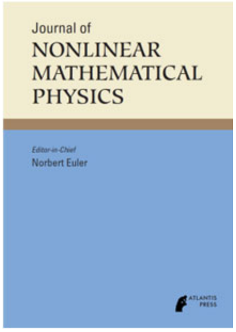


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# Some Examples of Algebraic Geodesics on Quadrics

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# SOME EXAMPLES OF ALGEBRAIC GEODESICS ON QUADRICS

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In this note we give the conditions for the existence of algebraic geodesics on some two-dimensional quadrics, namely, on hyperbolic paraboloids and elliptic paraboloids. It appears that in some cases, such geodesics are the rational space curves.

*Keywords*: Integrable systems; two-dimensional quadrics; algebraic geodesics.

## 1. Introduction

The problem of geodesics on the second order surfaces (quadrics) is a classical one. For a two-dimensional ellipsoid, an explicit description of geodesics was given by Jacobi [4] and Weierstrass [8]. For other quadrics, this problem was considered by Halphen [2] and Hadamard [3] (for the modern exposition of this topic, see [5–7]).

It is well known that the generic geodesic on a two-dimensional quadric is a transcendental space curve. However, in some cases, this geodesic becomes an algebraic space curve. Hence, such geodesics may be considered as the complete intersection (or a connected component of the intersection) of the two-dimensional quadric with the algebraic surface in the space $\mathbb{R}^3$.

In the paper [1], an approach was proposed for the description of such surfaces in the case of two-dimensional ellipsoid and some of them were described explicitly.

In this note we give the conditions for the existence of algebraic geodesics on other two-dimensional quadrics, namely, on hyperbolic paraboloids and elliptic paraboloids. It appears that in some cases, such geodesics are rational space curves.

## 2. Hyperbolic Paraboloid

The equation for the hyperbolic paraboloid in the three-dimensional Euclidean space $\mathbb{R}^3$ can be expressed in the form

$$\frac{x^2}{a} - \frac{y^2}{b} - 2z = 0, \quad a > 0, \quad b > 0. \tag{1}$$

Following [3] let us express coordinates $x, y, z$ in terms of elliptic coordinates $\lambda$, $\mu$:

$$x^2 = -\frac{a(\lambda+a)(\mu+a)}{a+b}, \tag{2}$$

$$y^2 = -\frac{b(\lambda-b)(\mu-b)}{a+b}, \tag{3}$$

$$2z = -(\lambda+\mu+a-b), \quad \lambda < -a, \quad \mu > b. \tag{4}$$

In these coordinates, the element of length takes the form

$$ds^2 = \frac{\lambda-\mu}{4}\left[\frac{\lambda\, d\lambda^2}{(\lambda+a)(\lambda-b)} - \frac{\mu\, d\mu^2}{(\mu+a)(\mu-b)}\right], \tag{5}$$

and the geodesic is given by the equation

$$\int d\lambda \sqrt{\frac{\lambda}{(\lambda+c)(\lambda+a)(\lambda-b)}} = \int d\mu \sqrt{\frac{\mu}{(\mu+c)(\mu+a)(\mu-b)}}, \tag{6}$$

where a constant $c$ characterizes the geodesic.

Recall that on the hyperbolic paraboloid there are two families of straight lines, and any such line is geodesic. Moreover, two principal parabolas (the intersection of planes $x = 0$ and $y = 0$ with paraboloid) are also geodesics.

As it was shown in [3], if $c \neq a$ or $c \neq -b$, then $\mu \to +\infty$ as $t \to +\infty$ or $t \to -\infty$ and any geodesic tends to the straight line given by the formulae

$$\begin{aligned} x &\sim \pm\sqrt{\frac{qa}{a+b}}\mu, \\ y &\sim \pm\sqrt{\frac{qb}{a+b}}\mu, \\ 2z &\sim (1-q)\mu, \quad q = \mathrm{const} = \lim_{t\to+\infty}\frac{|\lambda|}{\mu}. \end{aligned} \tag{7}$$

The main result of this note is the following theorem.

**Theorem 1.** *If $c = a$ and $\sqrt{\frac{a+b}{a}}$ is a rational number, the geodesic defined by Eq. (6) is an algebraic curve.*

**Proof.** In this case, the elliptic integral in (6) reduces to a more simple form:

$$\int d\lambda \frac{1}{\lambda+a}\sqrt{\frac{\lambda}{\lambda-b}} = \int d\mu \frac{1}{\mu+a}\sqrt{\frac{\mu}{\mu-b}}. \tag{8}$$

The integrand $I$ has two poles at $\lambda = -a$ and $\lambda = \infty$:

$$I \sim \frac{1}{r(\lambda+a)} \quad \text{as } \lambda \to -a$$

$$I \sim \frac{1}{\lambda} \quad \text{as } \lambda \to \infty.$$

Here

$$r = \sqrt{\frac{a+b}{a}}. \tag{9}$$

It is easy to see that if $r$ is a rational number $r = p/q$ (for $p$ and $q$ integer), then the integral in (8) is the logarithm of an algebraic function.

In fact, the integral in (8) can be calculated explicitly by means of the standard change of variables $(\lambda, \mu) \to (\xi, \eta)$:

$$\lambda = \frac{b/a}{1 - \xi^2}, \quad \mu = \frac{b/a}{1 - \eta^2}. \tag{10}$$

In this way we come to the algebraic equation

$$(1 - \xi)^p (1 - \eta)^p (r + \xi)^q (r + \eta)^q - c_2 (1 + \xi)^p (1 + \eta)^p (r - \xi)^q (r - \eta)^q = 0, \tag{11}$$

where $c_2$ is a constant of integration. Changing variables $(\xi, \eta)$ for $(x, y)$ and using Eqs. (2) and (3) we get the equation

$$F(x, y) = 0, \tag{12}$$

where $F(x, y)$ is a polynomial in $(x, y)$. Hence, this equation defines an algebraic plane curve. The variable $z$ may be found now from (1) or (4). Then we obtain an algebraic space curve.

So, we proved that our geodesic is an algebraic space curve. □

Note that our geodesic asymptotically approaches the principal parabola $(x = 0)$ as $t \to \infty$ and a straight line as $t \to -\infty$. It also has the property: $x > 0$, $y(t_0) = 0$, $y > 0$, $t > t_0$; $y < 0$, $t < t_0$. Note also that $(\lambda + a) \sim \mu^{-r}$ as $\lambda \to -a$, $\mu \to \infty$.

**Example 1.** Let us consider in more detail the case $a = 1$, $b = 3$, $r = 2$, $p = 2$, $q = 1$. Taking $c_2 = 1$ in (11) we get the algebraic equation for new variables $\xi$, $\eta$

$$(1 - \xi)^2 (1 - \eta)^2 (2 + \xi)(2 + \eta) - (1 + \xi)^2 (1 + \eta)^2 (2 - \xi)(2 - \eta) = 0. \tag{13}$$

Expanding the left-hand side we get

$$(\xi + \eta)(\xi^2 + \eta^2 - \xi\eta - 3) = 0. \tag{14}$$

Using Eqs. (2), (3), and (14) we obtain simple expressions for $x$ and $y$:

$$x^2 = \frac{1}{4} \frac{2 - \zeta}{\zeta + 1}, \quad y^2 = \frac{9}{4} \frac{\zeta^2}{(2 - \zeta)(\zeta + 1)}, \tag{15}$$

where

$$\zeta = \xi\eta, \quad -1 < \zeta < 2. \tag{16}$$

Eliminating $\zeta$ from these equations we get a relation between $x$ and $y$

$$x\left(x - \frac{y}{\sqrt{3}}\right) = \frac{1}{2} \quad \text{or } x\left(x + \frac{y}{\sqrt{3}}\right) = \frac{1}{2}. \tag{17}$$

From this we obtain a parametrization of our geodesic:

$$x = \frac{\tau}{2}, \quad y = \sqrt{3}\,\frac{\tau^2 - 2}{2\tau}, \quad z = \frac{\tau^2 - 1}{2\tau^2}. \tag{18}$$

Hence, the geodesic tends to the straight line in the plane $z = 1/2$ as $\tau \to \infty$ (in accordance with [3]), and to the principal hyperbola $(x = 0)$ as $\tau \to 0$.

Observe that if

$$\frac{x^2}{1} - \frac{y^2}{3} = 2z \tag{19}$$

we have

$$(\lambda + 1) \sim -\frac{1}{\mu^2} \quad \text{as } \mu \to \infty. \tag{20}$$

In a more general case

$$\frac{x^2}{1} - \frac{y^2}{b} = 2z, \quad b = n^2 - 1, \quad n \text{ is integer}, \tag{21}$$

we have

$$(\lambda + 1) \sim -\frac{\alpha_n}{\mu^n}, \quad \text{where } \alpha_n \text{ is constant}. \tag{22}$$

From this it follows that

$$x^2 \sim \frac{\alpha_n}{n^2}\frac{1}{\mu^{n-1}}, \quad y^2 \sim (n^2 - 1)\mu, \quad xy^{n-1} \to \text{const}. \tag{23}$$

## 3. Elliptic Paraboloid

The basic formulae for this case are similar to the formulae of previous section, so we give just few ones. The equation for elliptic non-degenerate paraboloid in the three-dimensional Euclidean space $\mathbb{R}^3$ has the form

$$\frac{x^2}{a} + \frac{y^2}{b} - 2z = 0, \quad a > b > 0. \tag{24}$$

In elliptic coordinates $\lambda$ and $\mu$, we have

$$\begin{aligned} x^2 &= -\frac{a(\lambda + a)(\mu + a)}{a - b}, \\ y^2 &= -\frac{b(\lambda + b)(\mu + b)}{b - a}, \end{aligned} \tag{25}$$

$$2z = -(\lambda + \mu + a + b), \quad \lambda < -a, \quad -a < \mu < -b. \tag{26}$$

The expression for the element of length is of the form

$$ds^2 = \frac{\lambda - \mu}{4}\left[\frac{\lambda\, d\lambda^2}{(\lambda + a)(\lambda + b)} - \frac{\mu\, d\mu^2}{(\mu + a)(\mu + b)}\right] \tag{27}$$

and the geodesic is represented by the equation

$$\int d\lambda \sqrt{\frac{\lambda}{(\lambda + c)(\lambda + a)(\lambda + b)}} = \int d\mu \sqrt{\frac{\mu}{(\mu + c)(\mu + a)(\mu + b)}}, \tag{28}$$

where a constant $c$ characterizes the geodesic.

**Theorem 2.** *If $c = a$, $r = \sqrt{(a - b)/a}$ is a rational number, the geodesic defined by Eq. (28) is an algebraic curve.*

**Proof.** It is completely analogous to the proof of Theorem 1. □

**Example 2.** In the simplest case $r = 1/2$, we have

$$\frac{x^2}{4} + \frac{y^2}{3} - 2z = 0. \tag{29}$$

Taking $c_2 = 1$ in (11) we come to the equation

$$(1-\xi)(1-\eta)\left(\frac{1}{2}+\xi\right)^2\left(\frac{1}{2}+\eta\right)^2 - (1+\xi)(1+\eta)\left(\frac{1}{2}-\xi\right)^2\left(\frac{1}{2}-\eta\right)^2 = 0. \tag{30}$$

After simplifications we get

$$(\xi+\eta)\left(\xi^2+\eta^2-\xi\eta-\frac{3}{4}\right) = 0. \tag{31}$$

This equation implies the following equations for coordinates $x$ and $y$:

$$\frac{x^2}{4} = 16\,\frac{\zeta+\frac{1}{4}}{\zeta-\frac{1}{2}}, \quad \frac{y^2}{3} = 9\,\frac{\zeta^2}{(\zeta-\frac{1}{2})^2}\,, \tag{32}$$

where

$$\zeta = \xi\eta, \quad -\frac{1}{4} < \zeta < \frac{1}{2}.$$

Eliminating $\zeta$ from these equations we obtain a simple relation between $x$ and $y$:

$$\frac{y}{\sqrt{3}} = \frac{x^2}{32} - 1. \tag{33}$$

So, our geodesic is the intersection of the elliptic paraboloid (29) with the parabolic cylinder (33).

Note that in both above examples the geodesic is a rational space curve. It would be interesting to find other examples of rational geodesics on hyperbolic and elliptic paraboloids.

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