks.

2. Separation Structures in the Minkowski Plane

Given a Hamiltonian

$$H(p_u, p_x, u, x) = \frac{1}{2}(-p_u^2 + p_x^2) + \Phi(u, x) \equiv \mathcal{E}, \quad (2.1)$$

the corresponding Hamilton–Jacobi equation is separable if there exists a 2nd-rank Killing tensor K_{ij} , $(i, j = 0, 1)$, for the pseudo-Riemannian metric

$$ds^2 = g_{ij}dx^i dx^j = 2\Gamma(u, x)(-du^2 + dx^2), \quad (2.2)$$

where $u = x^0, x = x^1$ and $\Gamma(u, x) = \Phi - \mathcal{E}$. The second independent integral of motion satisfying the commutation relation $\{H, I\} = 0$, is

$$I(p_u, p_x, u, x) = K^{ij}(u, x; \mathcal{E})p_i p_j|_{\mathcal{E} \rightarrow H}, \quad (2.3)$$

where indices are raised with the metric tensor of (2.2). It can be proven [3] that the conformal part of the Killing tensor [11], $P^{ij} = K^{ij} - \frac{1}{2}K^k{}_k g^{ij}$, is uniquely determined by two arbitrary functions $\Sigma = \Sigma(u+x)$ and $\hat{\Sigma} = \hat{\Sigma}(u-x)$ by

$$(P^{ij}) = \frac{1}{4} \begin{pmatrix} \Sigma + \hat{\Sigma} & \Sigma - \hat{\Sigma} \\ \Sigma - \hat{\Sigma} & \Sigma + \hat{\Sigma} \end{pmatrix}. \quad (2.4)$$

The Hamilton–Jacobi equation is separable for *arbitrary* values of $H = \mathcal{E}$ if these functions have the forms

$$\Sigma = k(u+x)^2 + b(u+x) + c, \quad \hat{\Sigma} = k(u-x)^2 + \hat{b}(u-x) + \hat{c}, \quad (2.5)$$

where $k, b, \hat{b}, c, \hat{c}$ are arbitrary real constants; this class of separable systems includes free motion on the flat hyperbolic plane [12]. They exist more general separable geodesic flows on a generic pseudo-Riemannian metric (2.2) and/or natural Hamiltonian systems separable for *fixed* values of $H = \mathcal{E}$ [10,12] in which the functions $\Sigma = \Sigma(u+x)$ and $\hat{\Sigma} = \hat{\Sigma}(u-x)$ are not restricted to the class (2.5). We observe that the procedure recalled in the present section applies to this general setting including fixed-energy (“weak”) separability and is based on a generalization of the conformal coordinate transformation introduced by Kolokoltsov [13]. In the subsequent sections we will stick with the arbitrary-energy (“strong”) separability.

The determinant of the conformal Killing tensor specifies the kind of separation structures: $\det(P^{ij}) > 0$ is associated with the *standard Liouville separation*, $\det(P^{ij}) < 0$ with the *harmonic* or *complex-Liouville separation* and $\det(P^{ij}) = 0$ with the *null separation*. The separating coordinates, U, X , originate from the transformations bringing P^{ij} to the standard (constant components) form $P = \frac{1}{2}\text{diag}(1, \epsilon)$, where $\epsilon = \text{sgn}(\det(P^{ij}))$. Using auxiliary “null” variables

$$\zeta = u + x, \quad \hat{\zeta} = u - x, \quad W = U + X, \quad \hat{W} = U - X, \tag{2.6}$$

the standardizing transformations are provided by [2–4,13]

$$W = U + X = \int \frac{d\zeta}{\sqrt{|\Sigma(\zeta)|}}, \tag{2.7}$$

$$\hat{W} = U - X = \int \frac{d\hat{\zeta}}{\sqrt{\epsilon|\hat{\Sigma}(\hat{\zeta})|}}. \tag{2.8}$$

In the standard case, U and X are separating variables and the metric can be put in the standard Liouville form,

$$ds^2 = \frac{A_0(U) + A_1(X)}{B_0(U) + B_1(X)}(dU^2 - dX^2), \tag{2.9}$$

with A_i arbitrary functions of the arguments and B_i specified by the particular pair of Σ and $\hat{\Sigma}$ chosen.

In the complex-Liouville case, separated expressions are obtained using complex conjugate pairs

$$Z = X + iU, \quad \bar{Z} = X - iU \tag{2.10}$$

and the metric takes the form

$$ds^2 = \frac{\Re\{Q(Z)\}}{\Re\{\Psi(Z)\}}(dZ^2 + d\bar{Z}^2), \tag{2.11}$$

where $Q(Z)$ and $\Psi(Z)$ are holomorphic functions: Q is arbitrary and Ψ is specified by the particular transformation chosen in the set with $\det(P^{ij}) < 0$.

In the linear/null case the metric is

$$ds^2 = [C(\hat{W})W + D(\hat{W})]dWd\hat{W}, \tag{2.12}$$

where $C(\hat{W})$ and $D(\hat{W})$ are arbitrary functions. It can be proven [3, 4] that, even if the metric does not take an explicitly separated form, the Hamilton–Jacobi equation is indeed separable.

3. Classification of Separable Systems

The separating coordinate systems for natural systems at arbitrary energy are classified by examining the inequivalent combinations of independent parameters appearing in (2.5): exploiting isometries and rescaling of variables, the leading order coefficient k can be assumed to take the values 0 or 1 and, in the first case, either b or c can be put equal to zero. Therefore, there are five distinct classes of transformations for each null variable. They are listed in Table 1 for the transformation $\zeta \rightarrow W$; analogous forms apply to the transformation $\hat{\zeta} \rightarrow \hat{W}$: the first column gives the number used in the classification; the second column gives $\zeta(W)$; the third $W(\zeta)$; the fourth $\Sigma(\zeta)$ and the fifth the values of the corresponding parameters, with $D = b^2 - 4kc$ and $\Delta = \sqrt{|D|}/4$. When combining the five cases, we must consider only cases with the same value of the separating constant k , since k appears both in Σ and $\hat{\Sigma}$. There are no other restrictions so this gives four cases with $k = 0$ and nine cases with $k \neq 0$, thirteen cases in total. However, it is reasonable not to distinguish systems which can be transformed into each other by the transformation $(\zeta, \hat{\zeta}) \rightarrow (\hat{\zeta}, \zeta)$ or equivalently $x \rightarrow -x$. This reduces the number of cases to three for $k = 0$ and six for $k \neq 0$, nine cases in total. Using the numbers 1–5 appearing in the first column of the table and the corresponding “hatted” figures $\hat{1}$ – $\hat{5}$, with an obvious notation, the set of possible independent separating coordinates is given by the combinations

$$k = 0 : \quad \hat{1}\hat{1}, \quad \hat{1}\hat{2}, \quad \hat{2}\hat{2}, \tag{3.1}$$

$$k = 1 : \quad \hat{3}\hat{3}, \quad \hat{3}\hat{4}, \quad \hat{3}\hat{5}, \quad \hat{4}\hat{4}, \quad \hat{4}\hat{5}, \quad \hat{5}\hat{5}. \tag{3.2}$$

Once the possible combinations of the functions Σ and $\hat{\Sigma}$ are established, we also need to analyze the sign of $\det(P)$: we see that the possibility of a negative sign appears when Σ_1, Σ_2 and Σ_4 are involved, whereas it may vanish when $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$ are involved. The general investigation with the complete list of all possible cases is given elsewhere [3]. Here we give simple but nontrivial examples of systems in which nonstandard separability appears. The explicit proof of integrability by quadrature of the nonstandard cases is given in [4, Sec. 2.2.2].

Table 1. The possible conformal transformation functions for (1 + 1)-dimensional integrable Hamiltonians with a second degree invariant.

1.	W	ζ	$\Sigma_1(\zeta) = 1$	$k = b = 0, c \neq 0$
2.	W^2	$\sqrt{\zeta}$	$\Sigma_2(\zeta) = 4\zeta$	$k = c = 0, b \neq 0$
3.	e^W	$\log \zeta$	$\Sigma_3(\zeta) = \zeta^2$	$k \neq 0, D = 0$
4.	$\Delta \cosh W$	$\cosh^{-1}(\zeta/\Delta)$	$\Sigma_4(\zeta) = \zeta^2 - \Delta^2$	$k \neq 0, D > 0$
5.	$\Delta \sinh W$	$\sinh^{-1}(\zeta/\Delta)$	$\Sigma_5(\zeta) = \zeta^2 + \Delta^2$	$k \neq 0, D < 0$

4. Examples

4.1. Example I: A Cartesian-parabolic case (12̂)

Let us consider the Hamiltonian

$$H_1 = \frac{1}{2}(-p_u^2 + p_x^2) + 3(u + x)^2 + u - x. \tag{4.1}$$

It is easy to check that

$$I_1 = \frac{1}{4}(p_u + p_x)^2 - 2(p_u - p_x)(up_x + xp_u) + 4((u + x)^3 + x^2 - u^2) \tag{4.2}$$

is such that $\{H_1, I_1\} = 0$. According to Table 1, the conformal coordinate transformation is generated by $\Sigma_1(\hat{\zeta}) = 1$ and $\hat{\Sigma}_2(\hat{\zeta}) = 4\hat{\zeta}$, so that the new variables are (see Fig. 1)

$$W = \zeta, \quad \hat{W} = \sqrt{\epsilon\hat{\zeta}}, \tag{4.3}$$

or, in non-null coordinates,

$$U = \frac{1}{2}(u + x + \sqrt{u - x}), \quad X = \frac{1}{2}(u + x - \sqrt{u - x}), \quad u > x, \tag{4.4}$$

$$U = \frac{1}{2}(u + x + \sqrt{x - u}), \quad X = \frac{1}{2}(u + x - \sqrt{x - u}), \quad u < x. \tag{4.5}$$

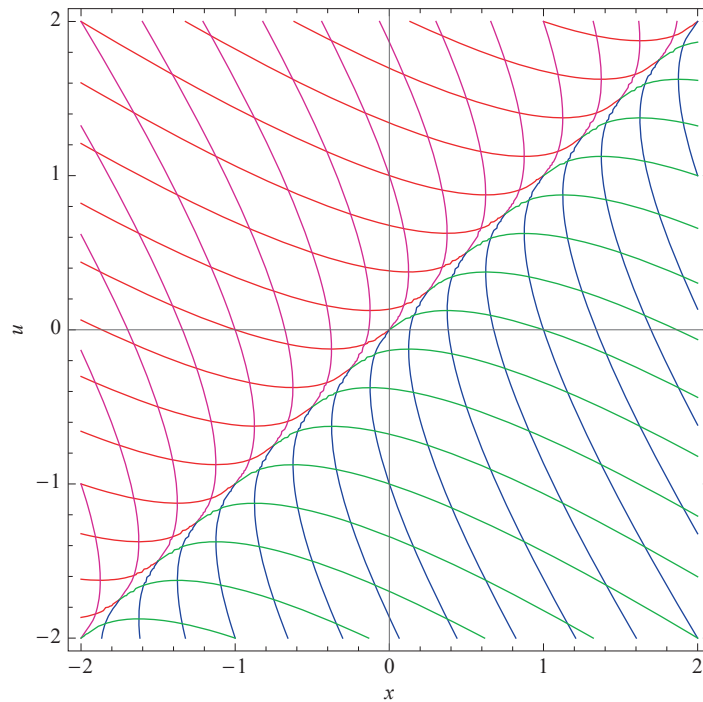


Fig. 1. Coordinate lines of the Cartesian-parabolic case (12̂): refer to definition (4.4) for coordinates in the upper-left half plane and to definition (4.5) for coordinates in the lower-right half plane.

Standard (Liouville) separation occurs in the $u > x$ half plane, where the separated form of the Hamiltonian and second integral are

$$H_1 = \frac{1}{U - X} \left[\frac{1}{4}(-p_U^2 + p_X^2) + 4(U^3 - X^3) \right], \tag{4.6}$$

$$I_1 = \frac{1}{U - X} [U(p_X^2 - 16X^3) - X(p_U^2 - 16U^3)]. \tag{4.7}$$

In the $u < x$ half plane, a direct application of the new transformation (4.5) puts the potential in the form

$$\Phi = 2(U^2 + X^2 + 4UX), \tag{4.8}$$

which is not separable. However, using complex variables (2.10), we get complex-Liouville/harmonic separated forms of the Hamiltonian and second integral

$$H_1 = 2 \frac{\Re\{p_Z^2 + (1 - i)Z^3\}}{\Re\{(1 + i)Z\}}, \tag{4.9}$$

$$I_1 = 2 \frac{\Im\{(1 - i)\bar{Z}(p_Z^2 + (1 - i)Z^3)\}}{\Re\{(1 + i)Z\}}. \tag{4.10}$$

4.2. Example II: A parabolic-parabolic case ($2\hat{2}$)

Let us consider the Hamiltonian

$$H_2 = \frac{1}{2}(-p_u^2 + p_x^2) + 4u^2 - x^2. \tag{4.11}$$

This Hamiltonian is superintegrable (actually “superseparable”), with integrals of motion

$$A_2 = \frac{1}{2}p_x^2 - x^2, \quad B_2 = H_2 - A_2 \tag{4.12}$$

and

$$I_2 = p_x(up_x + xp_u) + 2ux^2. \tag{4.13}$$

The existence of I_2 is associated with parabolic coordinates. According to Table 1, the conformal coordinate transformation is generated by $\Sigma_2(\zeta) = 4\zeta$ and $\hat{\Sigma}_2(\hat{\zeta}) = 4\hat{\zeta}$, so that the new variables are (see Fig. 2)

$$W = \sqrt{\zeta}, \quad \hat{W} = \sqrt{\epsilon\hat{\zeta}}, \tag{4.14}$$

or, in non-null coordinates,

$$U = \frac{1}{2}(\sqrt{u+x} + \sqrt{u-x}), \quad X = \frac{1}{2}(\sqrt{u+x} - \sqrt{u-x}), \quad u^2 > x^2, \tag{4.15}$$

$$U = \frac{1}{2}(\sqrt{u+x} + \sqrt{x-u}), \quad X = \frac{1}{2}(\sqrt{u+x} - \sqrt{x-u}), \quad u^2 < x^2. \tag{4.16}$$

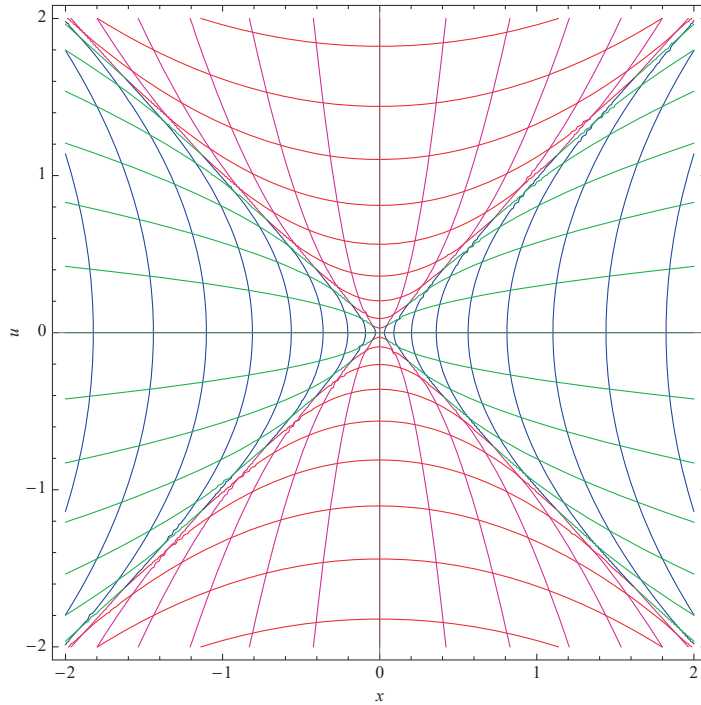


Fig. 2. Coordinate lines of the parabolic-parabolic case (22): refer to definition (4.15) for coordinates inside the light cone and to definition (4.16) for coordinates outside the light cone.

Standard (Liouville) separation occurs inside the “light cone”, where the separated form of the Hamiltonian and second integral are

$$H_2 = \frac{1}{4(U^2 - X^2)}[-p_U^2 + p_X^2 + 16(U^6 - X^6)], \tag{4.17}$$

$$I_2 = \frac{1}{U^2 - X^2}[U^2(p_X^2 - 32X^6) - X^2(p_U^2 - 32U^6)]. \tag{4.18}$$

Outside the light cone, the new transformation (4.16) puts the potential in the form

$$\Phi = 14U^2 X^2 - U^4 - X^4, \tag{4.19}$$

which is not separable. However, using complex variables (2.10), we get complex-Liouville/harmonic separated forms of the Hamiltonian and second integral

$$H_2 = \frac{\Re\{p_Z^2 - 2Z^6\}}{2\Re\{Z^2\}}, \tag{4.20}$$

$$I_2 = \frac{\Im\{\bar{Z}^2(p_Z^2 - 2Z^6)\}}{\Re\{Z^2\}}. \tag{4.21}$$

4.3. Example III: A polar-elliptical case (34)

Let us consider the Hamiltonian

$$H_3 = \frac{1}{2}(-p_u^2 + p_x^2) + 2(u + x)^2(4(u - x)^2 - 1). \tag{4.22}$$

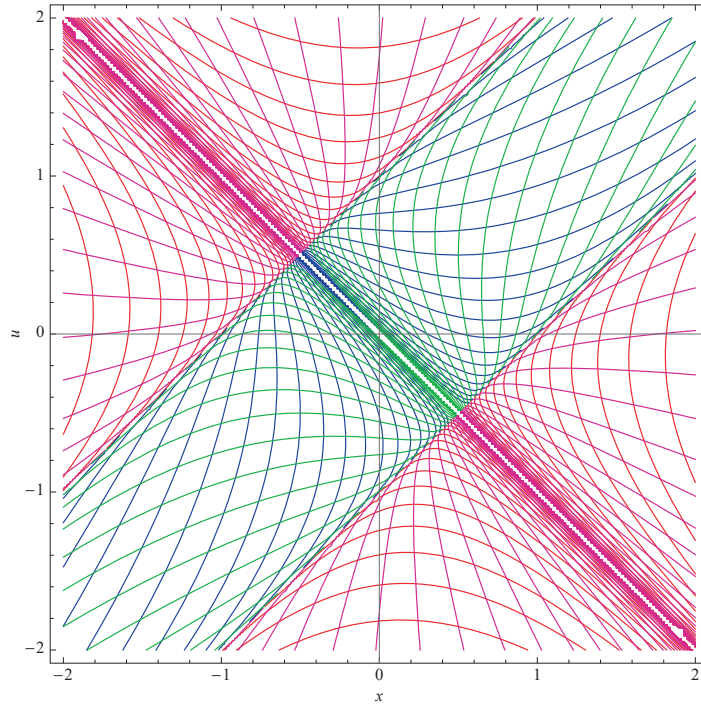


Fig. 3. Coordinate lines of the polar-elliptical case (34): refer to definitions (4.27–4.28) for coordinates inside the strip $|u - x| < 1$ and to definitions (4.25–4.26) for coordinates outside the strip.

This system admits the second integral of motion

$$I_3 = (up_x + xp_u)^2 - \frac{1}{4}(p_u - p_x)^2 + 4(u - x)(u + x)^3. \tag{4.23}$$

The existence of I_3 is associated with “polar-elliptical” coordinates (see Fig. 3). According to Table 1, the conformal coordinate transformation is generated by $\Sigma_3(\zeta) = \zeta^2$ and $\hat{\Sigma}_4(\hat{\zeta}) = \hat{\zeta}^2 - 1$, where, without loss of generality, the choice $\hat{\Delta} = 1$ has been made. The coordinate transformation is

$$\begin{aligned} \Sigma_3(\zeta) &= \zeta^2, \quad \hat{\Sigma}_4(\hat{\zeta}) = \hat{\zeta}^2 - 1, \quad W = A_3(\zeta) = \ln |\zeta|, \quad F_3(W) = e^W \\ \hat{W} = \hat{A}_4(\hat{\zeta}) &= \int \frac{d\hat{\zeta}}{\sqrt{\epsilon(\hat{\zeta}^2 - 1)}} \\ &= \begin{cases} \operatorname{arcosh} \hat{\zeta}, & \hat{F}_4(\hat{W}) = \cosh \hat{W}, \quad \epsilon = +1, \quad \text{Liouville separation} \\ \operatorname{arcsin} \hat{\zeta}, & \hat{F}_4(\hat{W}) = \sin \hat{W}, \quad \epsilon = -1, \quad \text{harmonic separation} \end{cases} \end{aligned} \tag{4.24}$$

or in non-null coordinates

$$U = \frac{1}{2}(\ln(u + x) + \ln(u - x + \sqrt{(u - x)^2 - 1})), \tag{4.25}$$

$$X = \frac{1}{2}(\ln(u + x) - \ln(u - x + \sqrt{(u - x)^2 - 1})), \quad \epsilon = +1, \quad |u - x| > 1 \tag{4.26}$$

and

$$U = \frac{1}{2}(\ln(u+x) + \arcsin(u-x)), \quad (4.27)$$

$$X = \frac{1}{2}(\ln(u+x) - \arcsin(u-x)), \quad \epsilon = -1, \quad |u-x| < 1. \quad (4.28)$$

Standard (Liouville) separation occurs *outside* the strip defined by $|u-x| < 1$, where separating variables are given by (4.25)–(4.26) and the separated form of the Hamiltonian and second integral are

$$H_3 = \frac{1}{e^{2U} - e^{2X}}[-p_U^2 + p_X^2 + e^{6U} - e^{6X}], \quad (4.29)$$

$$I_3 = \frac{1}{e^{2U} - e^{2X}}[e^{2U}(p_X^2 - 2e^{6X}) - e^{2X}(p_U^2 - 2e^{6U})]. \quad (4.30)$$

Inside the strip $|u-x| < 1$, the transformation (4.27)–(4.28) puts the potential in the form

$$\Phi = 2e^{2(U+X)}(4\sin^2(U-X) - 1), \quad (4.31)$$

which is not separable. However, using complex variables (2.10), we get complex-Liouville/harmonic separated forms of the Hamiltonian and second integral

$$H_3 = \frac{2\Re\{p_Z^2 - e^{3(1-i)Z}\}}{\Re\{e^{(1-i)Z}\}}, \quad (4.32)$$

$$I_3 = \frac{2\Im\{e^{(1+i)\bar{Z}}(p_Z^2 - e^{3(1-i)Z})\}}{\Re\{e^{(1-i)Z}\}}. \quad (4.33)$$

4.4. Example IV: A Cartesian–Cartesian ($1\hat{1}$) null case

Let us consider the Hamiltonian

$$H_4 = \frac{1}{2}(-p_u^2 + p_x^2) + 3u^2 - x^2 - 2ux. \quad (4.34)$$

The second integral of motion is

$$I_4 = \frac{1}{4}(p_u + p_x)^2 - (u-x)^2. \quad (4.35)$$

To obtain this integrable case, we can use pseudo-rotated Cartesian coordinates since they are themselves good coordinates. However, the choice $\Sigma_1(\zeta) = 1$ and $\hat{\Sigma}_1(\hat{\zeta}) = 0$ must be made in order to accommodate the (leading order) term in the momenta. This is an example of a vanishing determinant of the conformal Killing tensor. The separable forms are now expressed in terms of the null coordinates: they are

$$H_4 = -2p_\zeta p_{\hat{\zeta}} + 2\zeta\hat{\zeta} + \hat{\zeta}^2 \quad (4.36)$$

and

$$I_4 = p_\zeta^2 - \hat{\zeta}^2. \quad (4.37)$$

The integrability by quadrature follows by exploiting the recipe of references [3, Sec, III.E] and [4, Sec. 2.2.2].

5. Conclusions

We have presented examples of nonstandard separation of the natural Hamilton–Jacobi equation on the Minkowski plane \mathbb{M}^2 . The two new possibilities, the complex-Liouville (or harmonic) case and the linear/null (or Jordan block) case have been illustrated by means of explicit examples, showing that, in all cases, a suitable glueing of the orbits across the coordinate patches corresponding to the different structures allows us to separate indefinite natural systems all over \mathbb{M}^2 .

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