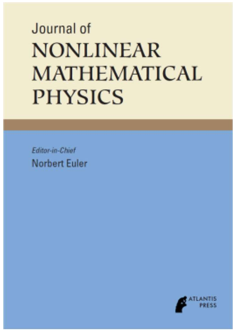

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ORBITAL LINEARIZATION IN THE QUADRATIC LOTKA–VOLTERRA SYSTEMS AROUND SINGULAR POINTS VIA LIE SYMMETRIES

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In this paper, we consider linearizability and orbital linearizability properties of the Lotka–Volterra system in the neighborhood of a singular point with eigenvalues 1 and $-q$. In this paper we give the explicit smooth near-identity change of variables that linearizes or orbital linearizes such Lotka–Volterra system with $q \in \mathbb{N}\backslash\{0, 1\}$ being seen that these changes are also valid for $q \in \mathbb{C}\backslash\{0, 1\}$.

Keywords: Lotka–Volterra system; Lie symmetries; linearization and orbitally linearization problem.

Mathematics Subject Classifications: 34C14, 34A26, 37C27, 34C25

1. Introduction

The Lotka–Volterra system

$$\dot{x}_i = x_i(a_i + b_{1i}x_1 + b_{2i}x_2 + \cdots + b_{ni}x_n) = x_iN_i(x), \quad i = 1, 2, \ldots, n \tag{1}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ was derived by Volterra in 1926 to describe the relationship between a predator and a prey, and independently by Lotka in 1920 to describe a chemical reaction. Ever since, Lotka–Volterra model has been applied in a large variety of problems in physics, biology and applied mathematics. A community for n interacting species is modeled by Eq. (1), the growth rate of the ith species is considered proportional to its species size x_i, while the interaction between species is reflected by the terms x_ix_j which may depend on the population size of the ith and jth species. For distinct i and j, the signs of $\partial N_i/\partial x_j$ and $\partial N_j/\partial x_i$ reflect the relationship between the ith and jth species. If both quantities are positive, then the growth of the each species promote the growth of the other. That is to say, they cooperate. If both quantities are negative, the two species compete. Finally, if two quantities have opposite sign, then, the species have a predator-prey relationship.

In [28] the Lotka–Volterra equations are used to modeled problems in population biology. Competition among species for resources is also discussed by using Lotka–Volterra system of nonlinear differential equations in [17]. Many socio-economic and biological processes can be modeled as systems of interacting individuals. In [31] the behavior of such systems are described within game-theoretic models. In [24] the asymptotic behavior of an n-dimensional Lotka–Volterra system with nonconstant coefficients that models the dynamics of an adaptive cellular network is analyzed. An epidemiological model whose dynamics are described by a pair of nonlinearly coupled Lotka–Volterra oscillators is shown in [26]. Sigmund in [27] gave a survey on mathematical models in ecology and evolution giving a review on the development for Lotka–Volterra system.

The Lotka–Volterra equations have also been used to model such diverse physics phenomena as mode coupling of waves in laser physics [20], the evolution of electrons and ions in plasma physics [19], and interactions of gases in a background host medium [23]. The work [2] deals with the convection in a rotating layer problem, the authors propose a three-mode Lotka–Volterra model to study the Kuppers–Lortz instability. Another subject in which Lotka–Volterra system has been used is in bifurcation theory; in a Lotka–Volterra system with two species competing or cooperating there are no periodic orbits. If species have a predator-prey relationship, then, in general the same result holds except for certain cases when the singular point located in the first quadrant $\{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 > 0, x_2 > 0\}$ is a center, that is, the singular point is surrounded by periodic orbits. For three or more species, this is no longer the case and it can be periodic orbits. In [32], the author find algebraic criteria on the parameters to predict Hopf bifurcation and, consequently, periodic orbits in competitive three-dimensional Lotka–Volterra systems. More about bifurcation theory in Lotka–Volterra systems can be found in [18, 21].

There is an extra motivation for the study of such a system, a large class of ordinary differential models in applied mathematics and physics can be transformed into a Lotka–Volterra system, see [1]. It is for that reason that it has been widely studied. In [10], Feng and Jifa provide a complete classification of the global phase portraits of two-dimensional quadratic Lotka–Volterra system as wells as provide sufficient and necessary conditions on the parameters to have closed orbits. Furthermore, an exhaustive classification of the finite singular points of the Lotka–Volterra system is given. Zeeman provided in [32] the classification of three-dimensional Lotka–Volterra systems. In summary, the two-dimensional Lotka–Volterra model appears in a large variety of problems in physics, biology and applied mathematics, as well as the three-dimensional version. Is for that reason that it has been widely studied, see for instance [3, 6].

The integrability problem in various meaning for Lotka–Volterra system have been recently investigated. Cairó and Llibre characterize all the polynomial first integrals of the two-dimensional Lotka–Volterra systems in [4] and study the existence of first integrals for three-dimensional Lotka–Volterra systems using the Darboux theory of integrability in [5]. In [6], the complete the classification of the two-dimensional Lotka–Volterra system having a Liouvillian first integral is given. The global analytic first integrals for the real planar Lotka–Volterra system is studied in [22].

Another subject of study is the classification of the normalizable and linearizable singular points of Lotka–Volterra system, see [7, 8, 18, 21]. In this paper we are mainly concerned about the classical problem of local linearization and orbital linearization in a neighborhood

$\mathcal{U} \subset \mathbb{C}^2$ of a singular point of the quadratic two-dimensional Lotka–Volterra family

$$\dot{x} = x(1 + ax + by), \quad \dot{y} = y(-q + cx + dy) \tag{2}$$

defined in $\mathbb{C}^2$ with $q \in \mathbb{C}\backslash\{0, 1\}$. We say that $\mathcal{X} = x(1 + ax + by)\partial_x + y(-q + cx + dy)\partial_y$ is the vector field associated to the differential system (2). In [8], necessary and sufficient conditions on the parameters (q, a, b, c, d) for analytic linearizability and orbital linearizability of system (2) are given for $q \in \mathbb{N}\backslash\{0, 1\}$. The case $q \in \{0, 1\}$ is studied in [7]. See also [18, 21] for some generalizations of the values of q. In [6] it is studied the Liovillian integrability of Lotka–Volterra systems (2) but this do not implies, in general, the local linearization and orbital linearization of system (2) in a neighborhood $\mathcal{U} \subset \mathbb{C}^2$ of the origin.

System (2) has at most four finite singular points. The origin is a hyperbolic saddle whose stable manifold is $x = 0$ and the unstable manifold is $y = 0$. In particular the first quadrant is invariant and can contains at most one singular point. In Sec. 3 we give the explicit smooth near-identity change of variables $\phi : \mathcal{U} \to \mathbb{C}^2$ of the form $\phi(x, y) = (u(x, y), v(x, y)) = (x + o(x, y)), y + o(x, y))$ that linearizes or orbital linearizes system (2) in a neighborhood $\mathcal{U} \subset \mathbb{C}^2$ of the trivial singular point, that is, the origin.

There exists the so-called Darboux linearization method for a polynomials systems by using invariants curves, see [8]. We want to comment that the innovations presented in this work come from an approach to the linearization (resp. orbital linearization) problem based on Lie symmetries. Thus, we use two different methods. We obtain the linearizing (resp. orbital linearizing) change of coordinates ϕ from a given commutator (resp. Lie symmetry), see [13, 16]. In some cases, in order to obtain the linearizing (resp. orbital linearizing) change of coordinates ϕ we use an improved version of the Darboux linearization method.

The *isochronicity problem*, i.e. to determine whether the periodic orbits around a center have the same period is equivalent to the existence of an analytic commutator and also to the existence of an analytic linearizing change of coordinates. The method applied in this work permits to construct the linearizing change of coordinates from the knowledge of a commutator, see [14].

Finally, to end the work we will show an example of linearization of system (2) in a neighborhood of a nontrivial singular point. The aim of this remark is to show that the procedure used to find the linearizing (resp. orbital linearizing) change of variables in a neighborhood of the trivial singular point can be applied in the same way to find the linearizing (resp. orbital linearizing) smooth near-identity change of variables in a neighborhood of a nontrivial singular point.

2. Some Preliminary Results

We consider the smooth differential system

$$\dot{x} = P(x, y) = \lambda x + o(x, y), \quad \dot{y} = Q(x, y) = \mu y + o(x, y) \tag{3}$$

defined around the origin. Hence, the system has a singular point with non-vanishing linear part at the origin. Let $\mathcal{X} = P(x, y)\partial_x + Q(x, y)\partial_y$ be its associated vector field.

Definition 1. The origin of (3) is linearizable if there exits an smooth change of coordinates $\phi(x, y) = (u(x, y), v(x, y)) = (x + o(x, y)), y + o(x, y))$ in the neighborhood of the origin transforming the system into the linear system

$$\dot{u} = \lambda u, \quad \dot{v} = \mu v. \tag{4}$$

Definition 2. The origin of (3) is orbitally linearizable if there exits an smooth change of coordinates $\phi(x, y) = (u(x, y), v(x, y)) = (x + o(x, y)), y + o(x, y))$ in the neighborhood of the origin transforming the system into

$$\dot{u} = \lambda u h(u, v), \quad \dot{v} = \mu v h(u, v) \tag{5}$$

where $h(u, v)$ is a smooth scalar function such that $h(0, 0) \neq 0$.

Definition 3. Let $\mathcal{X} = P(x, y)\partial_x + Q(x, y)\partial_y$ and $\mathcal{Y} = \xi(x, y)\partial_x + \eta(x, y)\partial_y$ be smooth vector fields. The Lie bracket of the smooth vector fields $\mathcal{X}$ and $\mathcal{Y}$ is $[\mathcal{X}, \mathcal{Y}] := \mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X}$. In other words we have

$$[\mathcal{X}, \mathcal{Y}] = \left(P\frac{\partial \xi}{\partial x} - \xi\frac{\partial P}{\partial x} + Q\frac{\partial \xi}{\partial y} - \eta\frac{\partial P}{\partial y}\right)\frac{\partial}{\partial x} + \left(P\frac{\partial \eta}{\partial x} - \xi\frac{\partial Q}{\partial x} + Q\frac{\partial \eta}{\partial y} - \eta\frac{\partial Q}{\partial y}\right)\frac{\partial}{\partial y}.$$

A smooth commuting vector field $\mathcal{Y}$ is a vector field such that $[\mathcal{X}, \mathcal{Y}] \equiv 0$.

Definition 4. The smooth vector field $\mathcal{Y}$ is a symmetry of smooth vector field $\mathcal{X}$ if the commutation relation

$$[\mathcal{X}, \mathcal{Y}] = \nu(x, y)\mathcal{X},$$

is satisfied for some smooth scalar function $\nu(x, y)$ such that $\nu(0, 0) = 0$.

Definition 5. The set $N(\mathcal{X})$ of Normalizers of $\mathcal{X}$ is defined as the set of all smooth vectors field $\mathcal{Y} = \xi(x, y)\partial_x + \eta(x, y)\partial_y$ of Lie symmetries of $\mathcal{X}$. In short, $N(\mathcal{X}) = \{\mathcal{Y} : [\mathcal{X}, \mathcal{Y}] = \nu(x, y)\mathcal{X}\}$.

The next proposition shows the structure of the set $N(\mathcal{X})$. It bring us the procedure to transform a symmetry into another symmetry of $\mathcal{X}$. For a proof see for instance [11].

Proposition 6. *$\mathcal{Y}, \bar{\mathcal{Y}} \in N(\mathcal{X})$ if and only if there are two C^1 scalar functions $f \not\equiv 0$ and g such that $\bar{\mathcal{Y}} = f(H)\mathcal{Y} + g\mathcal{X}$ where H is a first integral of $\mathcal{X}$. Hence, $[\mathcal{X}, \bar{\mathcal{Y}}] = \bar{\nu}(x, y)\mathcal{X}$ with $\bar{\nu}(x, y) = f(H)\nu(x, y) + \mathcal{X}g$.*

Definition 7. A C^1 function $V : \mathcal{U} \to \mathbb{R}$ such that $V \not\equiv 0$ and satisfying the linear partial differential equation

$$\mathcal{X}(V) = \text{div}\mathcal{X}V,$$

with $\text{div}\mathcal{X} = \partial P/\partial x + \partial Q/\partial y$, is called an inverse integrating factor of the vector field $\mathcal{X}$ on $\mathcal{U}$.

We recall that the rescaled vector field $\mathcal{X}/V$ is hamiltonian outside the set $\{V = 0\}$.

On the other hand, it is known that a vector field $\mathcal{X}$ which admits a symmetry $\mathcal{Y}$ has the following inverse integrating factor defined in $\mathcal{U}$

$$V(x, y) := \mathcal{X} \wedge \mathcal{Y} = P\eta - Q\xi, \tag{6}$$

provided $V(x,y) \not\equiv 0$, see [25]. Conversely given a inverse integrating factor V we can get a Lie symmetry $\mathcal{Y}$ of $\mathcal{X}$ as

$$\mathcal{Y} = \frac{1}{\operatorname{div}\mathcal{X}}\left(-\frac{\partial V}{\partial y}\partial_x + \frac{\partial V}{\partial x}\partial_y\right), \tag{7}$$

defined in $\mathcal{U}\backslash\{(x,y)\in\mathcal{U} : \operatorname{div}\mathcal{X} = 0\}$.

The following theorem is proved in [15] and it gives the equivalence between either the linearizability or orbital linearizability of a smooth vector field $\mathcal{X}$ and the fact of having a Lie symmetry of $\mathcal{X}$ of the form $\mathcal{Y} = (x + o(x,y))\partial_x + (y + o(x,y))\partial_y$.

Theorem 8. *Consider the smooth vector field $\mathcal{X}$ associated to system* (3)*. Then, $\mathcal{X}$ is linearizable* (*resp. orbitally linearizable*) *if, and only if, there exists a smooth vector field of the form $\mathcal{Y} = (x+o(x,y))\partial_x + (y+o(x,y))\partial_y$ such that $[\mathcal{X},\mathcal{Y}] \equiv 0$* (*resp. $[\mathcal{X},\mathcal{Y}] \equiv \nu(x,y)\mathcal{X}$ with $\nu(x,y)$ a smooth scalar function such that $\nu(0,0) = 0$*).

The following theorem is proved in [13] and [16]. It will allow us to construct a change of coordinates ϕ that either linearizes or orbitally linearizes system (3) from a given Lie Symmetry $\mathcal{Y}$ of the form $\mathcal{Y} = (x + \cdots)\partial_x + (y + \cdots)\partial_y$.

Theorem 9. *Let $\mathcal{X} = (\lambda x + o(x,y))\partial_x + (\mu y + o(x,y))\partial_y$ and $\mathcal{Y} = (x + o(x,y))\partial_x + (y+o(x,y))\partial_y$ be two smooth vector fields in a neighborhood $U \subset \mathbb{C}^2$ of the origin such that $[\mathcal{X},\mathcal{Y}] \equiv 0$* (*resp. $[\mathcal{X},\mathcal{Y}] \equiv \nu(x,y)\mathcal{X}$ with $\nu(x,y)$ a smooth scalar function such that $\nu(0,0) = 0$*)*. Then, a smooth near-identity change of variables $u = x + o(x,y)$, $v = y + o(x,y)$, that linearizes* (*resp. orbitally linearizable*) *$\mathcal{X}$ is obtained as follows*:

$$u = g(I)\left(\frac{f(H)}{g^{\mu}(I)}\right)^{\frac{1}{\mu-\lambda}}, \quad v = \left(\frac{f(H)}{g^{\mu}(I)}\right)^{\frac{1}{\mu-\lambda}}, \tag{8}$$

where H and I are first integrals of $\mathcal{X}$ and $\mathcal{Y}$, respectively, associated with the inverse integrating factor $\mathcal{X}\wedge\mathcal{Y}$ and f and g are two functions such that $f(H(x,y)) = (x+o(x,y))^{\mu}/(y+o(x,y))^{\lambda}$ and $g(I(x,y)) = (x + o(x,y))/(y + o(x,y))$.

The next theorem is a straightforward generalization of the version's one given in [8] about Darboux linearization. We give a previous definition to state the theorem.

Definition 10. A smooth function $F(x,y)$ satisfying $\mathcal{X}F = KF$ is called a Darboux factor and the smooth function $K(x,y)$ is called the cofactor.

Theorem 11. *System* (3) *is orbitally linearizable if there exist Darboux factors $F_i(x,y)$, for $i = 1,\ldots,m$, of system* (3) *with F_i defined in a neighborhood U of the origin and numbers $\alpha_j,\beta_j \in \mathbb{C}$, such that $F_1(x,y) = x + o(x,y)$, $F_m(x,y) = y + o(x,y)$, $F_i(0,0) \neq 0$ for $i = 2,\ldots,m-1$ in such a way that $\sum_{i=1}^{m-1}\alpha_i K_i(x,y) = \lambda h(x,y)$ and $\sum_{i=2}^{m}\beta_i K_i(x,y) = \mu h(x,y)$ where h is smooth on U and $h(0,0) \neq 0$. Here, K_i is the cofactor of F_i. Under these conditions*, *a change of variables $(x,y) \mapsto (u,v)$ that brings system* (3) *into its orbitally linearizable normal form is given by $(u,v) = (\prod_{i=1}^{m-1} F_i^{\alpha_i}, \prod_{i=2}^{m} F_i^{\beta_i})$.*

Proof. Since $\dot{u} = \mathcal{X}(u) = \sum_{i=1}^{m-1}\alpha_i\mathcal{X}(F_i)u/F_i$, taking into account that $F_i(x,y)$ are Darboux factors, it follows that $\dot{u} = u\sum_{i=1}^{m-1}\alpha_i K_i(x,y)$. Hence, $\dot{u} = \lambda u h(x,y)$. Analogously, $\dot{v} = \mu v h(x,y)$. □

3. Quadratic Lotka–Volterra Family

Theorem 12. *For* $q \in \mathbb{C}\backslash\{0,1\}$ *system* (2) *has a linearizable saddle at the origin if one of the conditions listed below are satisfied.*

(i) $c = 0$ *with* $q \neq 1/z$ *with* $z \in \mathbf{Z}$;
(ii) $b = d = 0$;
(iii) $a = c,\ b = d$;
(iv) $b = (q-1)\ a + c = 0$.

Moreover, if one of the following conditions are satisfied

(v) $qab - (q-1)\,ad - cd = 0$;
(vi) $ma + c = 0,\ m = 0, \ldots, q-2$;

the origin of (2) *is orbitally linearizable.*

Proof. We will compute either the analytic linearizing or the orbital analytic linearizing changes of variables $(x,y) \mapsto (u(x,y), v(x,y))$ in each case.

Case (i) $c = 0$. In this case system (2) has the following Darboux factors, $F_1(x,y) = x$, $F_2(x,y) = dy - q$, $F_3 = (1 - dy/q)^{b/d+1/q} + ax\ {}_2\mathcal{F}_1(-1/q, 1 - b/d - 1/q, 1 - 1/q, dy/q)$ and $F_4(x,y) = y$, with cofactors $K_1 = 1 + ax + by$, $K_2 = dy$, $K_3 = (dy + aqx + byq)/q$ and $K_4 = dy - q$, where ${}_2\mathcal{F}_1(a_1, a_2; b; x)$ is the hypergeometric function defined by

$${}_2\mathcal{F}_1(a_1, a_2; b; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{(b)_k} \frac{x^k}{k!}.$$

It is straightforward to check the existence of numbers $\alpha_1, \alpha_2, \alpha_3, \beta_2, \beta_3, \beta_4$ verifying $\alpha_1 K_1 + \alpha_2 K_2 + \alpha_3 K_3 = 1$, $\beta_2 K_2 + \beta_3 K_3 + \beta_4 K_4 = -q$, being $\alpha_1 = -\alpha_3 = 1$, $\alpha_2 = 1/q$, $\beta_4 = -\beta_2 = 1$, and $\beta_3 = 0$. Hence, from Theorem 11 we obtain the change of variables that, in this case, linearizes (2), given by $u(x,y) = x(dy-q)^{1/q}/F_3$, and $v(x,y) = y(dy-q)^{-1}$. The condition $q \neq 1/z$ with $z \in \mathbf{Z}$ guarantee the existence of ${}_2\mathcal{F}_1$.

Case (ii) $b = d = 0$. In this case, system (2) takes the form

$$\dot{x} = x(1 + ax) \quad \dot{y} = y(-q + cx). \tag{9}$$

The vector field associated to system (9) admits a quadratic polynomial commutator $\mathcal{Y}$, given by $\mathcal{Y} = x(1+ax)\partial_x + y(1+cx)\partial_y$. Taking into account that $V = \mathcal{X} \wedge \mathcal{Y}$ is an inverse integrating factor for both vector fields $\mathcal{X}$ and $\mathcal{Y}$, we can integrate them. We take the following first integrals in order to apply Theorem 9.

$$f(H) = \frac{(1+ax)^{q+c/a}\, x^{-q}}{y}, \quad g(I) = \frac{y(1+ax)^{1-c/a}}{x}.$$

Since $f(H) = u^{-q}/v$ and $g(I) = u/v$, from Eq. (8) we obtain that the change of variables which transforms system (9) into the form of system (5) is $u(x,y) = x/(1+ax)$, $v(x,y) = y/(1+ax)^{c/a}$. We notice that this change of variables also linearizes $\mathcal{Y}$.

Case (iii) $a = c$, $b = d$. This case is already solved in [13], but we include it for sake of completeness. In this case, system (2) reads for

$$\dot{x} = x(1 + cx + dy) \quad \dot{y} = y(-q + cx + dy). \tag{10}$$

The vector field associated to system (10) commute with $\mathcal{Y} = x(1 + cx - dy/q)\partial_x + y(1 + cx - dy/q)\partial_y$. Then, we obtain the inverse integrating factor $V = \mathcal{X} \wedge \mathcal{Y}$ of both vector fields $\mathcal{X}$ and $\mathcal{Y}$ that enables us to obtain the following first integrals

$$H(x, y) = xy^{\frac{1}{q}}(-dy + (1 + cx)q)^{-\frac{1+q}{q}}, \quad I(x, y) = \frac{x}{y},$$

of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Since $\mathcal{Y}$ is orbitally linearizable, then $I(x, y)$ is the trivial first integral u/v. Taking the following new first integral $f(H) = q^{-1-q}H^{-q} = u^{-q}/v$, the change of variables $(x, y) \mapsto (u(x, y), v(x, y))$ where $u(x, y) = x/(1 + cx - dy/q)$, $v(x, y) = y/(1 + cx - dy/q)$, linearizes both $\mathcal{X}$ and $\mathcal{Y}$. Notice that in the cases (ii) and (iii) for $q = -1$, $\mathcal{X} = \mathcal{Y}$ and consequently $V \equiv 0$. Hence, we cannot apply our method. However, as the change of variables in both cases also linearizes $\mathcal{Y}$, the same change linearizes $\mathcal{X}$ when $q = -1$.

Case (iv) $b = (q - 1)a + c = 0$. In this case, system (2) reads for

$$\dot{x} = x(1 + ax) \quad \dot{y} = y(-q + (1 - q)ax + dy). \tag{11}$$

The vector field $\mathcal{X}$ associated to system (11) has three algebraic invariant curves. This fact allows us to get an inverse integrating factor $V = y^2x^{q+1}$. Applying (7) we obtain from V a vector field, $\bar{\mathcal{Y}}$, such that it is a Lie symmetry of $\mathcal{X}$. Then, using the structure of Normalizers we look for a commutator $\mathcal{Y}$ of $\mathcal{X}$ with radial linear part given by $\mathcal{Y} = x(1 + cx)\partial_x + y(dy(cxq - 1) + (1 + cx)q(1 + c(x - xq)))/(q + cxq)\partial_y$. The first integrals $H(x, y)$ and $I(x, y)$ associated to V of $\mathcal{X}$ and $\mathcal{Y}$ (after deleting some logarithmic functions) are given by

$$H(x, y) = \frac{-x^{-q}(dy - (1 + cx)q)}{qy}, \quad I(x, y) = \frac{-qy(1 + cx)^{1+q}}{x(dy - (1 + cx)q)}.$$

Hence, it follows $f(H) = H(x, y) = u^{-q}/v$, $g(I) = I(x, y)^{-1} = u/v$ where

$$u(x, y) = \frac{x}{1 + cx}, \quad v(x, y) = \frac{-qy(1 + cx)^q}{(dy - (1 + cx)q)}.$$

Notice that the change of variables linearizes both, $\mathcal{X}$ and $\mathcal{Y}$.

Case (v) $qab - (q - 1)\,ad - cd = 0$. First, we consider the case $d \neq 0$, i.e., we take $c = a(qb - (q - 1)d)/d$. In this case, system (2) has three Darboux Factors, $F_1(x, y) = x$, $F_2(x, y) = -dy + q + aqx$, and $F_3(x, y) = y$. Their respective cofactors are $K_1 = 1 + ax + by$, $K_2 = ax + d$, and $K_3 = -q + cx + dy$. It is straightforward to check the existence of a numbers $\alpha_1, \alpha_2, \beta_2, \beta_3$ verifying $\alpha_1 K_1 + \alpha_2 K_2 = h$, $\beta_2 K_2 + \beta_3 K_3 = -qh$, being $\alpha_1 = \beta_1 = 1$, $\alpha_2 = -1$ $\beta_2 = -(d - dq + bq)/d$ and $h = 1 + by - dy$. Hence, from Theorem 11 we obtain the change of variables that, in this case, orbitally linearizes (2), given by

$$u(x, y) = qx(-dy + q + qax)^{-1}, \quad v(x, y) = q^{(d-dq+bq)/d}y(-dy + q + qax))^{-(d-dq+bq)/d}.$$

Let us to compute now the change of variables that orbitally linearizes (2) by using Lie Symmetries. $V = xy(-dy + q + qax)$ is an inverse integrating factor for system (2).

Then, proceeding as in the case (iv), we obtain a Lie Symmetry $\mathcal{Y}$, with radial linear part $\mathcal{Y} = \mathcal{K}dx(2dy(1+q)+q(1+2by-q+a(1+q)x))\partial_x+\mathcal{K}y(d^2y(q-1)+d(1+4ax-q)q+2abq^2x)\partial_y$, with $\mathcal{K} = 1/[q(2d^2y + d(1 + by - ax(q - 3) - q) + abqx)]$. Integrating both vector fields, $\mathcal{X}$ and $\mathcal{Y}$, we get

$$f(H) = \frac{(dy - (1 + ax)q)^{(d+bq)/d}}{x^q y}, \quad g(I) = \frac{x(dy - (1 + ax)q)^{(-3d-2bq-dq)/(d(q-1))}}{y}.$$

From Eq. (8) the change of variables that orbitally linearizes (2) and linearizes the Lie symmetry $\mathcal{Y}$, reads for

$$\bar{u}(x, y) = x(dy - (1 + ax)q)^{\frac{2d+bq}{d(1-q)}}, \quad \bar{v}(x, y) = y(dy - (1 + ax)q)^{\frac{d+(b+1)q}{d(q-1)}}.$$

We now consider the case when $d = 0$. Since the other cases are already solved, if $d = 0$ it follows $a = 0$. We shall use the same procedure than in the case $d \neq 0$. In this case system (2) has three Darboux factors, $F_1(x, y) = x$, $F_2(x, y) = \exp(-cx + by)$, and $F_3(x, y) = y$. Their respective cofactors are $K_1 = 1 + by$, $K_2 = -(cx + byq)$, and $K_3 = -q + cx$. There exist numbers $\alpha_1, \alpha_2, \beta_2, \beta_3$ verifying $\alpha_1 K_1 + \alpha_2 K_2 = h$, $\beta_2 K_2 + \beta_3 K_3 = -qh$. It is easy to see that $\alpha_1 = \alpha_2 = \beta_1 = 1$, $\beta_2 = 1 - \lambda$ and $h = 1 - cx + by - bqy$. Hence, applying Theorem 11 we obtain the change of variables that orbitally linearizes (2),

$$u(x, y) = x \exp(-cx + by), \quad v(x, y) = y \exp((1 - q)(-cx + by)).$$

In the following computations we obtain the change from a given symmetry $\mathcal{Y}$. $V = xy$ is an inverse integrating factor for system (2). Then, as in the former cases, we obtain a Lie symmetry $\mathcal{Y}$, with radial linear part

$$\mathcal{Y} = \frac{x(1 + 2cy - q)}{1 + cx + by - q}\partial_x + \frac{y(1 + 2bx - q)}{1 + cx + dy - q}\partial_y.$$

Integrating $\mathcal{X}$ and $\mathcal{Y}$ we obtain

$$f(H) = \frac{\exp(cx - by)}{x^q y}, \quad g(I) = \frac{x \exp\left(\frac{2(cx-by)}{1-q}\right)}{y}.$$

From Theorem 9 we obtain the change of variables that orbitally linearizes (2) and also linearizes the Lie symmetry $\mathcal{Y}$, that is

$$u(x, y) = x \exp\left(\frac{by - cx}{q - 1}\right), \quad v(x, y) = y \exp\left(\frac{by - cx}{1 - q}\right).$$

Case (vi) $ma + c = 0$ with $m = 0, \ldots, q - 2$. In this case system (2) has the following Darboux factors, $F_1(x, y) = x$, and $F_3(x, y) = y$, with cofactors $K_1 = (m - cx + bmy)/m$, and $K_3 = cx + dy - q$. There exists another Darboux factor $F_2(x, y)$ satisfying $qK_1 + K_2 + K_3 = 0$. This fact enables us to compute the cofactor of $F_2(x, y)$ without the explicit knowledge of it. Thus, solving the former expression for K_2 we obtain $K_2 = (cx(q - m) - ym(d + bq))/m$. Applying Theorem 11 we get the numbers $\alpha_1 = 1, \alpha_2 = 1/(q - m), \beta_2 = m/(m - q), \beta_3 = 1$. Hence, the change of variables that orbitally linearizes system (2) is $u(x, y) = xF_2(x, y)^{1/(q-m)}$, $v(x, y) = yF_2(x, y)^{m/(m-q)}$, where $F_2(0, 0) \neq 0$, see [12]. □

In fact, the necessary and sufficient conditions (cases (v) and (vi)) for analytic integrability, i.e., systems with an analytic first integral defined in a neighborhood of the origin, were given in [12]. The case (vi) is also linearizable for $q \in \mathbb{N}\backslash\{1\}$, see [8].

Remark. We want to notice that the procedure used in this work to find the linearizing or orbital linearizing smooth near-identity change of variables in a neighborhood of the trivial singular point can also be used to find such a change of variables in a neighborhood of a nontrivial singular point. We will show this fact with an illustrative example. Let us consider system (2) with $b = c = 0$, that is

$$\dot{x} = x(1 + ax), \quad \dot{y} = y(-q + dy). \tag{12}$$

The point $(1/a, q/d)$ with a and d not equal to zero is a nontrivial singular point for system (12). Translating such a point to the origin, system (12) reads for

$$\dot{u} = u(-1 + au), \quad \dot{v} = v(q + dv). \tag{13}$$

The vector field $\mathcal{Y} = -u(-1 + au)\partial_x + (v(q + dv)/q)\partial_y$ is a commutator for the vector field $\mathcal{X}$ associated to system (12). After compute an inverse integrating factor, we integrate both vector fields, $\mathcal{X}$ and $\mathcal{Y}$, getting

$$f(H) = \frac{u^q v}{(1 - au)^q (dv + q)}, \quad g(I) = \frac{u(dv + q)}{v(1 - au)}.$$

From Eq. (8) the change of variables that linearizes (13) and the commutator $\mathcal{Y}$ reads for

$$\bar{u}(x, y) = \frac{u}{1 - au}, \quad \bar{v}(x, y) = \frac{v}{dv + q}.$$

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