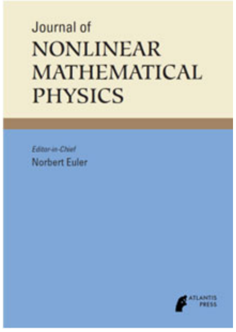

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GROUP ANALYSIS AND HEIR-EQUATIONS OF A MATHEMATICAL MODEL FOR THIN LIQUID FILMS

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Lie group analysis is applied to a mathematical model for thin liquid films, namely a nonlinear fourth order partial differential equation in two independent variables. A three-dimensional Lie symmetry algebra is found and reductions to fourth order ordinary differential equations are obtained by using its one-dimensional subalgebras. Two of these ordinary differential equations are studied by the reduction method and by the Jacobi last multiplier method, and found to be linearizable. Furthermore, the G-equation and η-equation, namely two of the heir-equations obtained by iterating the nonclassical symmetries method, are constructed and reductions to different ordinary differential equations are acquired by using two-dimensional and three-dimensional subalgebras, respectively.

Keywords: Group analysis; nonclassical symmetries.

MSC: 58J70, 35Q53

1. Introduction

In a letter dated London, November 7, 1773, and addressed to William Brownrigg, Benjamin Franklin wrote [8]:

> At length being at CLAPHAM where there is, on the common, a large pond, which I observed to be one day very rough with the wind, I fetched out a cruet of oil, and dropt a little of it on the water. I saw it spread itself with surprising swiftness upon the surface; but the effect of smoothing the waves was not produced; for I had applied it first on the leeward side of the pond, where the waves were the largest, and the wind drove my oil back upon the shore. I then went to the windward side, where they began to form; and there the oil, though not more than a teaspoonful, produced an instant calm over a space several yards square, which spread amazingly, and extending itself gradually till it reached the lee side, making all that quarter of the pond, perhaps half an acre, as smooth as a looking-glass.

This experiment was repeated by Lord Rayleigh 120 years later, who, using trioleum, cautiously concluded that there might be molecules with a length of 2 nm [11]. At about the same time Agnes

Pockels[a] [28] found that some contaminants form a thin layer as thick as the molecules are long and therefore called monomoleculars or monolayers. She observed the behaviour of contaminated water surfaces and measured their tension with "very homely appliances" as Rayleigh himself stated in the preface to [28]. In 1917, Irving Langmuir studied films on water and used a trough similar to that designed by Agnes Pockels, and nowadays it is referred to as the "Langmuir trough".[b] Surface tension is the driving mechanism of thin fluid films and, when introduced into standard lubrication theory, leads to a fourth order nonlinear parabolic equation [15]. Coating, film condensation, tear film, paint film, surfactants, etc. are some physical processes modelled by such an equation.

In this paper we have considered a mathematical model [31] for thin liquid films driven by surface tension and long range intermolecular force, namely the London-van der Waals force [2]. This model is a nonlinear fourth order partial differential equation in two independent variables, i.e.:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(c\frac{h^3}{3}\frac{\partial^3 h}{\partial x^3} + \frac{Ah_x}{h}\right) = 0 \tag{1.1}$$

where $h(x,t)$ is the film height, c is the inverse capillary number which measures the resistance of surface tension to viscous forces, and A is the van der Waals term. Of course, c and A are different from zero.

Using ad hoc REDUCE interactive programs [19] we have determined the three-dimensional Lie symmetries algebra L admitted by Eq. (1.1). Then we have reduced Eq. (1.1), its G-equation, and its η-equation, as defined in [16, 17], to ordinary differential equations by means of the subalgebras of L. We have used the classification of conjugacy classes of subalgebras of three-dimensional Lie algebras as given in [29]. We recall that the G-equation and η-equation are two of the so-called heir-equations of (1.1) which inherit the same Lie symmetry algebra of (1.1) [17].

In Sec. 2, after determining the Lie symmetry algebra admitted by Eq. (1.1), namely $A^a_{3,5}$, with $a = 2/5$, as in Patera–Winternitz [29] or Type VI as in Bianchi [3], we have used its four one-dimensional subalgebras [29] to reduce Eq. (1.1) to fourth order ordinary differential equations. The reduction method, as presented in [20] and used in several instances [7, 12, 14, 22–26, 32], has been applied to two of these equations yielding some integrable properties, namely first integrals and/or general solutions. In Sec. 3, after briefly recalling the concept of heir-equations [17], we have used the three two-dimensional subalgebras of $A^{2/5}_{3,5}$ [29] to reduce the G-equation of Eq. (1.1) to fourth order ordinary differential equations. Also in Sec. 3, we have used the three-dimensional algebra L itself to reduce the η-equation to a further fourth order ordinary differential equation. Some final remarks can be found in the last section.

2. Group Analysis of Eq. (1.1)

Using ad hoc REDUCE interactive programs [19] is easy to show that Eq. (1.1) admits a three-dimensional Lie symmetry algebra L spanned by the following operators:

$$X_1 = t\partial_t + \frac{2}{5}x\partial_x + \frac{h}{5}\partial_h \tag{2.1}$$

$$X_2 = \partial_t \tag{2.2}$$

$$X_3 = \partial_x. \tag{2.3}$$

[a] "These were landmark observations, and were all the more remarkable for the fact that Agnes Pockels had no formal training or advanced degree in science, and worked alone in her kitchen with an apparatus fashioned out of household items" [5].

[b] "One of the film-forming fatty acids Agnes Pockels used on the water surface was stearic acid. This particular substance is one that was used later by Irving Langmuir and Katharine B. Blodgett in their work. For his work in surface science, Langmuir was presented with the Nobel Prize in Chemistry in 1932; in one of his articles he gave recognition to Agnes Pockels and her work" [6].

The commutators are:

$$[X_2, X_1] = X_2, \quad [X_3, X_1] = \frac{2}{5}X_3, \quad [X_3, X_2] = 0 \tag{2.4}$$

and therefore L is solvable. If we take into consideration the notation used in [29] then the following correspondence can be made:

$$X_2 \to e_1$$
$$X_3 \to e_2$$
$$X_1 \to e_3$$

and the algebra $A^a_{3,5}$ with $a = 2/5$ in [29], *viz.* Type VI in [3], is recovered. Nontrivial conjugacy classes of its subalgebras are the following:

- two-dimensional

$$L_{1,2} = \langle e_1,\, e_3 \rangle$$
$$L_{2,2} = \langle e_2,\, e_3 \rangle$$
$$L_{3,2} = \langle e_1,\, e_2 \rangle$$

- one-dimensional

$$\begin{aligned} L_{1,1} &= \langle e_3 \rangle \\ L_{2,1} &= \langle e_2 \rangle \\ L_{3,1}(\varepsilon) &= \langle e_1 + \varepsilon e_2 \rangle \quad \text{with } \varepsilon = \pm 1. \\ L_{4,1} &= \langle e_1 \rangle. \end{aligned}$$

We now transform (1.1) into ordinary differential equations by using the invariants of each class of one-dimensional subalgebras.

2.1. *Subalgebra* $L_{1,1}$

This subalgebra is spanned by the operator:

$$X_1 = t\partial_t + \frac{2}{5}x\partial_x + \frac{h}{5}\partial_h.$$

Its corresponding invariant surface condition is:

$$th_t + \frac{2}{5}xh_x = \frac{h}{5}\,, \tag{2.5}$$

which is equivalent to the characteristic system:

$$\frac{\mathrm{d}t}{t} = \frac{5\mathrm{d}x}{2x} = \frac{5\mathrm{d}h}{h}. \tag{2.6}$$

The first equality in (2.6) yields a first integral which does not depend on h, i.e.:

$$\xi = \frac{x}{\sqrt[5]{t^2}} = \text{const.}, \tag{2.7}$$

while the second equality yields:

$$\widetilde{\xi} = \frac{h}{\sqrt[5]{t}} = \text{const.} \tag{2.8}$$

Therefore invariant functions for $L_{1,1}$ have the following form:

$$\Phi(\xi, \widetilde{\xi}) = 0, \tag{2.9}$$

viz.

$$h = \sqrt[5]{t}\Psi(\xi) \tag{2.10}$$

with Ψ an arbitrary function of ξ. Substituting (2.10) into (1.1) yields an ordinary differential equation (ODE) for Ψ, i.e.

$$\Psi_{\xi\xi\xi\xi} = \frac{3}{5c\Psi^5}(-5c\Psi_{\xi\xi\xi}\Psi_\xi\Psi^4 - 5A\Psi\Psi_{\xi\xi} + 5A\Psi_\xi^2 + 2\xi\Psi^2\Psi_\xi - \Psi^3), \tag{2.11}$$

which does not admit any Lie point symmetry.

2.2. *Subalgebra* $L_{2,1}$

This subalgebra is spanned by the operator:

$$X_3 = \partial_x.$$

The corresponding invariant surface condition is:

$$h_x = 0 \tag{2.12}$$

which means that t, h form a base of invariants of $L_{2,1}$. Hence:

$$h = \Psi(t), \tag{2.13}$$

with Ψ arbitrary function of t. Substituting into (1.1) yields

$$\Psi_t = 0 \Rightarrow h = \text{const.}, \tag{2.14}$$

and $h = \text{const.}$ is a trivial solution of (1.1).

2.3. *Subalgebra* $L_{3,1}(\varepsilon)$

These two subalgebras are spanned by the operators:

$$X_2 + \varepsilon X_3 = \partial_t + \varepsilon\partial_x \quad \text{with } \varepsilon = \pm 1.$$

Let us consider them separately:

- $\boldsymbol{\varepsilon} = 1$
  The operator

$$\partial_t + \partial_x \tag{2.15}$$

  admits the invariants $h, t + x$, and thus:

$$h = \Psi(\xi) \tag{2.16}$$

  with $\xi = t + x$ and Ψ an arbitrary function of ξ.
- $\boldsymbol{\varepsilon} = -1$
  The operator

$$\partial_t - \partial_x \tag{2.17}$$

  admits the invariants $h, t - x$, and thus

$$h = \Psi(\xi) \tag{2.18}$$

  with $\xi = t - x$ and Ψ an arbitrary function of ξ.

In both cases Eq. (1.1) is transformed into the following ODE:

$$\Psi_{\xi\xi\xi\xi} = \frac{3(-A\Psi\Psi_{\xi\xi} + A\Psi_{\xi}^2 - c\Psi^4\Psi_{\xi}\Psi_{\xi\xi\xi} + \Psi^2\Psi_{\xi})}{c\Psi^5}. \tag{2.19}$$

which is autonomous and therefore admits the Lie symmetry operator ∂_ξ. Group analysis does not give any other symmetry.

Now we apply the reduction method, as proposed in [20], to Eq. (2.19). The following change of dependent variables:

$$w_1(\xi) = \Psi, \quad w_2(\xi) = \Psi_\xi, \quad w_3(\xi) = \Psi_{\xi\xi}, \quad w_4(\xi) = \Psi_{\xi\xi\xi}$$

transform (2.19) into an autonomous system of four equations of first order, i.e.:

$$\begin{cases} \dot{w}_1 = w_2 \\ \dot{w}_2 = w_3 \\ \dot{w}_3 = w_4 \\ \dot{w}_4 = \dfrac{3(-Aw_1w_3 + Aw_2^2 - cw_1^4w_2w_4 + w_1^2w_2)}{cw_1^5} \end{cases} \tag{2.20}$$

in which the overdot denotes differentiation with respect to ξ. Now we can choose any of the four dependent variables as the new independent variable in order to reduce the order of system (2.20). Let us choose w_1 as the new independent variable. Then (2.20) yields:

$$\begin{cases} \dfrac{\mathrm{d}w_2}{\mathrm{d}w_1} = \dfrac{w_3}{w_2} \\ \dfrac{\mathrm{d}w_3}{\mathrm{d}w_1} = \dfrac{w_4}{w_2} \\ \dfrac{\mathrm{d}w_4}{\mathrm{d}w_1} = \dfrac{3(-Aw_1w_3 + Aw_2^2 - cw_1^4w_2w_4 + w_1^2w_2)}{cw_2w_1^5} \end{cases} \tag{2.21}$$

namely a system of three equations of first order. If we derive w_3 from the first equation in (2.21), i.e.:

$$w_3 = \frac{\mathrm{d}w_2}{\mathrm{d}w_1}w_2,$$

then we obtain the following system of two equations, one of the first order and one of the second order:

$$\begin{cases} \dfrac{\mathrm{d}w_4}{\mathrm{d}w_1} = \dfrac{3\left(Aw_2 - A\dfrac{\mathrm{d}w_2}{\mathrm{d}w_1}w_1 - cw_4w_1^4 + w_1^2\right)}{cw_1^5} \\ \dfrac{\mathrm{d}^2w_2}{\mathrm{d}w_1^2} = \dfrac{w_4 - w_2\left(\dfrac{\mathrm{d}w_2}{\mathrm{d}w_1}\right)^2}{w_2^2}. \end{cases} \tag{2.22}$$

Lie group analysis of (2.22) yields a linear parabolic equation with characteristic curve

$$\frac{cw_1^4w_4 + 3Aw_2}{cw_1^4}.$$

This suggests the following change of dependent variable:

$$w_4 = \frac{-3Aw_2 + cw_5w_1^4}{cw_1^4}$$

where $w_5 = w_5(w_1)$ is the new dependent variable, and then system (2.22) becomes:

$$\begin{cases} \dfrac{\mathrm{d}w_5}{\mathrm{d}w_1} = -\dfrac{3(cw_5w_1^2 - 1)}{cw_1^3} \\ \dfrac{\mathrm{d}^2w_2}{\mathrm{d}w_1^2} = \dfrac{-3Aw_2 + cw_5w_1^4 - cw_2\left(\dfrac{\mathrm{d}w_2}{\mathrm{d}w_1}\right)^2 w_1^4}{cw_2^2w_1^4}. \end{cases} \tag{2.23}$$

We can easily integrate the first equation in (2.23), i.e.:

$$w_5 = \frac{cD + 3w_1}{cw_1^3},$$

with D a constant, which corresponds to the following first integral of Eq. (2.19):

$$\Psi_{\xi\xi\xi}\Psi^3 + \frac{3(A\Psi_\xi - \Psi^2)}{c\Psi} = \text{const.} \tag{2.24}$$

Of course, (2.24) could have been derived from Eq. (2.19) itself by using the admitted Lie symmetry operator ∂_ξ. Finally, we are left with one second order equation, i.e.:

$$\frac{\mathrm{d}^2w_2}{\mathrm{d}w_1^2} = \frac{-3Aw_2 + cDw_1 - cw_2\left(\dfrac{\mathrm{d}w_2}{\mathrm{d}w_1}\right)^2 w_1^4 + 3w_1^2}{cw_2^2w_1^4} \tag{2.25}$$

which does not admit any Lie point symmetry even for $D = 0$.

We notice that $w_3 \equiv \Psi_{\xi\xi}$ does not appear in the first integral (2.24) as expected from the reduction method. Details on the reduction method and first integrals can be found in [14].

If we apply the Jacobi last multiplier method [27] to (2.25), namely if we use the Jacobi last multiplier of Eq. (2.25) to raise its order, then we obtain a second-order differential equation which admits an eight-dimensional Lie symmetry algebra if $D = 0$, and therefore is linearizable. Equation (2.25) can be written as the following system of two first-order equations:

$$\begin{cases} \dfrac{\mathrm{d}R_1}{\mathrm{d}w_1} = R_2 & (\equiv \Lambda_1) \\ \dfrac{\mathrm{d}R_2}{\mathrm{d}w_1} = \dfrac{-3AR_1 + cDw_1 - cR_1R_2^2w_1^4 + 3w_1^2}{cR_1^2w_1^4} & (\equiv \Lambda_2) \end{cases} \tag{2.26}$$

where $R_1 = w_2$, and $R_2 = \frac{\mathrm{d}w_2}{\mathrm{d}w_1}$. The Jacobi last multiplier M for system (2.26) has to satisfy the following equation [27]:

$$\frac{\mathrm{d}\log(M)}{\mathrm{d}w_1} + \frac{\partial\Lambda_1}{\partial R_1} + \frac{\partial\Lambda_2}{\partial R_2} = 0, \tag{2.27}$$

i.e.

$$\frac{\mathrm{d}\log(M)}{\mathrm{d}w_1} - 2\frac{R_2}{R_1} = 0. \tag{2.28}$$

Therefore if we introduce the new dependent variable[c] $J = -\frac{M}{2}$, and solve (2.28) with respect to R_2, i.e.

$$R_2 = -\frac{\mathrm{d}\log(J)}{\mathrm{d}w_1}R_1,$$

then the first equation of system (2.26) becomes:

$$\frac{\mathrm{d}J}{\mathrm{d}w_1}R_1 + \frac{\mathrm{d}R_1}{\mathrm{d}w_1}J = 0 \Rightarrow R_1 = \frac{C_0}{J},$$

[c]The choice $-\frac{1}{2}$ for the multiplicative constant is unessential and comes from the following calculations.

with C_0 an arbitrary constant different from zero, and finally a second-order differential equation satisfied by J is obtained, i.e.:

$$\frac{\mathrm{d}^2 J}{\mathrm{d}w_1^2} = \frac{3AC_0J^4 + 3cC_0^3 w_1^4 \left(\frac{\mathrm{d}J}{\mathrm{d}w_1}\right)^2 - cDJ^5 w_1}{cC_0^3 J w_1^4}. \tag{2.29}$$

If we apply Lie group analysis to this equation then if $D = 0$ Eq. (2.29) becomes

$$\frac{\mathrm{d}^2 J}{\mathrm{d}w_1^2} = 3\,\frac{AJ^4 + cC_0^2 w_1^4 \left(\frac{\mathrm{d}J}{\mathrm{d}w_1}\right)^2}{cC_0^2 J w_1^4}, \tag{2.30}$$

which admits an eight-dimensional Lie symmetry algebra spanned by the following operators:

$$\begin{aligned}
\Gamma_1 &= J^3 w_1 \partial_J \\
\Gamma_2 &= J^3 \partial_J \\
\Gamma_3 &= \frac{-2cC_0^2 w_1^2}{3A}\partial_{w_1} + \frac{3J^3 A + JcC_0^2 w_1^2}{3Aw_1}\partial_J \\
\Gamma_4 &= \frac{J(AJ^2 + cC_0^2 w_1^2)}{Aw_1^2}\partial_J \\
\Gamma_5 &= \frac{-cC_0^2}{A}\partial_{w_1} + \frac{J^3}{w_1^3}\partial_J \\
\Gamma_6 &= \frac{AJ^2 + cC_0^2 w_1^2}{3w_1^4 A^2 J^2}(-2cC_0^2 w_1^3 \partial_{w_1} + J(3AJ^2 + cC_0^2 w_1^2)\partial_J) \\
\Gamma_7 &= \frac{AJ^2 + cC_0^2 w_1^2}{w_1^5 A^2 J^2}(-cC_0^2 w_1^3 \partial_{w_1} + AJ^3 \partial_J) \\
\Gamma_8 &= w_1 \partial_{w_1} + J\partial_J.
\end{aligned} \tag{2.31}$$

This means that Eq. (2.29) is linearizable by means of a point transformation [13]. In order to find the linearizing transformation we have to look for a two-dimensional abelian intransitive subalgebra [13], and, following Lie's classification of two-dimensional algebras in the real plane [13], we have to transform it into the canonical form

$$\partial_u, \quad y\partial_u$$

with y and u the new independent and dependent variables, respectively. We found that one such subalgebra is that spanned by Γ_1 and Γ_2, i.e.

$$\Gamma_1 = J^3 w_1 \partial_J, \quad \Gamma_2 = J^3 \partial_J. \tag{2.32}$$

Then, it is easy to derive that

$$y = \frac{1}{w_1}, \quad u = -\frac{AJ^2 + cC_0^2 w_1^2}{2cC_0^2 J^2 w_1^3}$$

and Eq. (2.30) becomes:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = 0 \Rightarrow u = -a_2 y^2 - a_1 \tag{2.33}$$

with a_1, a_2 arbitrary constants. In the original variables this yields:

$$J = \pm C_0 w_1 \left(\frac{2a_1 cC_0^2 w_1^3 + 2a_2 cC_0^2 w_1^2 - A}{c}\right)^{-\frac{1}{2}} \tag{2.34}$$

and therefore:

$$\Psi_\xi = \pm\left(\frac{2a_1cC_0^2\Psi^3 + 2a_2cC_0^2\Psi^2 - A}{c\Psi^2}\right)^{\frac{1}{2}} \tag{2.35}$$

namely an implicit general solution of (2.19) in terms of elliptic functions.

2.4. *Subalgebra $L_{4,1}$*

This subalgebra is spanned by the operator:

$$X_2 = \partial_t,$$

then we have:

$$h = \Psi(x) \tag{2.36}$$

with Ψ an arbitrary function of x (steady state). Substituting (2.36) into (1.1) yields:

$$\Psi_{xxxx} = \frac{3(-A\Psi\Psi_{xx} + A\Psi_x^2 - c\Psi^4\Psi_x\Psi_{xxx})}{c\Psi^5}. \tag{2.37}$$

Equation (2.37) admits a two-dimensional Lie symmetry algebra spanned by the following operators:

$$Z_1 = \partial_x, \quad Z_2 = 2x\partial_x + \Psi\partial_\Psi.$$

A base of differential invariants of order ≤ 2 of this algebra is:

$$v = \Psi\Psi_x, \quad w = \Psi_{xx}.$$

Therefore we can reduce Eq. (2.37) to the following second order ODE:

$$\begin{aligned}\frac{\mathrm{d}^2 w}{\mathrm{d}v^2} = \Bigg(&-3Aw^2v^2 + 3A - 15cw^4v^6 - 12cw^3\frac{\mathrm{d}w}{\mathrm{d}v}v^7\\ &-24cw^3v^5 - cw^2\left(\frac{\mathrm{d}w}{\mathrm{d}v}\right)^2 v^8 - 27cw^2\frac{\mathrm{d}w}{\mathrm{d}v}v^6\\ &-9cw^2v^4 - 2cw\left(\frac{\mathrm{d}w}{\mathrm{d}v}\right)^2 v^7 - 18cw\frac{\mathrm{d}w}{\mathrm{d}v}v^5 - c\left(\frac{\mathrm{d}w}{\mathrm{d}v}\right)^2 v^6\\ &-3c\frac{\mathrm{d}w}{\mathrm{d}v}v^4\Bigg)\Bigg/(cv^5(w^3v^3 + 3w^2v^2 + 3wv + 1)).\end{aligned} \tag{2.38}$$

which does not admit any Lie point symmetry.

Now we apply the reduction method to the autonomous equation (2.37). The following change of dependent variables:

$$w_1(x) = \Psi, \quad w_2(x) = \Psi_x, \quad w_3(x) = \Psi_{xx}, \quad w_4(x) = \Psi_{xxx}, \tag{2.39}$$

transforms Eq. (2.37) into the following system of four first order ODEs:

$$\begin{cases}\dot{w}_1 = w_2\\ \dot{w}_2 = w_3\\ \dot{w}_3 = w_4\\ \dot{w}_4 = \dfrac{3(-Aw_1w_3 + Aw_2^2 - cw_1^4w_2w_4)}{cw_1^5}.\end{cases} \tag{2.40}$$

If we take w_1 as new independent variable, then system (2.40) becomes:

$$\begin{cases} \dfrac{\mathrm{d}w_2}{\mathrm{d}w_1} = \dfrac{w_3}{w_2} \\ \dfrac{\mathrm{d}w_3}{\mathrm{d}w_1} = \dfrac{w_4}{w_2} \\ \dfrac{\mathrm{d}w_4}{\mathrm{d}w_1} = \dfrac{3(-Aw_1w_3 + Aw_2^2 - cw_1^4 w_2 w_4)}{cw_2 w_1^5}. \end{cases} \tag{2.41}$$

Deriving w_3 from the first equation, i.e.:

$$w_3 = \frac{\mathrm{d}w_2}{\mathrm{d}w_1} w_2$$

yields the following system of two ODEs, one of the first order and one of the second order:

$$\begin{cases} \dfrac{\mathrm{d}w_4}{\mathrm{d}w_1} = \dfrac{3\left(Aw_2 - A\dfrac{\mathrm{d}w_2}{\mathrm{d}w_1}w_1 - cw_4w_1^4\right)}{cw_1^5} \\ \dfrac{\mathrm{d}^2w_2}{\mathrm{d}w_1^2} = \dfrac{w_4 - w_2\left(\dfrac{\mathrm{d}w_2}{\mathrm{d}w_1}\right)^2}{w_2^2}. \end{cases} \tag{2.42}$$

Applying Lie group analysis to (2.42) yields a linear parabolic equation with the following characteristic curve:

$$\frac{cw_1^4 w_4 + 3Aw_2}{cw_1^4}$$

which suggests the following change of dependent variable:

$$w_4 = \frac{-3Aw_2 + cw_5 w_1^4}{cw_1^4}$$

with $w_5 = w_5(w_1)$ the new dependent variable. Then system (2.42) becomes:

$$\begin{cases} \dfrac{\mathrm{d}w_5}{\mathrm{d}w_1} = -\dfrac{3w_5}{w_1} \\ \dfrac{\mathrm{d}^2w_2}{\mathrm{d}w_1^2} = \dfrac{-3Aw_2 + cw_5w_1^4 - cw_2\left(\dfrac{\mathrm{d}w_2}{\mathrm{d}w_1}\right)^2 w_1^4}{cw_2^2 w_1^4}. \end{cases} \tag{2.43}$$

We can easily integrate the first equation in (2.43), i.e.:

$$w_5 = \frac{C_2}{w_1^3},$$

with C_2 a constant, which yields the following first integral[d] of Eq. (2.37):

$$\Psi_{xxx}\Psi^3 + \frac{3A\Psi_x}{c\Psi} = \text{const.} \tag{2.44}$$

Finally, we are left with one second order equation, i.e.:

$$\frac{\mathrm{d}^2w_2}{\mathrm{d}w_1^2} = \frac{-3Aw_2 + cC_2w_1 - cw_2\left(\dfrac{\mathrm{d}w_2}{\mathrm{d}w_1}\right)^2 w_1^4}{cw_2^2 w_1^4}. \tag{2.45}$$

[d] Of course, (2.44) could have been derived from Eq. (2.37) itself by using the admitted operator ∂_x.

If $C_2 \neq 0$, then Eq. (2.45) does not admit any Lie point symmetry, but if $C_2 = 0$ then Eq. (2.45) becomes

$$\frac{\mathrm{d}^2 w_2}{\mathrm{d}w_1^2} = \frac{-3Aw_2 - cw_2\left(\frac{\mathrm{d}w_2}{\mathrm{d}w_1}\right)^2 w_1^4}{cw_2^2 w_1^4} \tag{2.46}$$

and admits an eight-dimensional Lie symmetry algebra spanned by the following operators:

$$\begin{aligned}
\Gamma_1 &= w_1^2\partial_{w_1} + \frac{3A + cw_2^2w_1^2}{2cw_2w_1}\partial_{w_2}\\
\Gamma_2 &= \frac{w_1}{w_2}\partial_{w_2}\\
\Gamma_3 &= \frac{A + cw_2^2w_1^2}{cw_1}\partial_{w_1} + \frac{3A^2 + 4Acw_2^2w_1^2 + c^2w_2^4w_1^4}{2c^2w_2w_1^4}\partial_{w_2}\\
\Gamma_4 &= \partial_{w_1} + \frac{A}{cw_2w_1^3}\partial_{w_2}\\
\Gamma_5 &= \frac{A + cw_2^2w_1^2}{cw_1^2}\partial_{w_1} + \frac{A(A + cw_2^2w_1^2)}{c^2w_2w_1^5}\partial_{w_2}\\
\Gamma_6 &= w_1^2\partial_{w_1} + \frac{5A + cw_2^2w_1^2}{4cw_2w_1^2}\partial_{w_2}\\
\Gamma_7 &= \frac{A + cw_2^2w_1^2}{cw_2w_1^2}\partial_{w_2}\\
\Gamma_8 &= \frac{1}{w_2}\partial_{w_2}.
\end{aligned} \tag{2.47}$$

This means that Eq. (2.45) is linearizable by means of a point transformation [13]. In order to find the linearizing transformation we have to look for a two-dimensional abelian intransitive subalgebra [13], and, following Lie's classification of two-dimensional algebras in the real plane [13], we have to transform it into the canonical form

$$\partial_u, \quad y\partial_u$$

with y and u the new independent and dependent variables, respectively. We found that one such subalgebra is that spanned by Γ_8 and Γ_2, i.e.

$$\Gamma_8 = \frac{1}{w_2}\partial_{w_2}, \quad \Gamma_2 = \frac{w_1}{w_2}\partial_{w_2}. \tag{2.48}$$

Then, it is easy to derive that

$$y = w_1, \quad u = \frac{w_2^2}{2}$$

and Eq. (2.46) becomes:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = -\frac{3A}{ct^4} \tag{2.49}$$

which can be easily integrated, i.e.:

$$u = \frac{2D_1ct^3 + 2D_2ct^2 - A}{2ct^2} \tag{2.50}$$

with $D_1 \mathrm{e} D_2$ arbitrary constants. In the original variables this yields:

$$w_2 = \pm\left(\frac{2D_1cw_1^3 + 2D_2cw_1^2 - A}{cw_1^2}\right)^{\frac{1}{2}} \tag{2.51}$$

and therefore:

$$\Psi_x = \pm \left(\frac{2D_1 c\Psi^3 + 2D_2 c\Psi^2 - A}{c\Psi^2} \right)^{\frac{1}{2}} \tag{2.52}$$

namely an implicit general solution of (2.37) in terms of elliptic functions. We note that similarity between Eq. (2.52) and Eq. (2.35), although the first yields a stationary solution while the latter does not.

3. Lie Symmetry Reduction of the Heir-Equations

The nonclassical symmetries method was introduced in 1969 by Bluman and Cole [4] to obtain new exact solutions of the linear heat equation, i.e. solutions not deducible from Lie group analysis. It consists in adding the invariant surface condition to the given equation, and then apply the Lie group analysis. The main difficulty of this approach is that the determining equations are no longer linear. In [17], we have found that iterations of the nonclassical symmetries method give rise to new nonlinear equations, which inherit the Lie point symmetry algebra of the given equation. Similarity solutions of those equations yields new solutions of the original equations, e.g. blow-up solutions as given in [9].

Next we recall how to construct the heir-equations. Let us consider a generic evolution equation in two independent variables and one dependent variable:

$$u_t = H(t, x, u, u_x, u_{xx}, u_{xxx}, \ldots). \tag{3.1}$$

Then its invariant surface condition is given by:

$$V_1(t, x, u)u_t + V_2(t, x, u)u_x = G(t, x, u). \tag{3.2}$$

Let us take the case with $V_1 = 0$ and $V_2 = 1$, so that (3.2) becomes:

$$u_x = G(t, x, u). \tag{3.3}$$

Then, an equation for G is easily obtained. We call this equation the G-equation [16]. Its invariant surface condition is given by:

$$\xi_1(t, x, u, G)G_t + \xi_2(t, x, u, G)G_x + \xi_3(t, x, u, G)G_u = \eta(t, x, u, G). \tag{3.4}$$

Let us consider the case $\xi_1 = 0$, $\xi_2 = 1$, and $\xi_3 = G$, so that (3.4) becomes:

$$G_x + GG_u = \eta(t, x, u, G). \tag{3.5}$$

Then, an equation for η is derived. We call this equation the η-equation. Clearly:

$$G_x + GG_u \equiv u_{xx} \equiv \eta. \tag{3.6}$$

We could keep iterating to obtain the Ω-equation, which corresponds to:

$$\eta_x + G\eta_u + \eta\eta_G \equiv u_{xxx} \equiv \Omega(t, x, u, G, \eta) \tag{3.7}$$

the ρ-equation, which corresponds to:

$$\Omega_x + G\Omega_u + \eta\Omega_G + \Omega\Omega_\eta \equiv u_{xxxx} \equiv \rho(t, x, u, G, \eta, \Omega) \tag{3.8}$$

and so on. Each of these equations inherits the Lie symmetry algebra of the original equation, with the right prolongation: first prolongation for the G-equation, second prolongation for the η-equation, and so on. Therefore, these equations were named heir-equations. More details, applications and further properties can be found in [1, 10, 17, 18, 21].

Now we derive the G-equation of Eq. (1.1), i.e.:

$$
\begin{aligned}
&-\frac{1}{3}cu^6\frac{\partial^4 G}{\partial x^4}+3AGu\frac{\partial G}{\partial x}-2cu^4G^2\frac{\partial G}{\partial u}\frac{\partial G}{\partial x}+\frac{\partial G}{\partial t}u^3-3\left(\frac{\partial G}{\partial u}\right)^2u^5cG\frac{\partial G}{\partial x}\\
&\quad-7cu^5G^2\frac{\partial^2 G}{\partial u^2}\frac{\partial G}{\partial x}-4\frac{\partial G}{\partial u}u^6c\frac{\partial^3 G}{\partial u^2\partial x}G^2-2Au^2\frac{\partial^2 G}{\partial u\partial x}G-\frac{4}{3}cu^6\frac{\partial^4 G}{\partial x^3\partial u}G\\
&\quad-Au^2\frac{\partial^2 G}{\partial x^2}-2cu^4G^3\left(\frac{\partial G}{\partial u}\right)^2-2\left(\frac{\partial G}{\partial u}\right)^3u^5cG^2-\frac{8}{3}cu^6\left(\frac{\partial^2 G}{\partial u\partial x}\right)^2G\\
&\quad-\frac{4}{3}cu^6\left(\frac{\partial^2 G}{\partial u^2}\right)^2G^3+2\frac{\partial G}{\partial u}uAG^2-8cu^5G\frac{\partial^2 G}{\partial u\partial x}\frac{\partial G}{\partial x}-8\frac{\partial G}{\partial u}u^5cG^3\frac{\partial^2 G}{\partial u^2}\\
&\quad-2\frac{\partial G}{\partial u}u^5cG\frac{\partial^2 G}{\partial x^2}-\frac{10}{3}\frac{\partial G}{\partial u}u^6cG\frac{\partial^2 G}{\partial u^2}\frac{\partial G}{\partial x}-2cu^6\frac{\partial^3 G}{\partial x^2\partial u}\frac{\partial G}{\partial x}-2\frac{\partial G}{\partial u}u^6c\frac{\partial^3 G}{\partial u^3}G^3\\
&\quad-\frac{4}{3}cu^6\frac{\partial^4 G}{\partial u^3\partial x}G^3-2cu^5G\frac{\partial^3 G}{\partial x^3}-cu^6\frac{\partial^2 G}{\partial u^2}\left(\frac{\partial G}{\partial x}\right)^2-\frac{4}{3}\frac{\partial G}{\partial u}u^6c\frac{\partial^2 G}{\partial u\partial x}G\frac{\partial G}{\partial x}\\
&\quad-cu^5\frac{\partial G}{\partial u}\left(\frac{\partial G}{\partial x}\right)^2-\frac{4}{3}\left(\frac{\partial G}{\partial u}\right)^2u^6c\frac{\partial^2 G}{\partial u\partial x}G-2AG^3-2cu^6G^2\frac{\partial^3 G}{\partial u^3}\frac{\partial G}{\partial x}\\
&\quad-10\frac{\partial G}{\partial u}u^5cG^2\frac{\partial^2 G}{\partial u\partial x}-4cu^6G\frac{\partial^3 G}{\partial u^2\partial x}\frac{\partial G}{\partial x}-4cu^6\frac{\partial^2 G}{\partial u\partial x}\frac{\partial^2 G}{\partial u^2}G^2-\frac{1}{3}cu^6\frac{\partial^4 G}{\partial u^4}G^4\\
&\quad-\frac{4}{3}cu^6G\frac{\partial^2 G}{\partial u^2}\frac{\partial^2 G}{\partial x^2}-2\frac{\partial G}{\partial u}u^6c\frac{\partial^3 G}{\partial x^2\partial u}G-\frac{7}{3}\left(\frac{\partial G}{\partial u}\right)^2u^6c\frac{\partial^2 G}{\partial u^2}G^2-2cu^6\frac{\partial^4 G}{\partial x^2\partial u^2}G^2\\
&\quad-\frac{4}{3}cu^6\frac{\partial^2 G}{\partial u\partial x}\frac{\partial^2 G}{\partial x^2}-2cu^4G^4\frac{\partial^2 G}{\partial u^2}-4cu^4G^3\frac{\partial^2 G}{\partial u\partial x}-Au^2\frac{\partial^2 G}{\partial u^2}G^2-6cu^5G^2\frac{\partial^3 G}{\partial x^2\partial u}\\
&\quad-2cu^4G^2\frac{\partial^2 G}{\partial x^2}-cu^5\frac{\partial^2 G}{\partial x^2}\frac{\partial G}{\partial x}-2cu^5G^4\frac{\partial^3 G}{\partial u^3}-6cu^5G^3\frac{\partial^3 G}{\partial u^2\partial x}=0. \qquad (3.9)
\end{aligned}
$$

This equation inherits the (once prolonged) Lie symmetry algebra L admitted by (1.1), i.e. the Lie symmetry algebra spanned by the following operators.

$$X_1 = t\partial_t + \frac{2}{5}x\partial_x + \frac{h}{5}\partial_h - \frac{G}{5}\partial_G \quad (e_3) \qquad (3.10)$$

$$X_2 = \partial_t \quad (e_1) \qquad (3.11)$$

$$X_3 = \partial_x \quad (e_2). \qquad (3.12)$$

Then we can use each class of two-dimensional subalgebras of L as given in Sec. 2 in order to reduce (3.9) to an ordinary differential equation.

3.1. *Subalgebra* $L_{1,2}$

This subalgebra is spanned by the operators:

$$X_1 = t\partial_t + \frac{2}{5}x\partial_x + \frac{h}{5}\partial_h - \frac{G}{5}\partial_G, \quad X_2 = \partial_t$$

and a base of its invariants is given by

$$\xi = \frac{h}{\sqrt{x}} \quad \text{and} \quad \widetilde{\xi} = hG. \qquad (3.13)$$

Thus

$$G = \frac{\Psi(\xi)}{h} \qquad (3.14)$$

with Ψ arbitrary function of ξ. Substituting (3.14) into the G-equation yields an ODE for Ψ, i.e.:

$$\begin{aligned}\Psi_{\xi\xi\xi\xi} = &-(-384c\Psi^3\Psi_\xi\xi^3\Psi_{\xi\xi} - 78c\xi^9\Psi_{\xi\xi}\Psi_\xi + 105c\xi^9\Psi_\xi + 36A\xi^5\Psi_\xi \\ &- 128c\Psi^2\Psi_\xi^3\xi^3 + 72A\Psi\Psi_\xi\xi^3 + 96c\xi^5\Psi^2\Psi_\xi - 18c\xi^8\Psi_\xi^2 + 88\xi^5c\Psi\Psi_\xi^3 \\ &- 176\xi^4c\Psi^2\Psi_\xi^2 + 88\xi^3c\Psi^3\Psi_\xi - 64c\Psi^4\Psi_{\xi\xi\xi}\xi^3 - 12c\xi^7\Psi_\xi^3 - 48\xi^5c\Psi_{\xi\xi\xi}\Psi^3 \\ &- 180c\xi^7\Psi\Psi_\xi - 12c\xi^{10}\Psi_\xi\Psi_{\xi\xi\xi} + 28\xi^8c\Psi_{\xi\xi}\Psi_\xi^2 + 72\xi^8c\Psi\Psi_\xi\Psi_{\xi\xi\xi} \\ &- 112\xi^6c\Psi\Psi_\xi^2\Psi_{\xi\xi} + 112c\Psi^2\Psi_\xi^2\xi^4\Psi_{\xi\xi} - 100c\xi^9\Psi\Psi_{\xi\xi\xi} + 72c\xi^5\Psi_{\xi\xi}\Psi^2\Psi_\xi \\ &+ 168\xi^7c\Psi^2\Psi_{\xi\xi\xi} - 300c\xi^8\Psi\Psi_{\xi\xi} + 288c\Psi^5 - 96c\xi^6\Psi_{\xi\xi}^2\Psi^2 + 64c\Psi^3\Psi_{\xi\xi}^2\xi^4 \\ &- 48A\xi^4\Psi_{\xi\xi}\Psi + 18c\xi^{11}\Psi_{\xi\xi\xi} + 48c\xi^8\Psi_{\xi\xi}^2\Psi + 48A\Psi^2\Psi_{\xi\xi}\xi^2 - 80\xi^4c\Psi^3\Psi_{\xi\xi} \\ &+ 87c\xi^{10}\Psi_{\xi\xi} + 272c\Psi^4\Psi_{\xi\xi}\xi^2 + 224\xi^6c\Psi^2\Psi_{\xi\xi} - 192A\Psi^2\Psi_\xi\xi - 8c\xi^10\Psi_{\xi\xi}^2 \\ &+ 288A\Psi^3 - 12\xi^6c\Psi\Psi_\xi^2 + 544c\Psi^3\Psi_\xi^2\xi^2 - 704c\Psi^4\Psi_\xi\xi + 12A\xi^6\Psi_{\xi\xi} \\ &- 144c\xi^6\Psi^2\Psi_\xi\Psi_{\xi\xi\xi} + 96c\Psi^3\Psi_\xi\xi^4\Psi_{\xi\xi\xi} + 216\xi^7c\Psi\Psi_\xi\Psi_{\xi\xi})/(c\xi^4(\xi^8 \\ &- 8\xi^6\Psi + 24\xi^4\Psi^2 + 16\Psi^4 - 32\xi^2\Psi^3))\end{aligned} \tag{3.15}$$

which does not admit any Lie point symmetry.

3.2. *Subalgebra* $L_{2,2}$

This subalgebra is spanned by the operators:

$$X_1 = t\partial_t + \frac{2}{5}x\partial_x + \frac{h}{5}\partial_h - \frac{G}{5}\partial_G, \quad X_3 = \partial_x$$

and a base of its invariants is given by

$$\xi = \frac{h}{\sqrt[5]{t}} \quad \text{and} \quad \widetilde{\xi} = hG$$

so that

$$G = \frac{\Psi(\xi)}{h} \tag{3.16}$$

is an invariant function, with Ψ an arbitrary function of ξ. Substituting (3.16) into (3.9) yields the following ODE:

$$\begin{aligned}\Psi_{\xi\xi\xi\xi} = &-(90c\Psi^5 - 120c\Psi^3\Psi_\xi\xi^3\Psi_{\xi\xi} + 170c\Psi^3\Psi_\xi^2\xi^2 + 30c\Psi^3\Psi_\xi\xi^4\Psi_{\xi\xi\xi} + 3\Psi_\xi\xi^6 + 90A\Psi^3 \\ &- 60A\Psi^2\Psi_\xi\xi + 85c\Psi^4\Psi_{\xi\xi}\xi^2 - 220c\Psi^4\Psi_\xi\xi + 15A\Psi^2\Psi_{\xi\xi}\xi^2 + 35c\Psi^2\Psi_\xi^2\xi^4\Psi_{\xi\xi} \\ &- 40c\Psi^2\Psi_\xi^3\xi^3 + 20c\Psi^3\Psi_{\xi\xi}^2\xi^4 - 20c\Psi^4\Psi_{\xi\xi\xi}\xi^3)/(5c\Psi^4\xi^4)\end{aligned} \tag{3.17}$$

which does not possess any Lie point symmetry.

3.3. *Subalgebra* $L_{3,2}$

This subalgebra is spanned by the operators:

$$X_2 = \partial_t, \quad X_3 = \partial_x.$$

which yields

$$G = \Psi(\xi) \quad \text{with} \quad \xi = h, \tag{3.18}$$

Ψ an arbitrary function of ξ. Substituting (3.18) into (3.9) yields the following ODE:

$$\begin{aligned}\Psi_{\xi\xi\xi\xi} = &-(6A\Psi + 4c\xi^6\Psi_{\xi\xi}^2\Psi + 3A\Psi_{\xi\xi}\xi^2 + 6c\xi^4\Psi^2\Psi_{\xi\xi} + 6c\xi^5\Psi^2\Psi_{\xi\xi\xi}\\ &+ 7\Psi_\xi^2\xi^6c\Psi_{\xi\xi} - 6A\Psi_\xi\xi + 6\Psi_\xi^3\xi^5c + 6c\xi^4\Psi\Psi_\xi^2 + 24\Psi_\xi\xi^5c\Psi\Psi_{\xi\xi}\\ &+ 6\Psi_\xi\xi^6c\Psi_{\xi\xi\xi}\Psi)/(c\xi^6\Psi^2) \end{aligned} \tag{3.19}$$

which admits an one-dimensional Lie symmetry algebra spanned by the following operator:

$$\Gamma = \xi\partial_\xi - \Psi\partial_\Psi.$$

Its differential invariants

$$s = \xi\Psi \quad \text{and} \quad v = \frac{\Psi_\xi}{\Psi^2}$$

transform (3.19) into the following third order ODE:

$$\begin{aligned}v_{sss} = &-((s^5v_s^3v + 4s^5v_sv_{ss}v^2 + s^4v_s^3 + 32s^4v_s^2v^2 + 8s^4v_sv_{ss}v + 20s^4v_{ss}v^3\\ &+ 51s^3v_s^2v + 4s^3v_sv_{ss} + 129s^3v_sv^3 + 46s^3v_{ss}v^2 + 19s^2v_s^2 + 201s^2v_sv^2\\ &+ 32s^2v_{ss}v + 90s^2v^4 + 78sv_sv + 6sv_{ss} + 90sv^3 + 6v_s + 18v^2)cs^4\\ &+ 3(s^3v_sv + s^2v_s + 2s^2v^2 - 2sv + 2)A)/(c(sv+1)^3s^6) \end{aligned} \tag{3.20}$$

which does not possess any Lie point symmetry.

3.4. *Reduction of the η-equation to ODE*

The η-equation[e] of (1.1) inherits the (twice prolonged) Lie symmetry algebra L admitted by (1.1), i.e. the Lie symmetry algebra spanned by the following operators:

$$X_1 = t\partial_t + \frac{2}{5}x\partial_x + \frac{h}{5}\partial_h - \frac{G}{5}\partial_G - \frac{3}{5}\eta\partial_\eta \tag{3.21}$$

$$X_2 = \partial_t \tag{3.22}$$

$$X_3 = \partial_x. \tag{3.23}$$

Then we can use the entire algebra to reduce the η-equation to an ODE. In fact, the invariants are

$$\eta = \frac{\Psi(\xi)}{h^3} \quad \text{and} \quad \xi = hG, \tag{3.24}$$

with Ψ an arbitrary function of ξ. Substituting (3.24) into the η-equation yields the following ODE:

$$\begin{aligned}\Psi_{\xi\xi\xi\xi} = &(5\Psi_\xi^3c\xi^3 + 18c\xi^5\Psi_\xi - 9c\xi^4\Psi_\xi^2 + 99c\xi^2\Psi^2 - 18c\xi^4\Psi - 99A\Psi\xi^2 + 3c\xi^7\Psi_{\xi\xi\xi}\\ &+ 9c\Psi^3\Psi_{\xi\xi} - 4c\Psi_{\xi\xi}^2\Psi^3 - 3A\Psi_{\xi\xi}\Psi^2 - 4c\xi^6\Psi_{\xi\xi}^2 - 9c\xi^6\Psi_{\xi\xi} - 3A\xi^4\Psi_{\xi\xi}\\ &+ 33\Psi_\xi A\xi^3 - 6\Psi_\xi c\Psi_{\xi\xi\xi}\Psi^3 + 11\Psi_\xi^2c\Psi^2 + 3c\xi\Psi_{\xi\xi\xi}\Psi^3 + 9c\xi^3\Psi^2\Psi_{\xi\xi\xi}\\ &+ 22c\xi^5\Psi_\xi\Psi_{\xi\xi} - 7c\Psi_{\xi\xi}\xi^4\Psi_\xi^2 - 6c\xi^6\Psi_{\xi\xi\xi}\Psi_\xi - 7\Psi_\xi^2c\Psi_{\xi\xi}\Psi^2 + 9A\Psi^2\\ &- 18c\Psi^2\Psi_{\xi\xi\xi}\xi^2\Psi_\xi - 14\Psi_\xi^2c\xi^2\Psi\Psi_{\xi\xi} + 5\Psi_\xi^3c\xi\Psi + 15\Psi_\xi A\Psi\xi + 2\Psi_\xi^2c\xi^2\Psi\\ &- 9c\xi^4\Psi\Psi_{\xi\xi} - 12c\xi^2\Psi_{\xi\xi}^2\Psi^2 - 60c\xi\Psi^2\Psi_\xi - 60c\xi^3\Psi_\xi\Psi\\ &- 18\Psi_\xi c\xi^4\Psi\Psi_{\xi\xi\xi} + 22\Psi_\xi c\xi\Psi_{\xi\xi}\Psi^2 + 44\Psi_\xi c\xi^3\Psi\Psi_{\xi\xi} - 6A\xi^2\Psi\Psi_{\xi\xi}\\ &+ 9c\xi^5\Psi\Psi_{\xi\xi\xi} + 9c\xi^2\Psi_{\xi\xi}\Psi^2 - 12c\xi^4\Psi\Psi_{\xi\xi}^2 + 18A\xi^4\\ &- 9c\Psi^3)/(c(\xi^8 + \Psi^4 + 4\xi^6\Psi + 6\xi^4\Psi^2 + 4\xi^2\Psi^3)) \end{aligned} \tag{3.25}$$

which does not possess any Lie point symmetry.

[e] Here we do not present the η-equation due to its long expression.

4. Final Remarks

In this paper we have applied Lie group analysis and the method of iteration of nonclassical symmetries (heir-equations) [10, 17, 18] to Eq. (1.1). We have shown that the same Lie symmetry algebra can be used to reduce a partial differential equation, i.e. (1.1), to several different ordinary differential equations. In fact none of the fourth-order ordinary differential equations of Sec. 3, namely Eqs. (3.15), (3.17), (3.19), (3.25) could have been obtained by the classical Lie symmetry reduction. Of course, it is essential to know the nontrivial conjugacy classes of the subalgebras of the admitted Lie symmetry algebra.

Finally, in Sec. 2 one can find the application of the reduction method [14, 20] and the application of the Jacobi last multiplier method [27] in order to find Lie symmetries of certain ordinary differential equations without Lie point symmetries. It is an open problem to be able to solve all the given fourth-order ordinary differential equations without recurring to a numerical approximate procedure.

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