# An Approximation of the Kinetic Energy of a Superfluid Film on a Riemann Surface

Chris Petersen Black

# AN APPROXIMATION OF THE KINETIC ENERGY OF A SUPERFLUID FILM ON A RIEMANN SURFACE

CHRIS PETERSEN BLACK

*Department of Mathematics*
*Central Washington University*
*Ellensburg, WA USA 98926*
*blackc@cwu.edu*

The flow of a superfluid film adsorbed on a porous medium can be modeled by a meromorphic differential on a Riemann surface of high genus. In this paper, we define the mixed Hodge metric of meromorphic differentials on a Riemann surface and justify using this metric to approximate the kinetic energy of a superfluid film flowing on a porous surface.

*Keywords*: Superfluid helium; helium-4; kinetic energy; Riemann surface; mixed Hodge metric.

## 1. Introduction and Background Information

A film of liquid helium-4 at temperatures near abolute zero behaves as a *superfluid*, meaning that it flows as a virtually frictionless two-dimensional film adsorbed on a surface. Riemann, not knowing that such a film could actually exist, referred to this phenomenon as an "ideal fluid". We refer to the flow of a superfluid helium film as a *superflow*. We have shown in previous work [2] that a superflow is determined by the positions of its vortices only up to the addition of a holomorphic form on the Riemann surface $\mathcal{M}$. Due to the principle of conservation of energy, any physical fluid flow with a fixed set of vortices must have the minimal kinetic energy of all other flows with that particular vortex configuration. If the superflows that exist physically exhibit strings of circulation, then it must be that these flows have minimal kinetic energy. Thus, the primary objective of this paper is to discuss a method of approximating the kinetic energy of a superfluid film on a Riemann surface $\mathcal{M}$ of genus $g$.

The remainder of this section provides some background information about superfluid helium films and some basic mathematical terms we will be using to approximate the kinetic energy of such a film. In Sec. 2 we will use the Mixed Hodge metric to approximate that energy, and we will devote Sec. 3 to justifying this usage.

### 1.1. *Some background information about superfluidity*

Liquid $^4$He undergoes a dramatic change at temperatures near $T_\lambda = 2.17\,\mathrm{K}$. Above this temperature, $^4$He is an ordinary low-viscosity liquid known as helium I, but at temperatures below $T_\lambda$, it becomes a remarkable fluid known as helium II. According to the *two-fluid theory* proposed by Tisza [13] in 1940, helium II is composed of two separate fluids. The *normal component* has normal viscosity, but

the *superfluid component* has very low viscosity and flows nearly frictionlessly and with high velocity through very narrow channels, leaving the normal component behind. In 1946, Andronikashvili demonstrated that as the temperature of liquid $^4$He is lowered from $T_\lambda$ to temperatures near absolute zero, the normal component of the fluid essentially disappears [1]. In this paper, we will consider the case when the fluid is entirely composed of helium-II, which is an idealization of the situation at temperatures near absolute zero.

The *Feynman–Osanger relation* [4, 11] is a consequence of quantum physics, which dictates that the total circulation of a superfluid along any closed loop is an integer multiple of the fundamental quantum of circulation $\frac{h}{m}$, where $h = 2\pi\hbar$ is Planck's constant and $m$ is the mass of one atom of $^4$He. This is one of the fundamental properties of superfluid flow, and it is an integral part of the discussion to follow.

Another fundamental property of helium II is that vortices appear in the fluid in pairs of equal and opposite polarity, called *vortex-antivortex pairs* [6]. Vortex-antivortex pairs are linked by *strings of circulation*, fixed paths between the vortices along which the circulation around any simple loop that intersects this path transversally remains a constant integer multiple of the fundamental quantum of circulation [8, 9]. For a more thorough presentation of the properties of superfluidity relevant to the upcoming discussion, see [2].

## 1.2. *Mathematical background*

The basic mathematical tools we will need are provided by Riemann surface theory. For further explanation of any of these terms, please refer to Nakahara [10] or Farkas and Kra [3]. Some of the definitions are provided here for convenience.

### 1.2.1. *Intersection numbers and homology bases*

**Definition 1.1.** Let $\alpha$ and $\beta$ be two smooth oriented paths on a Riemann surface $\mathcal{M}$ which intersect transversely. Then, the *intersection number* is defined by

$$\#(\alpha, \beta) = \sum_{p \in \alpha \cap \beta} \epsilon(p),$$

where

$$\epsilon(p) = \begin{cases} +1 & \text{if } \alpha, \beta \text{ is positively oriented} \\ -1 & \text{if } \alpha, \beta \text{ is negatively oriented} \end{cases}$$

as shown in Fig. 1.

In order to discuss a generic path on a Riemann surface, we need to be able to break a path down into its fundamental parts. The basic building blocks of a path on a Riemann surface comprise a set of two types of loops: those that encircle a hole in the surface, akin to the hole in a doughnut
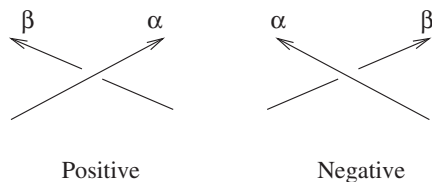


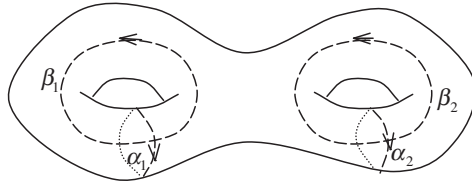Fig. 1. Positive and negative intersections.

Fig. 2. Two distinctly different types of paths on a genus 2 surface.

and denoted by $\beta$, and those that wrap around the surface through one of these holes, denoted by $\alpha$. These two types of simple paths are illustrated in Fig. 2.

Two paths are called *homotopic* if one can be smoothly deformed into the other. Informally, this means that one of the paths can be moved along the surface to coincide with the other, without cutting either path or jumping over a hole. Intersection numbers are invariant under homotopy and as a result, if two paths are homotopic we consider them to be essentially the same path. The $\alpha$ and $\beta$ loops described above form a set of building blocks of all possible paths on the surface, which we call a *canonical homology basis*, and we formally define in Definition 1.2. The four paths $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ shown in Fig. 2 comprise a canonical homology basis on a surface of genus 2. More specifically, we need the $\alpha$ loops to never intersect each other and the $\beta$ loops to never intersect each other, so that each $\alpha$ loop intersects only its corresponding $\beta$ loop exactly once. We keep track of these intersections using the intersection numbers, so we need to specify the orientation of the $\alpha$ and $\beta$ loops.

**Definition 1.2.** A *canonical homology basis* on a Riemann surface $\mathcal{M}$ of genus $g$ is a set of oriented loops $\{\alpha_i, \beta_i\}_{i=1}^g$ that satisfy

$$\#(\alpha_i, \beta_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\#(\alpha_i, \alpha_j) = 0$$

$$\#(\beta_i, \beta_j) = 0.$$

We are interested in studying fluid flows on a punctured Riemann surface. The punctures on the surface differ from the pre-existing holes, and introduce a new type of path on the surface; those that encircle the puncture. For the purposes of this paper, we are only interested in the case where the punctures come in pairs, corresponding to the vortex-antivortex pairs on the surface. The set of all three types of loops on the surface forms a canonical homology basis on the punctured surface, as defined in Definition 1.3 below.

**Definition 1.3.** Let $\mathcal{M}$ be a Riemann surface with paired points $\{(p_j, q_j)\}_{j=1}^r$, let $\gamma_{p_j}$ be a small loop around $p_j$, and let $\gamma_{q_j}$ be a small loop around $q_j$ oriented so that for any path $l$ from $q_j$ to $p_j$,

$$\#(\gamma_{p_j}, l) = 1 \quad \#(\gamma_{q_j}, l) = -1.$$

Then, the set

$$\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_{p_1}, \ldots, \gamma_{p_r}, \gamma_{q_1}, \ldots, \gamma_{q_{r-1}}\}$$

is a *canonical homology basis on the punctured Riemann surface* $\mathcal{M}\backslash\{p_j, q_j\}_{j=1}^r$.

An example of a canonical homology basis on a genus 2 Riemann surface with punctures at $p$ and $q$ is shown in Fig. 3.
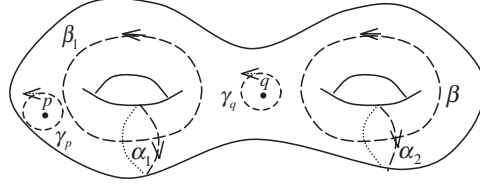
Fig. 3. A canonical homology basis for a punctured Riemann surface.

### 1.2.2. *Differential one-forms and the Hodge star operator*

**Definition 1.4.** A 1-*form* on a Riemann surface $\mathcal{M}$ is an assignment of two continuous functions $f$ and $g$ to each local coordinate $z = x + iy$ on $\mathcal{M}$ such that $f\,dx + g\,dy$ is invariant under coordinate changes. A 2-*form* on $\mathcal{M}$ is an assignment of a continuous function $f$ to each coordinate $z$ so that $f\,dx \wedge dy$ is invariant under coordinate changes.

In the above definition of a 2-form, we have used the exterior wedge product for multiplication of 1-forms. This multiplication satisfies the conditions $dx \wedge dx = 0$, $dy \wedge dy = 0$, and $dy \wedge dx = -dx \wedge dy$.

**Definition 1.5.** A 1-form $\omega$ is *holomorphic* provided that locally $\omega = df = f_x dx + f_y dy$ where $f$ is a holomorphic function. A 1-form $\omega$ is *meromorphic* if we can assign a meromorphic function $f$ to each local coordinate $z$ so that $\omega = f(z)dz$ is invariantly defined.

**Remark 1.** The class of holomorphic 1-forms is a subset of the class of meromorphic 1-forms. So, if a 1-form is holomorphic, it is automatically meromorphic. Meromorphic functions may have singularities, while holomorphic functions have a power series representation and thus have no singularities. It is the singularities of the meromorphic functions (and likewise the meromorphic differentials) that we will use to represent vortices in the superfluid flow.

On a Riemann surface $\mathcal{M}$ of genus $g$, denoted $(\mathcal{M}, g)$, an (*abelian*) *differential of the first kind* is a holomorphic 1-form, an (*abelian*) *differential of the second kind* is a meromorphic 1-form with all residues 0, and an (*abelian*) *differential of the third kind* is a meromorphic 1-form with at least one residue nonzero. We are concerned only with differentials of the first and third kinds.

To use complex notation for a 1-form, we let $dz = dx + i\,dy$ and $d\bar{z} = dx - i\,dy$. Then, a 1-form $\omega = u\,dz + v\,d\bar{z}$ is holomorphic if and only if $v = 0$ and $u$ is a holomorphic function of the local coordinate $z$. The differentials $dz$ and $d\bar{z}$ satisfy $dz \wedge dz = 0$, $d\bar{z} \wedge d\bar{z} = 0$ and $d\bar{z} \wedge dz = -dz \wedge d\bar{z}$.

**Definition 1.6.** We define the Hodge star operator $\star$ on a 1-form $\omega = f\,dx + g\,dy$ on a Riemann surface $\mathcal{M}$ by

$$\star\omega = -g\,dx + f\,dy.$$

For a complex representation of $\omega$, we write $\omega = u\,dz + v\,d\bar{z}$, where $f = u + v$ and $g = i(u - v)$. We then have

$$\star\omega = -iu\,dz + iv d\bar{z}.$$

In later computations, we will be interested in integrating the wedge product $\omega \wedge \star\bar{\omega}$.

## 2. Approximating Kinetic Energy

The process of computing the kinetic energy of a flow without vortices is well known. For a holomorphic flow $\xi$, the kinetic energy is given by the standard $L_2$ Hodge metric:

$$K_E^2 = \|\xi\|^2 = \int_{\mathcal{M}} \xi \wedge \star\bar{\xi},$$

where $\star$ is the Hodge star operator defined in Sec. 1.2.2. In this section, we combine what we know about superflows and meromorphic differentials on a Riemann surface to derive a good candidate for a metric on $\mathcal{M}$ that agrees with the use of the $L_2$ norm as the kinetic energy of a holomorphic flow.

Let $\mathcal{M}$ be a Riemann surface of genus $g$ with canonical homology basis $\{\alpha_k, \beta_k\}_{k=1}^g$. It is a well-known result [3] that for two given points $p$ and $q$ on $\mathcal{M}$, there exists a real harmonic function $u_{pq}$ on $\mathcal{M}$ so that

- $u_{pq} - \log(z - p)$ is holomorphic on $\mathcal{M} \backslash \{p, q\}$
- $u_{pq} + \log(z - q)$ is holomorphic on $\mathcal{M} \backslash \{p, q\}$
- $du_{pq}$ is a real harmonic 1-form
- $du_{pq} + i \star du_{pq}$ is a meromorphic differential with

$$\mathrm{res}_p(du_{pq} + i \star du_{pq}) = 1 = -\mathrm{res}_q(du_{pq} + i \star du_{pq}).$$

Additionally, for any open set $N$ on $\mathcal{M}$ containing $p$ and $q$,

$$\iint_{\mathcal{M} \backslash N} du_{pq} \wedge \star du_{pq} < \infty.$$

The meromorphic differential

$$\frac{1}{2\pi i} \frac{h}{m} (du_{pq} + i \star du_{pq}) = \frac{1}{2\pi} \frac{h}{m} (\star du_{pq} - i \, du_{pq})$$

is a complex representation of the quantized fluid flow with vortices at $p$ and $q$. This differential is the "meromorphic part" of a superflow with vortices at $p$ and $q$.

**Remark 2.** The function $u_{pq}$ may not be unique. However, $\star du_{pq} - i \, du_{pq}$ is the unique meromorphic differential with vortices at $p$ and $q$ and only real circulation.

Consequently, any superflow $\omega$ with paired vortices at $\{(p_j, q_j)\}_{j=1}^r$ is of the form

$$\omega = \frac{1}{2\pi} \frac{h}{m} (\star du_{pq} - i \, du_{pq}) + \nu, \tag{1}$$

where $\nu$ is a holomorphic differential so that the circulation of $\omega$ is quantized.

Let $D = \sum_{j=1}^r (p_j + q_j)$ be a divisor on $\mathcal{M}$. Then, a canonical homology basis of the punctured Riemann surface $\mathcal{M} \backslash D$ is $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_{p_1}, \ldots, \gamma_{p_r}, \gamma_{q_1}, \ldots, \gamma_{q_{r-1}}\}$, where $\gamma_{p_k}$ is a small loop around $p_k$ and $\gamma_{q_k}$ is a small loop around $q_k$. We define the subspace $\mathcal{H}^1(D, \mathbb{C})$ of differential forms on $\mathcal{M}$ by meromorphic 1-forms

$$\frac{1}{2\pi} \frac{h}{m} (\star du - i du)$$

where

$$du = \sum_{j=1}^r du_{p_j q_j}.$$

**Claim 2.1.** *The space $\mathcal{H}^1(D, \mathbb{C})$ is the set of meromorphic differentials with vortices at points of $D$ and only real circulation; this means that they have imaginary residues and real periods.*

**Proof.** We know that $\star du - i \, du$ is a meromorphic differential with vortices at points of $D$ by Remark 2. Since the differentials $du$ and $\star du$ are real and $du$ is exact, we have

$$\Im \left( \int_\alpha \star du - i \, du \right) = \int_\alpha du$$

$$= 0.$$

Similarly, the $\beta$-periods of $\star du - i\,du$ are also purely real. The residues of the differential $\star du - i\,du$ are $\pm\frac{1}{2\pi i}\frac{h}{m}$ at points of $D$, and are thus imaginary. Therefore, we have

$$\int_{\gamma_{p_j}}(\star du - i\,du) = \frac{h}{m} \quad \text{and} \quad \int_{\gamma_{q_j}}(\star du - i\,du) = -\frac{h}{m}.$$

Since $\{\alpha_1,\ldots,\alpha_g,\beta_1,\ldots,\beta_g,\gamma_{p_1},\ldots,\gamma_{p_r},\gamma_{q_1},\ldots,\gamma_{q_{r-1}}\}$ is a homology basis for $\mathcal{M}\backslash D$, any loop $\gamma$ is a linear combination of these elements over $\mathbb{Z}$, so for any loop $\gamma$ on $\mathcal{M}$ we have

$$\Im\left(\int_\gamma(\star du - i\,du)\right) = 0. \qquad \qquad \square$$

We define two other spaces of differentials on $\mathcal{M}$ by

$$H^1(\mathcal{M}\backslash D, \mathbb{C}) = \text{1-forms on } \mathcal{M} \text{ with simple poles at points of } D$$

$$H^1(\mathcal{M}, \mathbb{C}) = \text{Non-singular 1-forms on } \mathcal{M}.$$

Then, we have the canonical decomposition [3]

$$H^1(\mathcal{M}\backslash D, \mathbb{C}) = H^1(\mathcal{M}, \mathbb{C}) \oplus \mathcal{H}^1(D, \mathbb{C}). \tag{2}$$

This decomposition of vector spaces corresponds to the decomposition of superflows $\omega$ into a holomorphic part (in $H^1(\mathcal{M}, \mathbb{C})$) and a meromorphic part (in $\mathcal{H}^1(D, \mathbb{C})$) as in Eq. (1). From Remark 2, this decomposition is unique.

**Remark 3.** We want to ensure that the real part of a superflow $\omega$ conforms to the physical situation, and that the way we compute the kinetic energy has some basis in physical fact. First, we note that $\mathcal{H}^1(D, \mathbb{C})$ is just the complexification of $\mathcal{H}^1(D, \mathbb{R})$, where $\mathcal{H}^1(D, \mathbb{R})$ is the set of all harmonic differentials of the form $\star du$. Thus, we have

$$\mathcal{H}^1(D, \mathbb{C}) = \mathcal{H}^1(D, \mathbb{R}) \otimes \mathbb{C}.$$

We see now that the decomposition in Eq. (2) has a real analog

$$H^1(\mathcal{M}\backslash D, \mathbb{R}) = H^1(\mathcal{M}, \mathbb{R}) \oplus \mathcal{H}^1(D, \mathbb{R}),$$

where $H^1(\mathcal{M}, \mathbb{R})$ is the set of real non-singular harmonic 1-forms on $\mathcal{M}$. It is this real decomposition that describes the physical situation of a superfluid film on a Riemann surface.

If $\omega \in H^1(\mathcal{M}\backslash D, \mathbb{C})$, then we can write $\omega = \xi + \Omega$, for a holomorphic differential $\xi$ and $\Omega = (\star du - i\,du) \in \mathcal{H}^1(D, \mathbb{C})$. Then, the cohomology class of $\omega$ is just

$$[\omega] = [\star du]$$

since $\xi$ and $i\,du$ are exact. Thus, the cohomology of $H^1(\mathcal{M}\backslash D, \mathbb{C})$ is purely real; it is the same as the cohomology of $\mathcal{H}^1(D, \mathbb{R})$. This validates our choice of decomposition in Eq. (2) which we use to compute the kinetic energy of the superflow.

As mentioned previously, we have the standard $L_2$ Hodge metric on $H^1(\mathcal{M}, \mathbb{C})$:

$$\|\xi\|^2 = \iint_{\mathcal{M}} \xi \wedge \star\bar{\xi}.$$

However, we need to use what Kaplan and Pearlstein have named the mixed Hodge metric [5, 12] to approximate the kinetic energy of a superflow. This metric calculates the standard kinetic energy on the holomorphic part of the flow (using the $L_2$ norm) and approximates the "finite part" of the kinetic energy near the vortices. We define it here and devote the next section to justifying its use as an approximation to the kinetic energy of a superfluid flow.

**Definition 2.1.** The mixed Hodge norm $\|\omega\|_{mh}$ on the punctured Riemann surface $\mathcal{M}\backslash\{p_j, q_j\}_{j=1}^r$ is found by writing

$$\omega = \xi + \Omega,$$

where $\xi$ is a holomorphic form, and $\Omega$ is meromorphic with purely real circulation [5]. Then, the mixed Hodge norm is given by

$$\|\omega\|_{mh}^2 = \|\xi\|^2 + k \sum_{p \in \mathcal{M}} |\text{res}_p \Omega|^2,$$

for some positive real constant $k$.

**Remark 4.** Using the mixed Hodge metric on $H^1(\mathcal{M}\backslash D, \mathbb{C})$, the decomposition in Eq. (2) is orthogonal.

## 3. Justification for Using the Mixed Hodge Metric to Approximate Kinetic Energy of a Superflow

First of all, the total kinetic energy of a holomorphic form $\xi$ on $\mathcal{M}$ is given by the $L_2$ Hodge norm:

$$\|\xi\|^2 = \int_{\mathcal{M}} \xi \wedge \star\bar{\xi}.$$

Thus, in the case of a holomorphic superflow $\xi$, we have

$$\|\xi\|^2 = \|\xi\|_{mh}^2.$$

For a meromorphic form $\omega$, this $L_2$ Hodge norm is infinite near the poles of $\omega$, which is why we need to develop a new metric for approximating the kinetic energy.

Physically, vortices are not point singularities, contrary to how we treat them mathematically. In reality, there is a small disk around each vortex where there is no fluid flowing on the surface, called a *vortex core*. It is estimated that the radius of a vortex core is approximately $10\,\text{Å}$. Thus, the superfluid actually has finite energy, since the infinite regions have been removed (or never existed in the first place).

Consider the situation of a superflow $\omega$ on a sphere of radius $\rho$ with vortices at the north and south poles. According to physical intuition, if we change the radius of the sphere, the kinetic energy should change logarithmically with the radius $\rho$ [7] . Notice that increasing the radius $\rho$ of the sphere and keeping the vortex core radius $R$ fixed is mathematically equivalent to fixing the radius $\rho$ and shrinking the vortex core radius $R$. Thus, we would also expect that if we were to fix the sphere radius $\rho$ and change the vortex core radius $R$, the kinetic energy should change according to $\log(R)$.

Remove disks of radius $R$ centered at each vortex from the sphere. The top and bottom halves of the remaining surface are each conformally equivalent to an annulus $\Delta_R$ in $\mathbb{C}$, with inner radius $R$ and outer radius $\frac{\pi}{2}\rho$ (see Fig. 4). On $\Delta_R$ we use the standard Euclidean metric to compute the kinetic energy.

Since there are no holomorphic forms on the sphere, a superflow with poles at the north and south poles must be given by

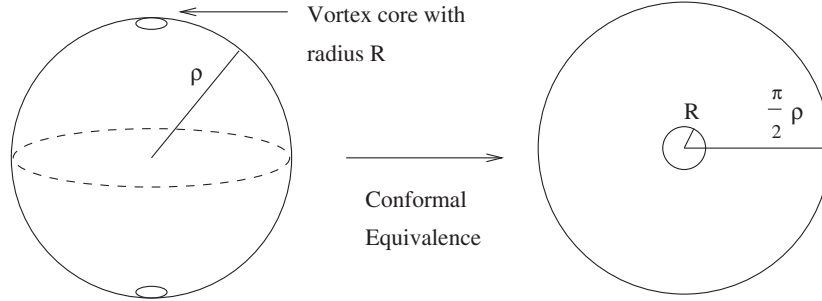$$\omega = \frac{1}{2\pi i} \frac{h}{m} \frac{1}{z} dz.$$

Fig. 4. The sphere minus disks (vortex cores).

The following calculation computes the kinetic energy of $\omega$ on $\Delta_R$.

$$
\iint_{\Delta_R} |\omega|^2 \, dx \, dy = \iint_{\Delta_R} \left| \frac{1}{2\pi i} \frac{h}{m} \frac{1}{z} \right|^2 dx \, dy
$$

$$
= \int_R^{\frac{\pi}{2}\rho} \int_0^{2\pi} \left| \frac{1}{2\pi i} \frac{h}{m} \frac{1}{re^{i\theta}} \right|^2 r \, d\theta \, dr
$$

$$
= \int_R^{\frac{\pi}{2}\rho} \int_0^{2\pi} \left( \frac{1}{2\pi} \frac{h}{m} \right)^2 \frac{1}{r} d\theta \, dr
$$

$$
= \frac{h^2}{2\pi m^2} \left[ \log\left( \frac{\pi}{2}\rho \right) - \log R \right]
$$

$$
= \frac{h^2}{2\pi m^2} \log \rho + C,
$$

for some constant $C \in \mathbb{R}$. Thus, as we increase the size of the sphere, the change in the kinetic energy is proportional to $\log \rho$. This agrees with our physical intuition, so we believe that we are on the right track for developing a metric to approximate the kinetic energy of a superflow with vortices.

For a surface of arbitrary genus, let $\mathcal{M}_R$ be the surface formed by removing small disks of radius $R$ from $\mathcal{M}$ around the poles of $\omega$. Then, we can find the kinetic energy of $\omega$ on the surface $\mathcal{M}_R$ by calculating

$$
\|\omega\|_R^2 = \int_{\mathcal{M}_R} \omega \wedge \star\bar{\omega}.
$$

Consider the flow of a superfluid in a small disk around a vortex. Let $\Delta$ be a fixed disk of radius 1 centered at a pole $p$ of $\omega$, and let $\Delta_R$ be the complement of a small disk of radius $R$ centered at $p$ in $\Delta$. On $\Delta$, we use the Euclidean metric. Locally, we can write

$$
\omega = \left( \frac{a}{z} + h(z) \right) dz,
$$

where $h(z)$ is a holomorphic and bounded function on all of $\Delta$, and $a = \mathrm{res}_p \omega$. Then, we can compute the kinetic energy of $\omega$ on the annulus $\Delta_R$, using the $L_2$ Hodge norm. (Notice that at this point, we are still using the known metric for a holomorphic form on a surface since we have removed the disks containing the poles of $\omega$.)

$$
\|\omega\|_R^2 = \iint_{\Delta_R} |\omega|^2 \, dx \, dy = \iint_{\Delta_R} \left| \frac{a}{z} + h(z) \right|^2 dx \, dy
$$

$$
= \int_R^1 \int_0^{2\pi} \left| \frac{a}{re^{i\theta}} + h(re^{i\theta}) \right|^2 r \, d\theta \, dr.
$$

We know that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\Re(\bar{z}_1 z_2)$ for any two complex numbers $z_1$ and $z_2$. Thus, we have

$$
\begin{aligned}
\|\omega\|_R^2 &= \int_R^1 \int_0^{2\pi} \left[ \left| \frac{a}{re^{i\theta}} \right|^2 + \left| h(re^{i\theta}) \right|^2 + 2\Re\left( \frac{\bar{a}}{re^{i\theta}} h(re^{i\theta}) \right) \right] r\, d\theta\, dr \\
&= |a|^2 \int_R^1 \int_0^{2\pi} \frac{1}{r}\, d\theta\, dr + \int_R^1 \int_0^{2\pi} r\, \left| h(re^{i\theta}) \right|^2 d\theta\, dr \\
&\quad + 2\bar{a} \int_R^1 \int_0^{2\pi} \Re(e^{-i\theta} h(re^{i\theta}))\, d\theta\, dr \\
&= 2\pi |a|^2 (-\log R) + b(R),
\end{aligned}
$$

where $b(R)$ is a bounded function of $R$, as $R \to 0$.

Thus, for the "meromorphic part" (the part with vortices and only real circulation) $\Omega$ of a superflow $\omega$ we have:

$$
\|\Omega\|^2 = 2\pi(-\log R) \sum_{p \in \mathcal{M}} |\mathrm{res}_p \Omega|^2 + b(R)
$$

$$
\frac{\|\Omega\|^2}{-\log R} = 2\pi \sum_{p \in \mathcal{M}} |\mathrm{res}_p \Omega|^2 + \frac{b(R)}{-\log R}.
$$

But, if we shrink the radius $R$ of the vortex cores, we have

$$
\begin{aligned}
\lim_{R \to 0} \frac{\|\Omega\|^2}{-\log R} &= 2\pi \sum_{p \in \mathcal{M}} |\mathrm{res}_p \Omega|^2 + \lim_{R \to 0} \frac{b(R)}{(-\log R)} \\
&= 2\pi \sum_{p \in \mathcal{M}} |\mathrm{res}_p \Omega|^2.
\end{aligned}
$$

For very small fixed $R$, we have

$$
\|\Omega\|^2 \approx 2\pi(-\log R) \sum_{p \in \mathcal{M}} |\mathrm{res}_p \Omega|^2.
$$

Thus, we use the metric

$$
\|\omega\|_{mh}^2 = \|\xi\|^2 + c \sum_{p \in \mathcal{M}} |\mathrm{res}_p \Omega|^2,
$$

where $c = 2\pi(-\log R)$ is a positive constant based on the vortex core radius $R$, the differential $\xi$ is holomorphic, and $\Omega$ is meromorphic with simple poles at $\{(p_j, q_j)\}_{j=1}^r$ and has only real circulation. This metric gives a good finite approximation to the kinetic energy.

## 4. Concluding Remarks

The only part of the mixed Hodge metric that depends on the choice of metric on the manifold $\mathcal{M}$ is the value of the constant $c$, which depends on the value $R$ we use for the radius of the vortex cores. Both the $L_2$ Hodge metric on $H^1(\mathcal{M}, \mathbb{C})$ and the sum of the squared magnitudes of the residues are independent of the choice of the metric on the manifold $\mathcal{M}$. The idea of harmonicity depends only on the conformal structure of $\mathcal{M}$. It is well known [3] that there exists a basis of $2g$ harmonic forms $\{\varphi_j\}_{j=1}^{2g}$ that satisfy

$$
\iint_{\mathcal{M}} \varphi_j \wedge \star\bar{\varphi}_k = \delta_{jk}.
$$

Using these, we can write any smooth differential $\xi$ on $\mathcal{M}$ as

$$\xi = \sum_{j=1}^{2g} \left[ \iint_{\mathcal{M}} \xi \wedge \star\bar{\varphi}_j \right] \varphi_j.$$

Then, we can write the $L_2$ Hodge norm as

$$\|\xi\|^2 = \sum_{j=1}^{2g} \left[ \iint_{\mathcal{M}} \xi \wedge \star\bar{\varphi}_j \right]^2 = \iint_{\mathcal{M}} \xi \wedge \star\bar{\xi}$$

and this representation as an integral is independent of the metric on $\mathcal{M}$. (We want to emphasize that this is peculiar to the two-dimensional case only.)

Thus, the only part of our new metric that fails to be independent of the metric chosen on $\mathcal{M}$ is the constant $c$. The dependence of $c$ on the variable $R$ may seem mathematically to invalidate the results in this paper. Quite the contrary: The fact that $c$ depends logarithmically on $R$ agrees with the physical expectations as discussed on page 157. Thus, instead of discouraging us, the fact that $c \approx (-\log R)$ validates the use of the mixed Hodge norm as an approximation of the kinetic energy of a superfluid flow.

We see here that the mixed Hodge norm for a holomorphic flow is the same as the kinetic energy. Additionally, we have shown that for a meromorphic flow, the mixed Hodge norm provides a good approximation to the kinetic energy of a superfluid by considering the finite size of the vortex cores.

## References

[1] E. Andronikashvili, *J. Physics* **X** (1946) 201.
[2] C. P. Black, *Rev. Math. Phys.* **15**(8) (2003) 925–947.
[3] H. Farkas and I. Kra, *Riemann Surfaces* (Springer-Verlag, 1992).
[4] R. P. Feynman, *Progress in Low Temperature Physics*, in Chap. II, ed. C. J. Gorter, Vol. I (Interscience Publishers Inc., New York, 1955).
[5] A. Kaplan, Notes on Hodge Theory, Personal communication (1996).
[6] J. M. Kosterlitz and D. J. Thouless, *J. Phys. C: Solid State Phys.* **6** (1973) 1181.
[7] J. Machta, Personal communication (1998).
[8] J. Machta and R. A. Guyer, Superfluid films in porous media, *Phys. Rev. Lett.* **60**(20) (1988) 2054–2057.
[9] T. Minoguchi and Y. Nagaoka, Vortices, superfluidity & phase transitions in $^4$He films adsorbed on porous material, *Progr. Theor. Phys.* **80**(3) (1988) 397–416.
[10] M. Nakahara, *Geometry, Topology and Physics* (Institute of Physics, 2003).
[11] L. Osanger, *Nuovo Cimento Suppl.* **6** (1949) 249.
[12] G. Pearlstein, The geometry of the deligne Hodge decomposition, Doctoral thesis, University of Massachusetts Amherst (1998).
[13] L. Tisza, *J. Phys. Radium* **I** (1940) 165, 350.