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A NOTE ON TRAVELING WAVE SOLUTIONS TO THE TWO COMPONENT CAMASSA–HOLM EQUATION

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In this paper we show that non-smooth functions which are distributional traveling wave solutions to the two component Camassa–Holm equation are distributional traveling wave solutions to the Camassa–Holm equation provided that the set $u^{-1}(c)$, where c is the speed of the wave, is of measure zero. In particular there are no new peakon or cuspon solutions beyond those already satisfying the Camassa–Holm equation. However, the two component Camassa–Holm equation has distinct from Camassa–Holm equation smooth traveling wave solutions as well as new distributional solutions when the measure of $u^{-1}(c)$ is not zero. We provide examples of such solutions.

Keywords: Camassa–Holm equation; traveling waves; peakons.

Mathematics Subject Classification 2000: 35Q35, 35Q53

The Camassa–Holm equation [1]

$$u_t + \kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad (1)$$

arises as a model for the unidirectional propagation of shallow water waves over a flat bottom, $u(x, t)$ representing the water's free surface, and $\kappa \in \mathbb{R}$ being a parameter related to the critical shallow water speed (see [13] and [12] for the derivation of this equation). Camassa and Holm [1] discovered that the equation has non-smooth solitary waves that retain their individual characteristics through the interaction and eventually emerge with their original shapes and speeds. The traveling wave solutions of the Camassa–Holm equation have been classified by Lenells [14]. The peaked traveling waves appear to have the characteristic of the traveling waves of greatest height similar to exact traveling wave solutions of the governing equations for water waves with a peak at their crest (see [17, 3, 4]). However, simpler approximate shallow water models like KdV do not admit traveling wave solutions with this feature (see [18]). Also, it should be noted that the Camassa–Holm peaked traveling waves are stable wave forms (see [5, 6, 15]). Therefore, they are physically recognizable. An alternative, and useful for generalizations form of this equation is

$$m_t + um_x + 2mu_x = 0, \quad (2)$$

where $m = u - u_{xx} + \frac{1}{2}\kappa$.

One such a generalization has been introduced by Chen, Liu and Zhang [2]:

$$\begin{cases} m_t + um_x + 2mu_x - \rho\rho_x = 0, \\ \rho_t + (\rho u)_x = 0. \end{cases} \quad (3)$$

This system is integrable in the sense that it has a bi-Hamiltonian structure with an associated Lax pair (see [16]). Recall that the Camassa–Holm equation is completely integrable as an infinite dimensional Hamiltonian system (see [7] and [8] for the characterization of the isospectral problem and for the direct scattering approach, see [9] for the inverse scattering approach). Also, see [10] for the discussion of well-posedness and blow-up phenomena for the two component Camassa–Holm equation. The traveling wave solutions are obtained by setting $u = u(x - ct)$ and $\rho = \rho(x - ct)$. In this case, easy manipulations show that Eq. (3) can be written as follows

$$\begin{cases} -2c(u' - u''') + 2\kappa u' + 3(u^2)' + ((u')^2)' - (u^2)''' = (\rho^2)', \\ -c\rho' + (\rho u)' = 0. \end{cases} \quad (4)$$

These equations are valid in the sense of distributions, if $u \in H_{loc}^1(\mathbb{R})$ and $\rho \in L_{loc}^2(\mathbb{R})$. Indeed, for a given function ρ , if $(\rho^2)' \in \mathcal{D}'(\mathbb{R})$, then $\rho \in L_{loc}^2(\mathbb{R})$.

Since every distribution has a primitive which is a distribution (see [11]), we can integrate and then rewrite

$$\begin{cases} (v^2)'' = (v')^2 + p(v) - \rho^2, \\ \rho v = B_1, \end{cases} \quad (5)$$

where $v = u - c$ and $p(v) = 3v^2 + (2\kappa + 4c)v + K$ for some constants K and B_1 .

Definition 1. A pair of functions (u, ρ) where $u \in H_{loc}^1(\mathbb{R})$ and $\rho \in L_{loc}^2(\mathbb{R})$, is called a traveling wave solution of Eq. (3) if u and ρ satisfy Eq. (5) in the sense of distributions.

The following Lemma is due to Lenells [14].

Lemma 1. *Let $p(v)$ be a polynomial with real coefficient. Assume that $v \in H_{loc}^1(\mathbb{R})$ satisfies*

$$(v^2)'' = (v')^2 + p(v) \quad \text{in } \mathcal{D}'(\mathbb{R}). \quad (6)$$

Then

$$v^k \in C^j(\mathbb{R}) \quad \text{for } k \geq 2^j. \quad (7)$$

In our case, we have the following generalization:

Lemma 2. *Let $p(v)$ be a polynomial with real coefficients. Assume that $v \in H_{loc}^1(\mathbb{R})$ and $\rho \in L_{loc}^2(\mathbb{R})$ satisfy the following system in $\mathcal{D}'(\mathbb{R})$:*

$$\begin{cases} (v^2)'' = (v')^2 + p(v) - \rho^2, \\ \rho v = B_1. \end{cases} \quad (8)$$

Then

$$v^k \in C^j(\mathbb{R}) \quad \text{for } k \geq 2^j \quad \text{and} \quad j \geq 0. \quad (9)$$

Proof. Since $v \in H_{loc}^1(\mathbb{R})$ and $\rho \in L_{loc}^2(\mathbb{R})$, (8) implies that $(v^2)'' \in L_{loc}^1(\mathbb{R})$. Therefore, $(v^2)'$ is absolutely continuous and $v^2 \in C^1(\mathbb{R})$. Also, since $v \in H_{loc}^1(\mathbb{R})$, then v is absolutely continuous and we can claim

$$(v^k)' = \frac{k}{2}(v^{k-2}(v^2)') \quad \text{for } k \geq 3.$$

To see why the claim is true, we first note that in fact, it is obviously true if k is an even number. Also, note that since the first derivative of an absolutely continuous function exists almost everywhere, in

taking the first derivative of the product of two absolutely continuous functions we can use the Leibniz Rule almost everywhere. Now, if k is an odd number, let's say $k = 2n + 1$, then we can write

$$\begin{aligned} (v^k)' &= (v^{2n}v)' = v(v^{2n})' + v'v^{2n} \\ &= v(nv^{2(n-1)})(v^2)' + \frac{1}{2}(v^2)'v^{2n-1} \\ &= \frac{k}{2}v^{k-2}(v^2)'. \end{aligned}$$

Thus, we have

$$\begin{aligned} (v^k)'' &= \frac{k}{2}(v^{k-2}(v^2)')' \\ &= \frac{k}{2}((v^{k-2})'(v^2)' + v^{k-2}(v^2)'') \\ &= k(k-2)v^{k-2}(v')^2 + \frac{k}{2}v^{k-2}(v^2)'' \quad \text{for } k \geq 3. \end{aligned}$$

Substituting from Eq. (8) we have

$$\begin{aligned} (v^k)'' &= k(k-2)v^{k-2}(v')^2 + \frac{k}{2}v^{k-2}((v')^2 + p(v) - \rho^2) \\ &= k\left(k - \frac{3}{2}\right)v^{k-2}(v')^2 + \frac{k}{2}v^{k-2}p(v) - \frac{k}{2}B_1v^{k-3}\rho. \end{aligned} \tag{10}$$

For $k \geq 3$ the right-hand side of the above equation belongs to $L^1_{loc}(\mathbb{R})$. Therefore

$$v^k \in C^1(\mathbb{R}) \quad \text{for } k \geq 2. \tag{11}$$

Thus, the assertion holds for $j = 1$. We proceed by induction on j . Suppose

$$v^k \in C^{j-1}(\mathbb{R}) \quad \text{for } k \geq 2^{j-1} \text{ and } j \geq 2.$$

Then for $k \geq 2^j$ we have

$$\begin{aligned} v^{k-2}(v')^2 &= \frac{1}{2^{j-1}}(2^{j-1}v^{2^{j-1}-1}v')\frac{1}{k-2^{j-1}}((k-2^{j-1})v^{k-2^{j-1}-1}v') \\ &= \frac{1}{2^{j-1}(k-2^{j-1})}(v^{2^{j-1}})'(v^{k-2^{j-1}})' \in C^{j-2}(\mathbb{R}). \end{aligned} \tag{12}$$

Also, we have $v^{k-2}p(v) \in C^{j-1}(\mathbb{R})$ and $v^{k-3}\rho = B_1v^{k-4} \in C^{j-2}(\mathbb{R})$. Therefore the right-hand side of Eq. (10) belongs to $C^{j-2}(\mathbb{R})$. Hence,

$$v^k \in C^j(\mathbb{R}) \quad \text{for } k \geq 2^j. \quad \square$$

Remark. Lemma 2 implies that v' is possibly discontinuous only at points where $v = 0$. In fact, a much stronger result is true:

Corollary 1. *If $v \in H^1_{loc}(\mathbb{R})$ and $\rho \in L^2_{loc}(\mathbb{R})$ satisfy (8) in $\mathcal{D}'(\mathbb{R})$, then*

$$v \in C^\infty(\mathbb{R} \setminus v^{-1}(0))$$

and

$$\rho \in C^\infty(\mathbb{R} \setminus v^{-1}(0)).$$

Proof. Suppose $k \geq 2$. Then $v^k \in C^1(\mathbb{R})$. Therefore

$$kv^{k-1}v' = (v^k)' \in C^1(\mathbb{R}).$$

This implies that $v' \in C(\mathbb{R} \setminus v^{-1}(0))$. Thus, $v \in C^1(\mathbb{R} \setminus v^{-1}(0))$.

Now, assume that $v \in C^j(\mathbb{R} \setminus v^{-1}(0))$ for $j \geq 1$. For $k \geq 2^{j+1}$, we have $v^k \in C^{j+1}(\mathbb{R})$. Therefore

$$kv^{k-1}v' = (v^k)' \in C^j(\mathbb{R}).$$

This shows that $v' \in C^j(\mathbb{R} \setminus v^{-1}(0))$. Hence, $v \in C^{j+1}(\mathbb{R} \setminus v^{-1}(0))$. Thus, u is in the desired space. Now the statement for ρ follows from the second equation of (5). \square

Remark. Since $v = u - c$, Corollary (1) shows that $u \in C^\infty(\mathbb{R} \setminus u^{-1}(c))$.

Since $\mathbb{R} \setminus u^{-1}(c)$ is an open set, we have

$$\mathbb{R} \setminus u^{-1}(c) = \bigcup_{i=1}^{\infty} (a_i, b_i).$$

So, u is smooth in every interval (a_i, b_i) where the following lemma holds (below $(a_i, b_i) = (a, b)$):

Lemma 3. *Let (u, ρ) be a traveling wave solution to Eq. (3). Suppose u is smooth in the interval (a, b) . Then in the interval (a, b) , u satisfies the following equation:*

$$(u - c)^2 u'^2 = P(u), \quad (13)$$

where

$$P(u) = (u^2 + \kappa u + A)(u - c)^2 + C(u - c) + B, \quad (14)$$

and A, B and C are some constants.

Proof. Since both u and ρ are smooth in (a, b) we use standard calculus rules. By the first equation of (5), we have

$$2(v')^2 + 2vv'' = (v')^2 + p(v) - \rho^2.$$

Therefore,

$$(v')^2 + 2vv'' = p(v) - \rho^2.$$

Multiplying by v' we have

$$(v')^3 + v((v')^2)' = v'p(v) - v'\rho^2.$$

Thus,

$$(v(v')^2)' = v'p(v) - v'\rho^2.$$

Hence,

$$(v(v')^2)' = (3v^2 + (2\kappa + 4c)v + K)v' - \frac{Bv'}{v^2},$$

where $B = B_1^2$.

Integration yields

$$v(v')^2 = v^3 + (\kappa + 2c)v^2 + Kv + \frac{B}{v} + C.$$

Now, multiplying this equation by v we get

$$v^2(v')^2 = (v^2 + (\kappa + 2c)v + K)v^2 + Cv + B.$$

Substituting $v = u - c$ and simplifying, we have

$$(u - c)^2(u')^2 = (u^2 + \kappa u + A)(u - c)^2 + C(u - c) + B,$$

for some constant A . \square

Theorem 1. *Suppose (u, ρ) is a non-smooth traveling wave solution to Eq. (3). If $u^{-1}(c)$ is a set of measure zero, then u is a solution to the Camassa–Holm equation.*

Proof. Suppose $\xi \in \mathbb{R} \setminus u^{-1}(c)$. Since, $u^{-1}(c) \neq \emptyset$, there exists an $\eta \in u^{-1}(c)$ such that either $\xi > \eta$ or $\xi < \eta$. Without loss of generality, assume that $\xi < \eta$. Let $\eta_0 = \inf\{\eta \in u^{-1}(c) : \eta > \xi\}$. Since $u^{-1}(c)$ is a closed set, $\eta_0 \in u^{-1}(c)$. So, $(\xi, \eta_0) \subseteq \mathbb{R} \setminus u^{-1}(c)$. Thus, we have proved that there exists an $\eta \in u^{-1}(c)$ such that either $(\xi, \eta) \subseteq \mathbb{R} \setminus u^{-1}(c)$ or $(\eta, \xi) \subseteq \mathbb{R} \setminus u^{-1}(c)$. Now, consider Eq. (13) and set $F(u) = \frac{P(u)}{(u-c)^2}$. We claim that B in Eq. (13) equals 0. Suppose $B \neq 0$. Since $B = B_1^2$, we have $B > 0$. Then Eq. (14) implies that

$$\frac{1}{\sqrt{F(u)}} = \frac{1}{\sqrt{B}}|u - c| + \mathcal{O}((u - c)^2) \quad u \rightarrow c.$$

On the other hand, we have

$$\frac{d\xi}{du} = \pm \frac{1}{\sqrt{F(u)}}.$$

Since $u \in C(\mathbb{R})$, for ξ close enough to η , integration yields

$$|\xi - \eta| = \frac{1}{2\sqrt{B}}(u - c)^2 + \mathcal{O}((u - c)^3) \quad u \rightarrow c. \quad (15)$$

Therefore,

$$|\xi - \eta| = \frac{1}{2\sqrt{B}}(u - c)^2(1 + \mathcal{O}(u - c)) \quad u \rightarrow c.$$

So,

$$|\xi - \eta|^{\frac{1}{2}} = \frac{1}{\sqrt{2\sqrt{B}}}|u - c|\sqrt{1 + \mathcal{O}(u - c)} \quad u \rightarrow c.$$

Thus,

$$|\xi - \eta|^{\frac{1}{2}} = \frac{1}{\sqrt{2\sqrt{B}}}|u - c|(1 + \mathcal{O}(u - c)) \quad u \rightarrow c.$$

Hence,

$$|\xi - \eta|^{\frac{1}{2}} = \frac{1}{\sqrt{2\sqrt{B}}}|u - c| + \mathcal{O}((u - c)^2) \quad u \rightarrow c.$$

This implies that

$$(u - c) = \mathcal{O}((\xi - \eta)^{\frac{1}{2}}) \quad \xi \rightarrow \eta.$$

Therefore,

$$(u - c)^2 = \mathcal{O}(\xi - \eta) \quad \xi \rightarrow \eta.$$

Thus, we have

$$|u - c| = \sqrt{2\sqrt{B}}|\xi - \eta|^{\frac{1}{2}} + \mathcal{O}(\xi - \eta) \quad \xi \rightarrow \eta. \quad (16)$$

Hence,

$$\begin{aligned} |\xi - \eta|^{-\frac{1}{2}} - \sqrt{2\sqrt{B}}|u - c|^{-1} &= \mathcal{O}\left(\frac{|\xi - \eta|^{\frac{1}{2}}}{u - c}\right) \\ &= \mathcal{O}(1) \quad \xi \rightarrow \eta. \end{aligned}$$

So,

$$|u - c|^{-1} = \frac{1}{\sqrt{2\sqrt{B}}}|\xi - \eta|^{-\frac{1}{2}} + \mathcal{O}(1) \quad \xi \rightarrow \eta. \quad (17)$$

On the other hand, from Eq. (14) we have

$$|u'| = \sqrt{B}(u - c)^{-1} + \mathcal{O}(1) \quad \xi \rightarrow \eta. \quad (18)$$

Now combining Eqs. (18) and (17), we have

$$|u'| = \frac{\sqrt[4]{B}}{\sqrt{2}}|\xi - \eta|^{-\frac{1}{2}} + \mathcal{O}(1) \quad \xi \rightarrow \eta. \quad (19)$$

Hence, $u' \notin L^2_{loc}(\mathbb{R})$. This contradiction shows that $B = 0$. Therefore, the second equation of (5) implies that $\rho = 0$ almost everywhere. \square

Now, we provide an example of a smooth solution of Eq. (3) that is not a solution of Camassa–Holm equation.

Example 1. Let $P(u)$ be as in the previous Theorem. Observe that $P(u) = (u - G)^2(u - L)^2$ if and only if

$$\begin{cases} \kappa = 2(c - (L + G)), \\ A = 2c\kappa - c^2 + (L + G)^2 + 2LG, \\ C = 2cA - \kappa c^2 - 2LG(L + G), \\ B = Cc - Ac^2 + L^2G^2. \end{cases} \quad (20)$$

Suppose $|u| < 1$ and $c > 1$. Therefore, if $G = -1$ and $L = 1$, integration yields

$$(1 - u)^{1-c}(1 + u)^{1+c} = e^{2(\xi - \xi_0)}. \quad (21)$$

Let us say $c = 2$ and $\xi_0 = 0$. We observe that the equation

$$\frac{(1 + u)^3}{1 - u} = e^{2\xi},$$

provides a smooth solution of Eq. (3) which is not a solution of Camassa–Holm equation. See Fig. 1.

The following Lemma provides necessary and sufficient conditions for a piecewise smooth function to be a distributional solution to Eq. (3).

Lemma 4. *Suppose u is a piecewise smooth function. The pair (u, ρ) is a distributional solution to Eq. (3) in the sense of definition 1 if and only if all of the following conditions hold:*

1. $u \in H^1_{loc}(\mathbb{R})$ and $\rho \in L^2_{loc}(\mathbb{R})$.
2. $(u - c)^2 \in W^{2,1}_{loc}(\mathbb{R})$.
3. u and ρ satisfy Eq. (5) locally with the same constant K on every interval where u is smooth.

Proof. The part (\Rightarrow) is easy. For the converse (\Leftarrow) , we note that since $(u - c)^2 \in W^{2,1}_{loc}(\mathbb{R})$, then $((u - c)^2)'$ is absolutely continuous and has no jumps. Therefore, $((u - c)^2)''$ defines a regular distribution [11]. Thus, every term in Eq. (5) can be represented by an integral that defines a distribution on the space of test functions and we are allowed to write each integral as a finite sum of integrals over local intervals and use condition 3 to prove that u and ρ satisfy Eq. (5) in the sense of distributions. \square

Remark. We note that if the measure of $u^{-1}(c)$ is not zero, then Eq. (5) implies that $\rho^2 = K$ on $u^{-1}(c)$. However in the Camassa–Holm equation if the measure of $u^{-1}(c)$ is not zero, then $K = 0$

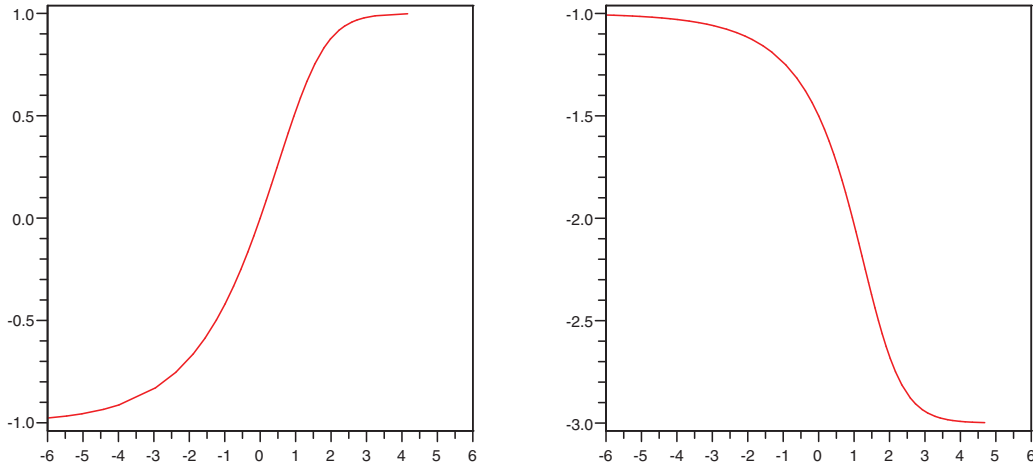


Fig. 1. (u on the left and ρ on the right) A smooth solution of Eq. (3) which is not a solution of Camassa–Holm equation.

because $\rho = 0$. This implies that solutions of the form given in the following example cannot arise from the Camassa–Holm equation.

Example 2. Set $\kappa = 0$. The pair of functions (u, ρ) given by

$$u(x) = \begin{cases} ce^{1-|x|} & \text{if } |x| > 1, \\ c & \text{if } |x| < 1, \end{cases}$$

$$\rho(x) = \begin{cases} c & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1, \end{cases}$$

is a solution to Eq. (3) but u is not a solution of Camassa–Holm equation. To see this, observe that the left-hand side derivative of u at -1 and the right-hand side derivative of u at 1 are nonzero and finite in contrast with the Camassa–Holm equation for which Lenells [14] showed that if the measure of $u^{-1}(c)$ is not zero, then these limits cannot be finite. See Fig. 2.

Definition 2. Suppose f is a continuous function on \mathbb{R} .

1. We say f has a peak at x if f is smooth locally on both sides of x and

$$0 \neq \lim_{y \downarrow x} f'(y) = - \lim_{y \uparrow x} f'(y) \neq \pm\infty.$$

Traveling wave solutions of Eq. (3) with peaks are called peakons.

2. We say f has a cusp at x if f is smooth locally on both sides of x and

$$\lim_{y \downarrow x} f'(y) = - \lim_{y \uparrow x} f'(y) = \pm\infty.$$

Traveling wave solutions of Eq. (3) with cusps are called cuspons.

3. We say that f has a stump if there is an interval $[a, b]$ on which f is a constant and f is smooth locally to the left of a and to the right of b and

$$0 \neq \lim_{x \uparrow a} f'(x) = - \lim_{x \downarrow b} f'(x).$$

Traveling wave solutions of Eq. (3) with stumps are called stumpons. Note that, in the definition of a stump the limits can be either finite or infinite.

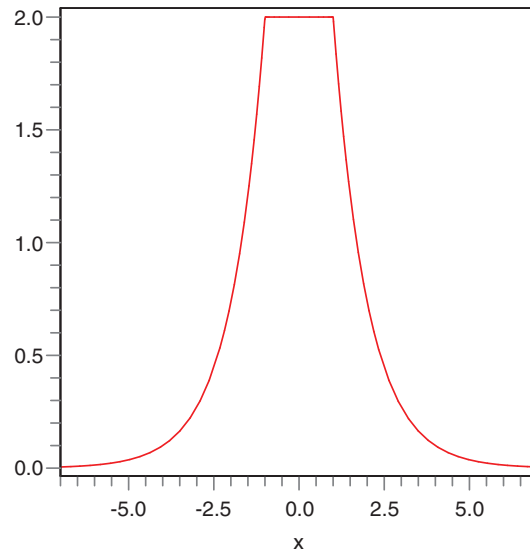


Fig. 2. $u(x)$ is a solution of Eq. (3) but it is not a solution of Camassa–Holm equation.

Theorem 1 limits the existence of new distributional peakon or cuspon solutions to Eq. (3). However, the stumpon solutions exist according to the example above.

Corollary 2. *Every peakon or cuspon traveling wave solution to Eq. (3) is a traveling wave solution to the Camassa–Holm equation.*

Finally we would like to comment on the peaked solution reported in [2]. For reasons explained below, that solution is not a distributional solution. First, we note that by Corollary (1) the non-smooth points of a distributional solution u can only appear when $u = c$. Also, Lemma 2 shows that if (u, ρ) is a traveling wave solution to Eq. (3), then $(u - c)^2 \in C^1(\mathbb{R})$. Now, consider the peaked

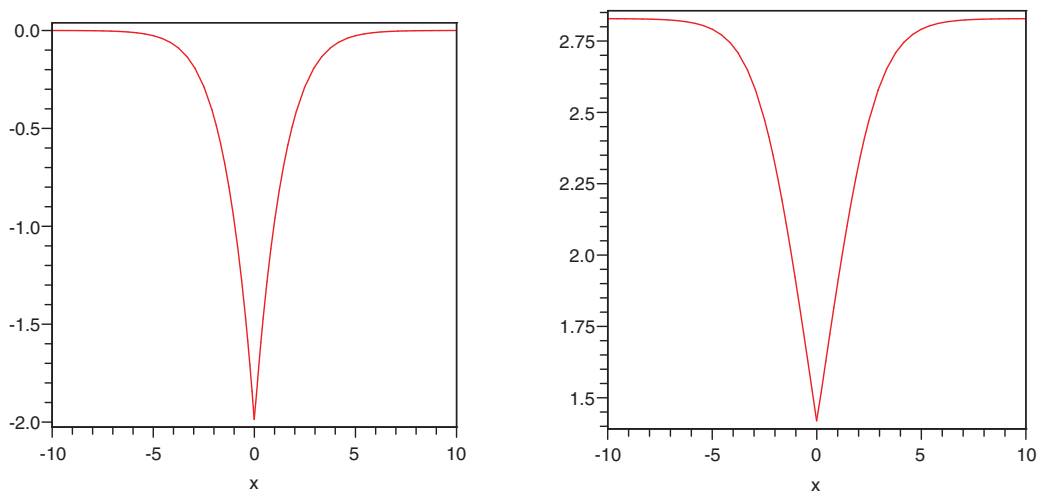


Fig. 3. (u on the left and ρ on the right) This pair is not a distributional traveling wave solution of Eq. (3). $c = 2$ and $K_1 = -1$.

function (see [2])

$$u = \chi + \sqrt{\chi^2 - c^2}, \quad \rho = \sqrt{-cK_1} \left(1 + \sqrt{\frac{\chi + c}{\chi - c}} \right), \quad \chi = -(c + K_1) \cosh(x - ct) + K_1,$$

where $K_1 = -\frac{1}{4}\kappa$, $K_1 < 0$ and $c > |K_1| > 0$. Away from its non-smooth point, u is a solution to Eq. (3). However, it is clear that u is not smooth at $\xi = 0$ and $u(0) = -c$. Furthermore, $(u - c)^2 \notin C^1(\mathbb{R})$ because

$$\lim_{\xi \downarrow 0} ((u - c)^2)' - \lim_{\xi \uparrow 0} ((u - c)^2)' = -8c\sqrt{c(c + K_1)},$$

where $\xi = x - ct$. Therefore, u is not a distributional solution to Eq. (3) even though it superficially looks like a peakon solution (see Fig. 3).

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