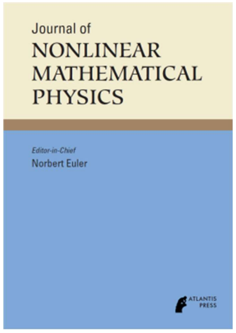


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# EULER–LAGRANGE EQUATIONS FOR FUNCTIONALS DEFINED ON FRÉCHET MANIFOLDS

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We prove a version of the variational Euler–Lagrange equations valid for functionals defined on Fréchet manifolds, such as the spaces of sections of differentiable vector bundles appearing in various physical theories.

*Keywords*: Spaces of mappings; calculus of variations; infinite-dimensional manifolds; Euler–Lagrange equations; Einstein's equations.

## 1. Introduction

Calculus of variations is usually applied in Physics as follows: consider a particle moving in $\mathbb{R}^n$ from a point $P$ to a point $Q$. Among all possible paths $c : \mathbb{R} \to \mathbb{R}^n$, the particle will follow the one that makes stationary the action functional

$$A[c] = \int_{t_P}^{t_Q} L(c(t), \dot{c}(t))dt, \tag{1.1}$$

where $L \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is called the Lagrange function or Lagrangian, subject to the end points conditions

$$c(t_P) = P, c(t_Q) = Q. \tag{1.2}$$

This path can be characterized as a solution (with initial conditions (1.2)) to the Euler–Lagrange equations

$$\frac{\partial L}{\partial \dot{x}^i}(c(t), \dot{c}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}(c(t), \dot{c}(t)) = 0, \quad 1 \le i \le n, \tag{1.3}$$

with $\{x^i, y^i\}_{i=1}^n$ a set of coordinates on $\mathbb{R}^n \times \mathbb{R}^n$.

This is the setting of particle mechanics, but in Physics there are more general constructions appropriate to deal with fields. A (vector) field is a differentiable mapping $\psi : M \to \Gamma^\infty E$ where $M$ is a manifold (usually Minkowski's spacetime) and $E$ is some vector bundle on $M$, with $\Gamma^\infty E$ its space of differentiable sections. For example, in quantum field theory $E$ is a spinor bundle on $M$ (see [5, 16, 19]), in gauge field theory $E$ is a bundle of connections on a principal bundle over $M$ (see [3, 4, 13]), and in general relativity $E$ is the bundle of Riemannian metrics on $M$ (see [10, 14, 17]). The dynamical equations for these fields are the Euler–Lagrange equations for a functional of the type (1.1), where $L$ depends now on the fields $\psi(x^\mu)$ and its derivatives $\partial_\nu \psi(x^\mu)$.

From the point of view of global analysis, these fields $\psi$ are viewed as mappings $\psi : M \to N$ where $N$ is a tensor manifold, so we are interested in studying critical points of functionals of the form

$$A[\psi] = \int_M \mathcal{L}(\psi, \partial\psi) dx$$

where $\mathcal{L} \in T\mathcal{C}^\infty(M, N)$. The space of differentiable mappings, $\mathcal{C}^\infty(M, N)$, can be viewed as an infinite-dimensional manifold, modeled on Fréchet spaces if $M$ is compact (a case[a] which encompasses situations of the most interest in Physics, such as dynamics on a torus or a sphere, although it is possible to extend the theory presented here to an arbitrary $M$, see e.g. Example 3.3 below). So, we could as well consider a Fréchet infinite-dimensional manifold $\mathcal{M}$ *ab initio* and study the functionals defined on it.

More generally, in order to deal with dynamical situations we will consider curves $c : J \subset \mathbb{R} \to \mathcal{M}$ and Lagrangians depending on the curve and its derivatives. Our main goal is to obtain an expression for the Euler–Lagrange equations in this setting as close as possible to that of the "finite-dimensional" calculus of variations (1.3). Indeed, the main result of this paper states that if $\mathcal{L} \in C^\infty(T\mathcal{M}, \mathbb{R})$ is such a Lagrangian, then, a curve $c \in C^\infty(J, \mathcal{M})$ is critical for $\mathcal{L}$ if and only if it verifies the Euler–Lagrange equations

$$(D_1 L)(u(t), u'(t)) - \left.\frac{d}{d\xi}\right|_{\xi=t} (D_2 L)(u(\xi), u'(\xi)) = 0,$$

in a local chart where $L$ and $u(t)$ are, respectively, the local expressions for $\mathcal{L}$ and $c(t)$, and $D_i L$ ($i \in \{1, 2\}$) are the partial derivatives of $L$ (all the notions appearing here will be explicitly defined in Sec. 6).

Note that this is not a trivial task, as many of the commonly used results in the calculus of variations on, say, Banach spaces does not hold when directly translated to the Fréchet setting.

In Sec. 2 we offer some motivation to our problem by considering a variational description of Einstein's equations. Then, in Sec. 3 we fix notations and list several facts from the theory of infinite-dimensional manifolds for reference. Sections 4 and 5 contain some preliminary technical results that will be used later. Finally, in the last section we state and prove the main result.

## 2. Einstein's Equations as a Variational System

Let $M$ be a differential manifold and $(E, \pi, M)$ a vector bundle. Denote by $\Gamma^\infty E$ the set of differentiable sections of $E$, by $\Gamma_K^\infty E$ the set of differentiable sections with compact support

[a] There are other reasons for considering Fréchet spaces. For example, an infinite dimensional manifold is metrizable if and only if is paracompact modeled on Fréchet spaces.

$K \subset M$, and by $\Gamma_0^\infty E$ the set of all differentiable sections with compact support. If $M$ itself is compact, then $\Gamma^\infty E$ with the $C^\infty$-topology is a Fréchet space (that is, a locally convex topological vector space which is metrizable and complete). For an arbitrary $M$, if $K \subset M$ is a compact subset then $\Gamma_K^\infty E$ is a Fréchet space contained in $\Gamma_0^\infty E$ and if we take a fundamental sequence of compact subsets in $M$, $\{K_i\}_{i\in I}$ (so $M = \bigcup_{i\in I} K_i$), $\Gamma_0^\infty E$ can be endowed with the inductive limit topology and it becomes an LF-space (see [18]). We have a disjoint union decomposition $\Gamma^\infty E = \bigsqcup_{s\in\Lambda}(\Gamma_0^\infty E + s)$, where $\Lambda$ is a certain set of sections with support not necessarily compact. A particularly interesting case is that in which $E = T^{(r,s)}M$ is a tensor manifold over $M$; its sections are denoted by $\Gamma^\infty(T^{(r,s)}M)$.

The sets above can be used to endow the set of mappings $\mathcal{C}^\infty(M,N)$ (with $M, N$ differential manifolds) with a manifold structure. To build an atlas for $\mathcal{C}^\infty(M,N)$ take a Riemannian metric $g$ on $N$; then, there exists an open set $Z \subset TN$ containing the 0-section and such that, if we write $Z_q = Z \cap T_qN$, $Z_q$ is contained in the domain in which the exponential mapping $\exp_q$ is a diffeomorphism. Now, given an $f \in \mathcal{C}^\infty(M,N)$ we construct a set $V_f \subset \Gamma_0^\infty(f^*TN)$ as

$$V_f = \{\zeta \in \Gamma_0^\infty(f^*TN) : \zeta_q \in Z_{f(q)}\}$$

(here $f^*TN$ is the pullback bundle of $TN$ by $f : M \to N$). This $V_f$ is an open set. We also construct a subset of $\mathcal{C}^\infty(M,N)$ containing $f$ through

$$U_f = \{\widetilde{f} : \exists K \subset M \text{ compact with } \widetilde{f}|_{M-K} = f|_{M-K} \text{ and } \widetilde{f}(p) \in \exp_{f(p)}(Z_{f(p)})\}.$$

The charts for $\mathcal{C}^\infty(M,N)$ are the bijections

$$\begin{aligned} \Psi_f : U_f &\to V_f \\ \widetilde{f} &\mapsto \Psi_f(\widetilde{f}) = \zeta_{\widetilde{f}} \end{aligned}$$

where $\zeta_{\widetilde{f}}(p) = \exp^{-1}_{f(p)}(\widetilde{f}(p))$. It can be proved that the resulting differential structure is independent of the chosen metric $g$ and open set $Z$.

Once an atlas of $\mathcal{C}^\infty(M,N)$ is known, it results that some particularly important subsets as the immersions $\mathrm{Imm}(M,N)$, the embeddings $\mathrm{Emb}(M,N)$, or diffeomorphisms $\mathrm{Diff}(M)$ are open, and so inherit a manifold structure.

**Example 2.1.** Consider the tensor bundle $S^2M$, of 2-covariant symmetric tensors on a manifold $M$. If $\mathcal{M} = S^2_+M$ denotes the subspace of positive definite elements, then it admits a differential manifold structure (the manifold of Riemannian metrics on $M$) modeled on $\Gamma_0(S^2M)$ (the sections of compact support). This structure is such that given a metric $g \in \mathcal{M}$, its tangent space is $T_g\mathcal{M} \simeq \Gamma_0(S^2M$, as $\mathcal{M}$ is open in $S^2M$. A detailed study of the properties of $\mathcal{M}$, including its Riemannian structure, can be found in [7].

By using this example, we can give a variational description on Einstein's equations of General Relativity. Recall that a spacetime is a four dimensional Lorentzian manifold $(\widetilde{N}, \widetilde{g})$ and that Einstein's vacuum equations are

$$\rho(\widetilde{g}) - \frac{1}{2}\tau(\widetilde{g})\widetilde{g} = 0, \tag{2.1}$$

where $\rho$ is the Ricci tensor and $\tau$ the scalar curvature. When $\widetilde{N}$ is globally hyperbolic, it can be viewed as a foliation $\widetilde{N} = \bigcup_{t\in\mathbb{R}} M_t$ by Cauchy's 3-dimensional hypersurfaces,[b] all of them diffeomorphic to a manifold $M$ (see [2] and references therein). Thus, we have a curve of immersions

$$\begin{aligned} \varphi : I \subset \mathbb{R} &\to \mathrm{Imm}(M, N) \\ t &\mapsto \varphi(t) = \varphi_t \end{aligned},$$

with $\varphi_t(M) = M_t$ the leaves of the foliation. Each $M_t$ has an induced Riemannian structure, so we can also consider a curve of metrics

$$\begin{aligned} g : I \subset \mathbb{R} &\to \mathcal{M} \\ t &\mapsto g(t) \equiv g_t = \varphi_t^* g \end{aligned}, \tag{2.2}$$

with $\mathcal{M}$ the manifold of Riemannian metrics of $M$. With the aid of the $3+1$ or ADM formalism (see [1]), it is possible to transfer conditions (2.1) on $\widetilde{g}$ to a set of conditions for the curve (2.2). Without entering into details (which will appear elsewhere, [8]), let us note that $\widetilde{g}$ is Ricci flat (that is, it verifies (2.1)) if and only if the curve $g_t$ satisfies

$$g_t'' = \Gamma_{g_t}^{-}(g_t, g_t') - 2\lambda\, \mathrm{Grad}^{-} S(g_t) \tag{2.3}$$

and a set of other subsidiary conditions which do not interest us at this point. In (2.3), $g_t'$ is the derivative of $g_t$ with respect to $t$, $\Gamma^-$ is the Christoffel symbol associated to a certain metric $G^-$ on $\mathcal{M}$, and $\mathrm{Grad}^-$ is the gradient determined by $G^-$. $S(g_t)$ is the total scalar curvature of $g_t$.

Thus, a variational treatment of General Relativity can be achieved if we write equations (2.3) as the Euler–Lagrange's equations for a suitable Lagrangian $\mathcal{L} \in \mathcal{C}^\infty(T\mathcal{M})$, to be satisfied by the extremals of an action of the form

$$A[g_t] = \int_{t_A}^{t_B} \mathcal{L}(g_t, g_t')dt. \tag{2.4}$$

This correspond to the image of a 4-dimensional universe obtained through the time evolution of a 3-dimensional "slice" $M$ in such a way that it makes (2.4) stationary.

Note that $\mathcal{L}$ takes its arguments in the tangent bundle of an infinite dimensional Fréchet manifold when $M$ is compact. To find a set of equations we can call Euler–Lagrange's equations, we must make use of the techniques of differential calculus in Fréchet spaces, which presents some differences with the finite dimensional case (there is no direct analog of the inverse function theorem, for instance, see [9]). Also, if we want to proceed as in the classical calculus of variations, we must give a sense to the integral of functionals defined on duals of Fréchet spaces,[c] as explained in Sec. 4. These spaces are even "worse" than the Fréchet ones (in particular, they are never Fréchet), but they still are locally convex, a property which will suffice for our purpose. These remarks should illustrate the differences between the finite and infinite dimensional settings.

[b] Connected and spacelike.

[c] In the finite dimensional case, the integration takes place in the dual of $\mathbb{R}^n$ which is canonically identified as an euclidean space with $\mathbb{R}^n$ itself.

In this paper only a part of this programme will be presented, namely, the deduction of Euler–Lagrange's equations for a functional of the form (2.4), which is an important result *per se* and could be of interest in other situations of relevance in Physics, as stated in the Introduction.

## 3. Infinite Dimensional Manifolds

For the sake of generality, we will work in the category of convenient vector spaces, which are $c^\infty$-complete locally convex vector spaces (also called bornologically complete). This setting is slightly more general than that of Fréchet and LF-spaces, and both are equivalent when applicable. However, as stated, the class of convenient vector spaces is wider. For background definitions and proofs in this section, we refer the reader to the comprehensive book by A. Kriegl and P. W. Michor [12].

**Definition 3.1.** Let $E$ be a locally convex vector space. A net $(x_\gamma)_{\gamma\in\Gamma} \subset E$ is called Mackey–Cauchy if there exists a bounded (absolutely convex) set $B \subset E$ and a net $(\mu_{(\gamma,\gamma')})_{(\gamma,\gamma')\in\Gamma\times\Gamma} \subset \mathbb{R}$ converging to $0 \in \mathbb{R}$ such that $x_\gamma - x_{\gamma'} \in \mu_{(\gamma,\gamma')}B$. The space $E$ is Mackey complete if every Mackey–Cauchy net in $E$ converges.

Obviously, Mackey–Cauchy sequences are Cauchy, so sequentially complete locally convex spaces are Mackey complete.

**Proposition 3.2.** *If $E$ is metrizable, sequential completeness and Mackey completeness are equivalent. Moreover, Mackey completeness of $E$ is equivalent to the following condition: each curve $c : \mathbb{R} \to \mathbb{E}$ with the property that for each linear mapping $l : E \to \mathbb{R}$ then $l \circ c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$, belongs to the class $\mathcal{C}^\infty$.*

Note that curves defined on a locally convex vector space have a natural definition of differentiability. The spaces in which these equivalent properties hold are the convenient vector spaces in the terminology of Kriegl and Michor.

**Example 3.3.** Given a compact finite-dimensional differential manifold $M$ (not necessarily compact) and a vector bundle $\pi : E \to M$, consider a compact subset $K \subset M$. Then, the space of differentiable sections with support on $K$, $\Gamma^\infty_K(E)$, is a Fréchet space contained in $\Gamma^\infty_0(E)$, the space of differentiable sections with compact support. If we take a fundamental sequence of compacts in $M$, $\{K_i\}_{i=1}^\infty$, then $\Gamma^\infty_0(E)$ can be endowed with the inductive limit topology and is a convenient vector space.

Convenient vector spaces carry on the so called $c^\infty$-topology, which is the final topology with respect to all smooth curves $c : \mathbb{R} \to E$. Its open sets are called $c^\infty$-open sets.

Now, $C^\infty$-manifolds (of arbitrary dimensions) can be defined by modeling them on convenient vector spaces and requiring that the transition functions $u_{\alpha\beta} : u_\beta(U_\alpha \cap U_\beta) \to u_\alpha(U_\alpha \cap U_\beta)$ be smooth mappings[d] between $c^\infty$-open sets of a convenient vector space (which means that these $u_{\alpha\beta}$ map smooth curves in $u_\beta(U_\alpha \cap U_\beta)$ to smooth curves in $u_\alpha(U_\alpha \cap U_\beta)$). A particular example has been considered in the previous section, where the model space was Fréchet. Other constructions work exactly as in the finite dimensional case.

[d] We use the symbol $C^\infty$ to denote the class of such smooth mappings, reserving $\mathcal{C}^\infty$ for smooth mappings between finite dimensional spaces.

**Definition 3.4.** A mapping $f : \mathcal{M} \to \mathcal{N}$ between manifolds modeled on convenient vector spaces is smooth (denoted $f \in C^\infty(\mathcal{M}, \mathcal{N})$) if, for each $x \in \mathcal{M}$ and each chart $(\widetilde{U}, v)$ around $f(x) \in \mathcal{N}$, there is a chart $(U, u)$ on $\mathcal{M}$ around $x$ such that $f(U) \subset \widetilde{U}$ and $v \circ f \circ u^{-1} : U \to \widetilde{U}$ is smooth as a mapping between $c^\infty$-open sets in convenient vector spaces.

Tangent spaces and tangent mappings are defined as usual.

**Example 3.5.** Let $E$ be a convenient vector space and $U \subset E$ a $c^\infty$-open set. Then, viewed as an open submanifold (with the chart given by the inclusion $j : U \hookrightarrow E$) at any $x \in U$ we have $T_x U \simeq E$.

In particular, tangent vectors can be viewed as tangents to smooth curves as well as derivations over evaluation morphisms.

**Proposition 3.6.** *If $f : \mathcal{M} \to \mathcal{N}$ is a smooth mapping between manifolds modeled on convenient vector spaces, it induces a mapping $T_x f : T_x\mathcal{M} \to T_{f(x)}\mathcal{N}$ for each $x \in \mathcal{M}$ through the formula*

$$(T_x f(A))(h) = A(h \circ f)$$

*for $A \in T_x\mathcal{M}$, $h \in C^\infty(\mathcal{N}, \mathbb{R})$.*

For an infinite-dimensional manifold $\mathcal{M}$, an analogue of the exponential mapping is introduced.

**Definition 3.7.** Let $T\mathcal{M}$ be the tangent bundle of $\mathcal{M}$, with projection $\pi_{\mathcal{M}}$. Then, a mapping $\alpha : U \subset T\mathcal{M} \to \mathcal{M}$ defined on an open neighborhood of the zero section in $T\mathcal{M}$, which satisfies:

(i) $(\pi_{\mathcal{M}}, \alpha) : U \subset T\mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is a diffeomorphism onto a $c^\infty$-open neighborhood of the origin,
(ii) $\alpha(0_x) = x$ for all $x \in \mathcal{M}$,

is called a local addition on $\mathcal{M}$.

**Example 3.8.** (a) The affine structure on each convenient vector space gives a local addition.
(b) Let $\pi : V \to M$ be a finite rank vector bundle over a finite dimensional differential manifold $M$. Then, the space of sections $\Gamma_0^\infty(V)$ admits a local addition. Moreover, $\Gamma_0^\infty(V)$ can be viewed as a splitting submanifold of $\mathcal{C}^\infty(M, V)$. Even more, the total space $V$ admits a local addition.

**Remark 3.9.** When $\mathcal{M}$ is an infinite-dimensional manifold admitting a local addition, we have that the space $C^\infty(J, \mathcal{M})$, where $J \subset \mathbb{R}$ is open, is a smooth manifold modeled on spaces $\Gamma_0^\infty(c^*T\mathcal{M})$ of smooth sections (with compact support) of pullback bundles along curves $c : J \to \mathcal{M}$. In view of the preceding example, this is the case when $\mathcal{M}$ is a space of sections of a tensorial bundle over some compact finite-dimensional manifold $M$, already discussed in the Introduction.

## 4. Integration on Duals of Fréchet Spaces

In this section, we refer the reader to the text [15] for definitions and proofs unless otherwise explicitly stated.

Let $V$ be a real topological vector space. We denote by $V^*$ the space of continuous linear functionals $l : V \to \mathbb{R}$ and call it the dual space of $V$. This dual is a real vector space too and it can be topologized in a variety of ways. In particular, the weak* topology makes it a locally convex topological vector space.

Recall that a family $\mathcal{F} = \{l_\alpha\}_{\alpha\in\Lambda} \subset V^*$ is said to separate points on $V$ if, for each $v_1, v_2 \in V$ such that $v_1 \neq v_2$, there exists an $l_\alpha \in \mathcal{F}$ such that

$$l_\alpha(v_1) \neq l_\alpha(v_2).$$

The following is an easy consequence of Hahn–Banach's theorem(s).

**Lemma 4.1.** *If $V$ is a locally convex space then $V^*$ separates points.*

We also recall the construction of a certain notion of integral for vector valued functions.

**Definition 4.2.** Let $\mu$ be a measure on a measure space $X$, $V$ a topological vector space such that $V^*$ separates points, and $f : X \to V$ a function such that the scalar functions $l^*f$, for every $l \in V^*$, are integrable with respect to $\mu$ (where $l^*f : X \to \mathbb{R}$ is defined by $(l^*f)(x) = l(f(x))$). If there exists a vector $v \in V$ such that, for every $l^* \in V$

$$l(v) = \int_X (l^*f)d\mu,$$

then we define

$$\int_X f d\mu = v.$$

This integral exists under very general conditions.

**Proposition 4.3.** *Let $V$ be a topological vector space. Suppose that*:

(i) *$V^*$ separates points on $V$.*
(ii) *$\mu$ is a Borel probability measure on a compact Hausdorff space $X$.*

*If $f : X \to V$ is continuous and the closed convex hull $\overline{co}(f(X))$ is compact in $V$, then the integral*

$$v = \int_X f d\mu$$

*exists in the sense of Definition* 4.2.

We wish to adapt this construction to the setting we will use later, in which $V = E^*$ is the dual of a Fréchet space $E$, and $X = [a, b]$ is a compact interval in the real line $\mathbb{R}$. We already know (by Lemma 4.1) that the dual of $E^*$ separates points on $E^*$. Also, the Lebesgue measure on $[a, b]$ obviously satisfies condition (ii) in Proposition 4.3. Thus, what we need is to prove that given a continuous $f : [a, b] \to E^*$, the closed convex hull $\overline{co}(f([a, b]))$ is compact in $E^*$.

**Proposition 4.4.** *Let $[a,b] \subset \mathbb{R}$ and $f : [a,b] \to E^*$ be a continuous function, with $E$ a Fréchet space. Then, $\overline{co}(f([a,b]))$ is compact in $E^*$.*

**Proof.** As $[a,b]$ is compact and $f$ continuous, $K = f([a,b])$ is compact in $E^*$. Now, by Krein's theorem (see [11, §24.5(5)]), the closed convex hull $\overline{co}(K)$ of the compact set $K$ in the locally convex space $E^*$ is compact if and only if $\overline{co}(K)$ is $\tau_k$-complete, where $\tau_k$ is the Mackey topology on $E^*$. But $E$ is Fréchet, so $E^*$ is automatically $\tau_k$-complete (see [11, §21.6(4)]), so the closed set $\overline{co}(K)$ is also $\tau_k$-complete and thus compact. □

**Corollary 4.5.** *Let $[a,b] \subset \mathbb{R}$ and $f : [a,b] \to E^*$ be a continuous function with $E$ a Fréchet space. Then $l = \int_{[a,b]} f dt$ exists as a vector integral and $l \in E^*$. For every $e \in E$,*

$$l(e) = \int_{[a,b]} (f(t))(e)dt.$$

**Proof.** Immediate from Proposition 4.4 and Definition 4.2. □

**Remark 4.6.** Of course, the same construction of the integral works for functions $f : [a,b] \to E$, with $E$ Fréchet. In this case the situation is even simpler, as for a Fréchet space it is well known that if $K \subset E$ is compact, then $\overline{co}(K)$ is compact too.

## 5. DuBois–Reymond Lemma

In this section, we generalize the classical DuBois–Reymond lemma from the theory of the calculus of variations, for later use.

**Lemma 5.1.** *Let $E$ be a Fréchet space and $f : J \subset \mathbb{R} \to E^*$ a continuous mapping with $J = [a,b]$. Then*

$$\int_J f(t)(\mu'(t))dt = 0,$$

*for all $\mu \in C^1(J,E)$ with support $\sup \mu \subset J$, if and only if $f$ is constant.*

**Proof.** For the nontrivial implication, note that if $\phi \in C^1(J,\mathbb{R})$ and $y \in E$ is a fixed element, then the mapping $\mu : J \to E$ given by $\mu(t) = y\phi(t)$ verifies $\mu \in C^1(J,E)$ (and has support $\sup \mu \subset J$ if $\sup \phi \subset J$). Thus, for any fixed $l \in E^*$, $y \in E$, $\phi \in C^1(J,\mathbb{R})$,

$$\int_J (f(t) - l)(y\phi'(t))dt = 0.$$

Take now

$$l = \int_J f(t)dt$$

(this integral in the sense of the preceding section) and

$$\phi(t) = \int_a^t (f(s) - l)(y)ds.$$

We have $\phi \in C^1(J, \mathbb{R})$ and, applying Corollary 4.5 and the hypothesis,

$$\int_J ((f(t) - l)(y))^2 dt = 0,$$

so $f(t) = l$, for all $t \in J$. $\square$

**Proposition 5.2 (DuBois–Reymond Lemma).** *Let $f, g : J \subset \mathbb{R} \to E^*$ be continuous functions with $J = [a, b]$. Then*

$$\int_J \{f(t)(\mu(t)) + g(t)(\mu'(t))\} dt = 0,$$

*for all $\mu \in C^1(J, E)$ with support in $J$, if and only if the mapping $h : J \to E^*$ given by*

$$h(t) = g(t) - \int_a^t f(s) ds$$

*is constant on $J$, that is, $f(t) = -g'(t)$.*

**Proof.** Define

$$F(t) = \int_a^t f(s) ds \ (t \in J)$$

so that $f(t) = F'(t)$ and the statement we want to prove is equivalent to

$$\int_J \{F'(t)(\mu(t)) + g(t)(\mu'(t))\} dt = 0.$$

Now, noting that $\frac{d}{dt}(t \mapsto (F(t)(\mu(t))) = (F'(t))(\mu(t)) + (F(t))(\mu'(t))$ and integrating by parts in the first member of the integrand (taking into account that $\sup \mu \subset J$),

$$\int_J \{-F(t)(\mu'(t)) + g(t)(\mu'(t))\} dt = 0,$$

and the statement follows applying Lemma 5.1. $\square$

## 6. The Euler–Lagrange Equations

Given a Fréchet manifold $\mathcal{M}$, there is defined a canonical lifting of curves to the tangent bundle $\lambda : C^\infty(J, \mathcal{M}) \to C^\infty(J, T\mathcal{M})$, where $J \subset \mathbb{R}$. For each $\mathcal{L} \in C^\infty(T\mathcal{M}, \mathbb{R})$ we will denote by $\overline{\mathcal{L}} : C^\infty(J, T\mathcal{M}) \to \mathcal{C}^\infty(J)$ the mapping

$$\overline{\mathcal{L}}(\gamma) = \mathcal{L} \circ \gamma.$$

**Definition 6.1.** A smooth function $\mathcal{L} \in C^\infty(T\mathcal{M}, \mathbb{R})$ is called a Lagrangian. For each open set $J \subset \mathbb{R}$ with compact closure, we can construct the action functional

$$F^J_{\mathcal{L}} : C^\infty(J, \mathcal{M}) \to \mathbb{R}$$
$$c \mapsto \textstyle\int_J (f_{\mathcal{L}} c)(t) dt,$$

where the action density associated to $\mathcal{L}$, $f_{\mathcal{L}}$, is given by

$$f_{\mathcal{L}} = \overline{\mathcal{L}} \circ \lambda$$

(note that $f_{\mathcal{L}} c : J \to \mathbb{R}$ and, indeed, $f_{\mathcal{L}} \in \mathcal{C}^\infty(J)$).

In view of the results of O. Gil-Medrano (see [6, Proposition 3]), we define critical points of a density action as follows.

**Definition 6.2.** A curve $c \in C^\infty(J, \mathcal{M})$ is critical for the Lagrangian $\mathcal{L}$ (or for the action density $f_{\mathcal{L}}$) if and only if for each open set $J \subset \mathbb{R}$ with compact closure,

$$T_c F^J_{\mathcal{L}}(A) = 0,$$

for all $A \in \Gamma^\infty(c^* T\mathcal{M})$ with support contained in $J$.

Now, we can state and prove the main result of the paper. Recall that if $U$ is a $c^\infty$-open set of a Fréchet space $E$, the partial derivatives of a mapping $L : U \times E \to \mathbb{R}$, are defined by

$$D_1 L(u,e)(f) = \lim_{\xi \to 0} \frac{1}{\xi}(L(u + \xi f, e) - L(u,e))$$

and

$$D_2 L(u,e)(f) = \lim_{\xi \to 0} \frac{1}{\xi}(L(u, e + \xi f) - L(u,e)),$$

for $(u,e) \in U \times E$ and $f \in E$.

**Theorem 6.3.** *Let $\mathcal{L} \in C^\infty(T\mathcal{M}, \mathbb{R})$ be a Lagrangian. Then a curve $c \in C^\infty(J, \mathcal{M})$ is critical for $\mathcal{L}$ if and only if it verifies the Euler–Lagrange equations*

$$(D_1 L)(u(t), u'(t)) - \left.\frac{d}{d\xi}\right|_{\xi = t} (D_2 L)(u(\xi), u'(\xi)) = 0,$$

*in a local chart where $L$ and $u(t)$ are, respectively, the local expressions for $\mathcal{L}$ and $c(t)$, and $D_i L$ $(i \in \{1,2\})$ are the partial derivatives of $L$.*

**Proof.** Let $\mathcal{M}$ be a differential Fréchet manifold modeled on the Fréchet space $E$ and let $\varphi : \mathcal{U} \subset \mathcal{M} \to U \subset E$ be a local chart, with $U$ a $c^\infty$-open set in $E$. Let $c \in C^\infty(J, \mathcal{M})$ be a critical point of $\mathcal{L}$ and suppose (for simplicity) that $c(J) \subset \mathcal{U}$. Finally, let $A \in \Gamma^\infty(c^* T\mathcal{M})$ be such that $\sup A \subset J$. We have that $A$ is of the form

$$A = \left.\frac{d}{ds}\right|_{s=0} \alpha_s$$

for some curve

$$\begin{aligned} \alpha : \mathbb{R} &\to C^\infty(J, \mathcal{M}) \\ s &\mapsto \alpha_s \end{aligned}$$

such that $\alpha_0 = c : J \to \mathcal{M}$. Then (recall Proposition 3.6)

$$0 = T_c F^J_{\mathcal{L}}(A) = \left.\frac{d}{ds}\right|_{s=0} \left( s \mapsto \int_J (f_{\mathcal{L}} \alpha_s)(t) dt \right).$$

Applying the rule for differentiation under the integral to the $\mathcal{C}^\infty$ integrand $f_{\mathcal{L}}\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (where $(f_{\mathcal{L}}\alpha)(s,t) = (f_{\mathcal{L}}\alpha_s)(t)$), we get

$$0 = T_c F^J_{\mathcal{L}}(A) = \int_J \left( s \mapsto \left.\frac{\partial}{\partial s}\right|_{s=0} (f_{\mathcal{L}} \alpha_s)(t) \right) dt. \tag{6.1}$$

Let us write $\varphi \circ \alpha_s = u_s : \mathbb{R} \to U$ for the local representatives of the curves $\alpha_s$, so $u_s \in C^\infty(J, U)$. In particular, we will put $u = u_0 = \varphi \circ c$. Also, let us write $L$ for the local representative of $\mathcal{L}$ in the chart $(\mathcal{U}, \varphi)$ of $\mathcal{M}$ (and the identity on $\mathbb{R}$), so $L = id \circ \mathcal{L} \circ (T\varphi)^{-1}$. Then (6.1) can be written locally as

$$0 = T_c F^J_{\mathcal{L}}(A) = \int_J \left( s \mapsto \frac{\partial}{\partial s}\bigg|_{s=0} L(u_s(t), u'_s(t)) \right) dt, \tag{6.2}$$

where

$$u'_s(t) = \frac{\partial}{\partial t} u_s(t).$$

Note that our setting can be represented as the integration of the derivative evaluated at zero with respect to the second argument of a composite map

$$\mathbb{R} \times \mathbb{R} \xrightarrow{(u,u')} U \times E \subset E \times E \xrightarrow{L} \mathbb{R},$$

where $U$ is a $c^\infty$-open set of a Fréchet space $E$. Applying the chain rule (see Theorem 3.18 in [12] and Theorem 3.3.4 in [9]), we get

$$\begin{aligned} 0 &= T_c F^J_{\mathcal{L}}(A) \\ &= \int_J \left\{ D_1 L(u(t), u'(t)) \left( \frac{\partial u_s(t)}{\partial s}\bigg|_{s=0} \right) + D_2 L(u(t), u'(t)) \left( \frac{\partial^2 u_s(t)}{\partial s \partial dt}\bigg|_{s=0} \right) \right\} dt \end{aligned} \tag{6.3}$$

where $D_1 L$ and $D_2 L$ are the partial derivatives of $L : U \times E \to \mathbb{R}$. Note that, under the assumption $L \in C^\infty(U \times E, \mathbb{R})$, the function $L$ has total derivative $dL$ and, in this case,

$$dL(e, u)(f, g) = (D_1 L)(e, u)(f) + (D_2 L)(e, u)(g).$$

Moreover, we have linearity in the argument $f \in E$. Thus, we can interpret the expression (6.3) as an integral of the form

$$\int_J \{ f(t)(\mu(t)) + g(t)(\mu'(t)) \} dt = 0,$$

where $f(t) = D_1 L(u(t), u'(t))$, $g(t) = D_2 L(u(t), u'(t))$ and $\mu(t) = \frac{\partial u_s(t)}{\partial s}\big|_{s=0}$. Applying the DuBois–Reymond Lemma (Proposition 5.2), we get the desired result. $\square$

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