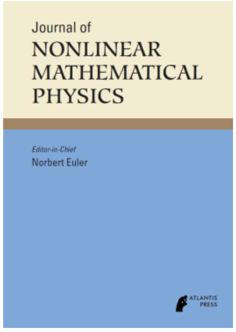


Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 ISSN (Print): 1402-9251
Journal Home Page: https://www.tandfonline.com/loi/tnmp20

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**To cite this article**: Pasha Zusmanovich (2010) Invariants of Lie Algebras Extended Over Commutative Algebras Without Unit, Journal of Nonlinear Mathematical Physics 17:Supplement 1, 87–102, DOI: https://doi.org/10.1142/S1402925110000817

**To link to this article**: https://doi.org/10.1142/S1402925110000817

Published online: 04 January 2021

Article

World Scientific
www.worldscientific.com

ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics, Vol. 17, Suppl. 1 (2010) 87–102

DOI: 10.1142/S1402925110000817

# INVARIANTS OF LIE ALGEBRAS EXTENDED OVER COMMUTATIVE ALGEBRAS WITHOUT UNIT

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Received 14 February 2009
Accepted 14 March 2009

We establish results about the second cohomology with coefficients in the trivial module, symmetric invariant bilinear forms, and derivations of a Lie algebra extended over a commutative associative algebra without unit. These results provide a simple unified approach to a number of questions treated earlier in completely separated ways: periodization of semisimple Lie algebras (Anna Larsson), derivation algebras, with prescribed semisimple part, of nilpotent Lie algebras (Benoist), and presentations of affine Kac–Moody algebras.

*Keywords*: Current Lie algebra; Kac–Moody algebras; second cohomology; invariant bilinear form; derivation.

Mathematics Subject Classification: 17B20, 17B40, 17B55, 17B56, 17B67

## 0. Introduction

In this paper we consider current Lie algebras, i.e., Lie algebras of the form $L \otimes A$, where $L$ is a Lie algebra, $A$ is a commutative associative algebra, and the multiplication in $L \otimes A$ is defined by the formula

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab$$

for any $x, y \in L$, $a, b \in A$. Note that $A$ can be considered as the algebra of functions on the spectrum of $A$, and $L \otimes A$ can therefore be interpreted as the algebra of "currents", as physicists say, on this spectrum.

Berezin and Karpelevich [3] were among the first to study certain class of representations of current algebras over local finite-dimensional commutative algebras $A$. In that, somewhat obscure, paper they showed that cohomology of such algebras can be reduced to cohomology of $L$. In what follows we will consider other types of commutative algebras $A$ and the description of cohomology in our case is more involved and interesting. We are primarily interested in the second cohomology of $L \otimes A$ with trivial coefficients, the space of symmetric invariant bilinear forms on $L \otimes A$, and the algebras of derivations of $L \otimes A$.

These invariants were determined for numerous particular cases of current Lie algebras (see, for example, [28]), the general formulae for the second homology with trivial coefficients in terms of invariants of $L$ and $A$ were obtained in [12], [29] and [24], and similar formulae for the space of symmetric invariant bilinear forms and derivation algebras were obtained in [29] and [30], respectively.

So why return to these settled questions? In all considerations until now, the algebra $A$ was supposed to have a unit. However, there are many interesting examples of current algebras where $A$ is not unital. For example, in [17], the so-called periodization of semisimple Lie algebras $\mathfrak{g}$ was considered, which is nothing but $\mathfrak{g} \otimes t\mathbb{C}[t]$. It is known that the second homology of any nilpotent Lie algebra with trivial coefficients has interpretation in terms of presentation of the algebra, so allowing $A$ to be nilpotent allows us to obtain presentation of $L \otimes A$ irrespective of the properties of $L$.

It turns out that elementary arguments similar to those in [30] allow us to extend the above mentioned results to the case of non-unital $A$. In particular, concerning the second cohomology and symmetric invariant bilinear forms, we provide another proof, considerably shorter than all the previous ones even in the case of unital $A$.

The contents of this paper are as follows. In Secs. 1–3 we establish the general formulae for 2-cocycles, symmetric invariant bilinear forms, and get partial results about derivations of the current Lie algebras, respectively. This is followed by applications: in Sec. 4 we reprove the result from [17] about presentations of periodizations of the semisimple Lie algebras. In passing, we also mention how to derive from our results the theorem from [2] about semisimple components of the derivation algebras of certain current Lie algebras, and the Serre defining relations between Chevalley generators of the non-twisted affine Kac–Moody algebra.

In all these cases, the absence of unit in $A$ is essential. All these proofs are significantly shorter than the original ones, and reveal various almost trivial, but so far unnoticed or unpublished, links between different concepts and results. These links are, perhaps, the main virtue of this paper.

It seems that most if not everything considered here can be extended in a straightforward way to twisted, Leibniz and super settings, but we will not venture into this, at least for now.

## Notation and Conventions

All algebras and vector spaces are defined over an arbitrary field $K$ of characteristic different from 2 and 3, unless stated otherwise (some of the results are valid in characteristic 3, but we will not go into this).

In what follows, $L$ denotes a Lie algebra, $A$ is an associative commutative algebra.

Given an $L$-module $M$, let $B^n(L, M)$, $Z^n(L, M)$, $C^n(L, M)$ and $H^n(L, M)$ denote the space of $n$th degree coboundaries, cocycles, cochains, and cohomology of $L$ with coefficients in $M$, respectively (we will be mainly interested in the particular cases of degree 2 and the trivial module $K$, or degree 1 and the adjoint module or its dual). Note that $C^2(L, K)$ is the space of all skew-symmetric bilinear forms on $L$. The space of all symmetric bilinear forms on $L$ will be denoted as $S^2(L, K)$.

Let $\mathcal{Z}(L)$, $[L, L]$ and $\mathrm{Der}(L)$ denote the center, the commutant (the derived algebra), and the Lie algebra of derivations of $L$, respectively. Similarly, let $\mathrm{Ann}(A) = \{a \in A \mid Aa = 0\}$

and $AA$ denote the annulator and the square of $A$, respectively, and let $HC^*(A)$ denotes its cyclic cohomology.

A bilinear form $\varphi : L \times L \to K$ is said to be *cyclic* if

$$\varphi([x, y], z) = \varphi([z, x], y)$$

for any $x, y, z \in L$. Note that if $\varphi$ is symmetric, this condition is equivalent to the *invariance* of the form $\varphi$:

$$\varphi([x, y], z) + \varphi(y, [x, z]) = 0,$$

while if $\varphi$ is skew-symmetric, the notions of cyclic and invariant forms differ. Let $\mathcal{B}(L)$ denote the space of all symmetric bilinear invariant (=cyclic) forms on $L$.

Similarly, a bilinear form $\alpha : A \times A \to K$ is said to be *cyclic* if

$$\alpha(ab, c) = \alpha(ca, b),$$

and *invariant* if

$$\alpha(ab, c) = \alpha(a, bc)$$

for any $a, b, c \in A$. If the form $\alpha$ is symmetric, it is cyclic if and only if it is invariant.

## 1. The Second Cohomology

**Theorem 1.1.** *Let $L$ be a Lie algebra, $A$ an associative commutative algebra, and at least one of $L$ and $A$ be finite-dimensional. Then each cocycle in $Z^2(L \otimes A, K)$ can be represented as the sum of decomposable cocycles $\varphi \otimes \alpha$, where $\varphi : L \times L \to K$ and $\alpha : A \times A \to K$ are of one of the following 8 types:*

(i) $\varphi([x, y], z) + \varphi([z, x], y) + \varphi([y, z], x) = 0$ *and $\alpha$ is cyclic,*
(ii) *$\varphi$ is cyclic and* $\alpha(ab, c) + \alpha(ca, b) + \alpha(bc, a) = 0$,
(iii) $\varphi([L, L], L) = 0$,
(iv) $\alpha(AA, A) = 0$,

*where each of these 4 types splits into two subtypes: with $\varphi$ skew-symmetric and $\alpha$ symmetric, and with $\varphi$ symmetric and $\alpha$ skew-symmetric.*

**Proof.** Each cocycle $\Phi \in Z^2(L \otimes A, K)$, being an element of

$$\mathrm{End}(L \otimes A \otimes L \otimes A, K) \simeq \mathrm{End}(L \otimes L, K) \otimes \mathrm{End}(A \otimes A, K),$$

can be written in the form $\Phi = \sum_{i \in I} \varphi_i \otimes \alpha_i$, where $\varphi_i : L \times L \to K$ and $\alpha_i : A \times A \to K$ are bilinear maps, and $I$ is a finite set of indices (this is the place where the assumption of finite-dimensionality is needed). Using this representation, write the cocycle equation for an arbitrary triple $x \otimes a$, $y \otimes b$, $z \otimes c$, where $x, y, z \in L$, $a, b, c \in A$:

$$\sum_{i \in I} \varphi_i([x, y], z) \otimes \alpha_i(ab, c) + \varphi_i([z, x], y) \otimes \alpha_i(ca, b) + \varphi_i([y, z], x) \otimes \alpha_i(bc, a) = 0. \quad (1.1)$$

Symmetrizing this equality with respect to $x, y$, we get:

$$\sum_{i \in I} (\varphi_i([x, z], y) + \varphi_i([y, z], x)) \otimes (\alpha_i(bc, a) - \alpha_i(ca, b)) = 0.$$

On the other hand, cyclically permuting $x, y, z$ in (1.1) and summing up the 3 equalities obtained, we get:

$$\sum_{i \in I} (\varphi_i([x,y],z) + \varphi_i([z,x],y) + \varphi_i([y,z],x)) \otimes (\alpha_i(ab,c) + \alpha_i(bc,a) + \alpha_i(ca,b)) = 0.$$

Applying Lemma 1.1 from [30] to the last two equalities, we get a partition of the index set $I = I_1 \cup I_2 \cup I_3 \cup I_4$ such that

$$\begin{array}{lll}
\varphi_i([x,z],y) + \varphi_i([y,z],x) = 0, & \varphi_i([x,y],z) + \varphi_i([z,x],y) + \varphi_i([y,z],x) = 0 & \text{for } i \in I_1, \\
\varphi_i([x,z],y) + \varphi_i([y,z],x) = 0, & \alpha_i(ab,c) + \alpha_i(bc,a) + \alpha_i(ca,b) = 0 & \text{for } i \in I_2, \\
\varphi_i([x,y],z) + \varphi_i([z,x],y) + \varphi_i([y,z],x) = 0, & \alpha_i(bc,a) - \alpha_i(ca,b) = 0 & \text{for } i \in I_3, \\
\alpha_i(bc,a) - \alpha_i(ca,b) = 0, & \alpha_i(ab,c) + \alpha_i(bc,a) + \alpha_i(ca,b) = 0 & \text{for } i \in I_4.
\end{array}$$

It is obvious that if the characteristic of $K$ is different from 3, then $\varphi_i([L,L],L) = 0$ for $i \in I_1$, and $\alpha_i(AA,A) = 0$ for $i \in I_4$. It is obvious also that $\varphi_i \otimes \alpha_i$ satisfies the cocycle equation (1.1) for each $i \in I_1, I_2, I_3, I_4$.

Now write the condition of skew-symmetry of $\Phi$:

$$\sum_{i \in I} \varphi_i(x,y) \otimes \alpha_i(a,b) + \varphi_i(y,x) \otimes \alpha_i(b,a) = 0 \tag{1.2}$$

and symmetrize it with respect to $x, y$:

$$\sum_{i \in I} (\varphi_i(x,y) - \varphi_i(y,x)) \otimes (\alpha_i(a,b) - \alpha_i(b,a)) = 0$$

$$\sum_{i \in I} (\varphi_i(x,y) + \varphi_i(y,x)) \otimes (\alpha_i(a,b) + \alpha_i(b,a)) = 0.$$

From the last two equalities, using again Lemma 1.1 from [30], we see that each set $I_1, I_2, I_3, I_4$ can be split further into two subsets, one having skew-symmetric $\varphi_i$ and symmetric $\alpha_i$, and the other one having symmetric $\varphi_i$ and skew-symmetric $\alpha_i$. □

**Remark 1.1.** As all our bilinear maps are $K$-valued, the cocycles of the form $\varphi \otimes \alpha$ are, of course, just products of bilinear maps $\varphi\alpha$. However, we have retained the symbol $\otimes$, to make it easier to track dependence on the more general situation of [30].

**Remark 1.2.** The second subdivision in the statement of Theorem 1.1, which follows from equality (1.2), is merely a manifestation of the vector space isomorphism

$$C^2(L \otimes A, K) \simeq (S^2(L,K) \otimes C^2(A,K)) \oplus (C^2(L,K) \otimes S^2(A,K)). \tag{1.3}$$

Let $d\Omega$ be the 2-coboundary defined by a given linear map $\Omega : L \otimes A \to K$. The latter can be written in the form $\Omega = \sum_{i \in I} \omega_i \otimes \beta_i$ for some linear maps $\omega_i : L \to K$ and $\beta_i : A \to K$. Then

$$d\Omega(x \otimes a, y \otimes b) = \sum_{i \in I} \omega_i([x,y]) \otimes \beta_i(ab),$$

i.e., coboundaries always lie in the direct summand $C^2(L,K) \otimes S^2(A,K)$. Consequently, nonzero cocycles from different direct summands in (1.3) can never be cohomologically dependent, and the cocycles from the direct summand $S^2(L,K) \otimes C^2(A,K)$ are cohomologically independent if and only if they are linearly independent.

One may try to formulate Theorem 1.1 as a statement about $H^2(L \otimes A, K)$, but in full generality this will lead only to cumbersome complications. In each case of interest, one can easily obtain an information about the cohomology. For example, assuming $A$ contains a unit, one immediately sees that cocycles of type (i) necessarily have $\varphi$ skew-symmetric and $\alpha$ symmetric, cocycles of type (ii) necessarily have $\varphi$ symmetric and $\alpha$ skew-symmetric, and cocycles of type (iv) vanish. This leads to a known formula for $H^2(L \otimes A, K)$, where the cocycles of type (i) contribute to the term $H^2(L,K) \otimes A^*$, the cocycles of type (ii) contribute to the term $\mathcal{B}(L) \otimes HC^1(A)$, and the cocycles of type (iii) are non-essential (in terminology of [29]).

Another, more concrete, application is given in Sec. 4.

## 2. Symmetric Invariant Bilinear Forms

**Theorem 2.1.** *Let $L$ be a Lie algebra, $A$ an associative commutative algebra, and at least one of $L$ and $A$ be finite-dimensional. Then each symmetric invariant bilinear form on $L \otimes A$ can be represented as a sum of decomposable forms $\varphi \otimes \alpha$, $\varphi : L \times L \to K$, $\alpha : A \times A \to K$ of one of the following* 6 *types*:

(i) *both $\varphi$ and $\alpha$ are cyclic*,
(ii) $\varphi([L,L],L) = 0$,
(iii) $\alpha(AA,A) = 0$,

*where each of these* 3 *types splits into two subtypes*: *with both $\varphi$ and $\alpha$ symmetric*, *and with both $\varphi$ and $\alpha$ skew-symmetric.*

**Proof.** The proof is absolutely similar to that of Theorem 1.1. As in the proof of Theorem 1.1, we may write a symmetric invariant bilinear form $\Phi$ on $L \otimes A$ as $\sum_{i\in I} \varphi_i \otimes \alpha_i$ for suitable bilinear maps $\varphi_i : L \times L \to K$ and $\alpha_i : A \times A \to K$. The invariance condition, written for a given triple $x \otimes a$, $y \otimes b$, $z \otimes c$, reads:

$$\sum_{i\in I} \varphi_i([x,y],z) \otimes \alpha_i(ab,c) + \varphi_i([x,z],y) \otimes \alpha_i(ca,b) = 0. \tag{2.1}$$

Symmetrizing this with respect to $x, y$, we get:

$$\sum_{i\in I} (\varphi_i([x,z],y) + \varphi_i([y,z],x)) \otimes \alpha_i(ca,b) = 0.$$

Hence the index set can be partitioned $I = I_1 \cup I_2$ in such a way that

$$\varphi_i([x,z],y) + \varphi_i([y,z],x) = 0$$

for any $i \in I_1$, and $\alpha_i(AA,A) = 0$ for any $i \in I_2$. Then (2.1) can be rewritten as

$$\sum_{i\in I_1} \varphi_i([x,y],z) \otimes (\alpha_i(ab,c) - \alpha_i(ca,b)) = 0.$$

Hence there is a partition $I_1 = I_{11} \cup I_{12}$ such that $\varphi_i([L,L],L) = 0$ for any $i \in I_{11}$, and $\alpha_i(ab,c) = \alpha_i(ac,b)$ for any $i \in I_{12}$.

The condition of symmetry of $\Phi$:

$$\sum_{i \in I} \varphi_i(x,y) \otimes \alpha_i(a,b) - \varphi_i(y,x) \otimes \alpha_i(b,a) = 0,$$

being symmetrized with respect to $x, y$, allows us to partition further each of the sets $I_{11}, I_{12}, I_2$ into two subsets, one having both $\varphi_i$ and $\alpha_i$ symmetric, and the other one having both $\varphi_i$ and $\alpha_i$ skew-symmetric. □

This generalizes [29, Theorem 4.1], where a similar statement is proved for $A$ unital.

## 3. Derivations

Naturally, one may try to apply the same approach to description of the derivations of a given current algebra $L \otimes A$ (for unital $A$, see [30, Corollary 2.2]). Indeed, each derivation $D$ of $L \otimes A$, being an element of

$$\mathrm{End}(L \otimes A, L \otimes A) \simeq \mathrm{End}(L, L) \otimes \mathrm{End}(A, A),$$

can be expressed in the form $D = \sum_{i \in I} \varphi_i \otimes \alpha_i$, where $\varphi_i : L \to L$ and $\alpha_i : A \to A$ are linear maps. The condition that $D$ is a derivation, written for an arbitrary pair $x \otimes a$ and $y \otimes b$, where $x, y \in L$ and $a, b \in A$, reads:

$$\sum_{i \in I} \varphi_i([x,y]) \otimes \alpha_i(ab) - [\varphi_i(x), y] \otimes \alpha_i(a)b - [x, \varphi_i(y)] \otimes a\alpha_i(b) = 0.$$

Symmetrizing this equality with respect to $a$, $b$ (this is equivalent to symmetrization with respect to $x$, $y$):

$$\sum_{i \in I} ([\varphi_i(x), y] - [x, \varphi_i(y)]) \otimes (a\alpha_i(b) - b\alpha_i(a)) = 0,$$

we get a partition of the index set $I$ into two subsets with conditions $[\varphi_i(x), y] = [x, \varphi_i(y)]$ and $a\alpha_i(b) = b\alpha_i(a)$, respectively. But as there are only two variables in each of $L$ and $A$, no other symmetrization is possible, so the last equality is all what we can get in this way.

The failure of this method can be also explained by looking at the simple example of the Lie algebra $sl(2) \otimes tK[t]$. In $sl(2)$, fix a basis $\{e_-, h, e_+\}$ with multiplication

$$[h, e_-] = -e_-, \quad [h, e_+] = e_+, \quad [e_-, e_+] = h. \tag{3.1}$$

It is easy to check that the map defined, for any $f(t) \in tK[t]$, by the formulae

$$e_- \otimes f(t) \mapsto e_- \otimes t\left(\frac{df(t)}{dt} - f(t)\right)$$

$$e_+ \otimes f(t) \mapsto e_+ \otimes t\left(\frac{df(t)}{dt} + f(t)\right)$$

$$h \otimes f(t) \mapsto h \otimes t\frac{df(t)}{dt}$$

is a derivation of $sl(2) \otimes tK[t]$. It is obvious that this map is not a decomposable one, i.e., of the form $\varphi \otimes \alpha$ for some $\varphi : sl(2) \to sl(2)$ and $\alpha : tK[t] \to tK[t]$. But for this approach to succeed, all the maps in question should be representable in such a way in the end.

However, under additional assumption on $L \otimes A$, we can derive information about $\mathrm{Der}(L \otimes A)$ from the results of the preceding sections, using the relationship between $H^1(L, L^*)$, $H^2(L, K)$, and $\mathcal{B}(L)$. In the literature, this relationship was noted many times in a slightly different form, and goes back to the classical works of Koszul and Hochschild–Serre [14]. Namely, there is an exact sequence

$$0 \to H^2(L, K) \xrightarrow{u} H^1(L, L^*) \xrightarrow{v} \mathcal{B}(L) \xrightarrow{w} H^3(L, K) \tag{3.2}$$

where for the representative $\varphi \in Z^2(L, K)$ of a given cohomology class, we have to take the class of $u(\varphi)$, the latter being given by

$$(u(\varphi)(x))(y) = \varphi(x, y)$$

for any $x, y \in L$, $v$ is sending the class of a given cocycle $D \in Z^1(L, L^*)$ to the bilinear form $v(D) : L \times L \to K$ defined by the formula

$$v(D)(x, y) = D(x)(y) + D(y)(x),$$

and $w$ is sending a given symmetric bilinear invariant form $\psi : L \times L \to K$ to the class of the cocycle $\omega \in Z^3(L, K)$ defined by

$$\omega(x, y, z) = \psi([x, y], z)$$

(see, for example, [5], where a certain long exact sequence is obtained, of which this one is the beginning, and references therein for many earlier particular versions; this exact sequence was also established in [24, Proposition 7.2] with two additional terms on the right).

In the case where $L \simeq L^*$ as $L$-modules, the sequence (3.2) provides a way to evaluate $H^1(L, L)$ given $H^2(L, K)$ and $\mathcal{B}(L)$. The $L$-module isomorphism $L \simeq L^*$ implies the existence of a symmetric invariant non-degenerate form $\langle \cdot, \cdot \rangle$ on $L$. In terms of this form, $u$ is sending the class of a given cocycle $\varphi \in Z^2(L, K)$ to the class of the cocycle $u(\varphi) \in Z^1(L, L)$ defined by

$$\langle (u(\varphi))(x), y \rangle = \varphi(x, y),$$

and $v$ is sending the class of a given cocycle $D \in Z^1(L, L)$ to the bilinear form $v(D) : L \times L \to K$ defined by the formula

$$v(D)(x, y) = \langle D(x), y \rangle + \langle x, D(y) \rangle.$$

Turning to current Lie algebras, we will make even stronger assumption: that $L \simeq L^*$ and $A \simeq A^*$. Then, utilizing the results of preceding sections about $H^2(L \otimes A, K)$ and $\mathcal{B}(L \otimes A)$, we will derive results about $\mathrm{Der}(L \otimes A)$.

In the literature, given $H^1(L, L^*)$, the space $H^2(L, K)$ was computed for various Lie algebras $L$ (see, for example, [28], [5] and references therein). Here we utilize this connection in the other direction.

**Theorem 3.1.** *Let $L$ be a non-Abelian Lie algebra, $A$ an associative commutative algebra, both $L$ and $A$ finite-dimensional and with symmetric invariant non-degenerate bilinear form.*

*Then each derivation of $L \otimes A$ can be represented as the sum of decomposable linear maps $d \otimes \beta$, where $d : L \to L$ and $\beta : A \to A$ are of one of the following types*:

(i) $d([x,y]) = \lambda([d(x),y] + [x,d(y)])$, $\beta(ab) = \mu\beta(a)b$ *for certain* $\lambda, \mu \in K$ *such that* $\lambda\mu = 1$,
(ii) $d([x,y]) = \lambda[d(x),y]$, $\beta(ab) = \mu(\beta(a)b + a\beta(b))$ *for certain* $\lambda, \mu \in K$ *such that* $\lambda\mu = 1$,
(iii) $[d(x),y] + [x,d(y)] = 0$, $\beta(AA) = 0$, $\beta(a)b = a\beta(b)$,
(iv) $d([L,L]) = 0$, $[d(x),y] + [x,d(y)] = 0$, $\beta(a)b = a\beta(b)$,
(v) $d([L,L]) = 0$, $[d(x),x] = 0$, $\beta(a)b + a\beta(b) = 0$,
(vi) $[d(x),x] = 0$, $\beta(AA) = 0$, $\beta(a)b + a\beta(b) = 0$,
(vii) $d([L,L]) = 0$, $d(L) \subseteq \mathcal{Z}(L)$,
(viii) $d([L,L]) = 0$, $\beta(A) \subseteq \mathrm{Ann}(A)$,
(ix) $d(L) \subseteq \mathcal{Z}(L)$, $\beta(AA) = 0$,
(x) $\beta(AA) = 0$, $\beta(A) \subseteq \mathrm{Ann}(A)$.

**Proof.** By abuse of notation, let $\langle \cdot\,, \cdot \rangle$ denote a symmetric invariant non-degenerate bilinear form both on $L$ and $A$. Obviously, the tensor product of these forms defines a symmetric invariant non-degenerate bilinear form on $L \otimes A$, for which by even bigger abuse of notation we will use the same symbol:

$$\langle x \otimes a, y \otimes b \rangle = \langle x, y \rangle \langle a, b \rangle.$$

We have $L^* \simeq L$ as $L$-modules, $A^* \simeq A$ as $A$-modules, and $(L \otimes A)^* \simeq L \otimes A$ as $L \otimes A$-modules.

As a vector space, $H^1(L \otimes A, L \otimes A)$ can be represented as the direct sum of $\mathrm{Ker}\, v$ and $\mathrm{Im}\, v$, and the exact sequence (3.2) tells that $\mathrm{Ker}\, v = \mathrm{Im}\, u$ and $\mathrm{Im}\, v = \mathrm{Ker}\, w$.

By Theorem 1.1, $H^2(L \otimes A, K)$ is spanned by cohomology classes which can be represented by decomposable cocycles $\varphi \otimes \alpha$ for appropriate $\varphi : L \times L \to L$ and $\alpha : A \times A \to K$. For each such pair $\varphi$ and $\alpha$, there are unique linear maps $d : L \to L$ and $\beta : A \to A$ such that

$$\langle d(x), y \rangle = \varphi(x, y) \tag{3.3}$$

for any $x, y \in L$, and

$$\langle \beta(a), b \rangle = \alpha(a, b) \tag{3.4}$$

for any $a, b \in A$. Hence the decomposable linear map $d \otimes \beta : L \otimes A \to L \otimes A$ satisfies

$$\langle (d \otimes \beta)(x \otimes a), y \otimes b \rangle = (\varphi \otimes \alpha)(x \otimes a, y \otimes b),$$

i.e., coincides with $u(\varphi \otimes \alpha)$. Thus, $\mathrm{Im}\, u$ is spanned by cohomology classes whose representatives are decomposable derivations.

Similarly, by the proof of Theorem 2.1, $\mathcal{B}(L \otimes A)$ is spanned by decomposable elements $\varphi \otimes \alpha$, and $\mathrm{Ker}\, w$ is spanned by such elements of types (ii) and (iii), i.e., either $\varphi([L,L],L) = 0$ or $\alpha(AA,A) = 0$. Again, for each such element we can find $d : L \to L$ and $\beta : A \to A$ satisfying (3.3) and (3.4) respectively. Furthermore, we may assume that for each such $\varphi \otimes \alpha$, the maps $\varphi$ and $\alpha$ are either both symmetric, or both skew-symmetric, and hence both $d$ and $\beta$ are either self-adjoint or skew-self-adjoint, respectively, with respect