Analogs of the Orthogonal, Hamiltonian, Poisson, and Contact Lie Superalgebras in Characteristic 2

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ANALOGS OF THE ORTHOGONAL, HAMILTONIAN, POISSON, AND CONTACT LIE SUPERALGEBRAS IN CHARACTERISTIC 2

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Over algebraically closed fields of characteristic 2, the analogs of the orthogonal, symplectic, Hamiltonian, Poisson, and contact Lie superalgebras are described. The number of non-isomorphic types, and several properties of these algebras are unexpected, for example, interpretation in terms of exterior differential forms preserved is not applicable to one of these types. The divided powers of differential forms and related (co)homology are introduced.

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1. Introduction

1.1. *Setting*

In this paper I consider several problems with seemingly obvious or well-known answers which in reality are different. Take, e.g., the following statements:

> "Physicists and mathematicians mostly deal with symmetries embodied by real or complex Lie (super)algebras. Among these algebras, the simple ones are of prime interest, for mathematicians as well."

Sometimes other ground fields and other types of Lie (super)algebras are no less natural. For example, Witten suggested to consider all p-adic fields for a precise description of physical reality. In this paper the ground field $\mathbb{K}$ is an (algebraically closed) one of characteristic 2.

The spinor and oscillator representations of Lie algebras (to say nothing about superalgebras) are most natural to interpret in terms of quantization of the Poisson Lie superalgebras, see [20]. In the process we need not so much *simple* Lie (super)algebras, but rather Lie (super)algebras of nontrivial central extensions of orthogonal and symplectic Lie algebras. The Lie (super)algebras of outer derivations also naturally appear.

The complete description of deformations of the Poisson bracket is needed to define the spinor and oscillator representations. These representations[a] turned out to be the most vital ingredients in dealing with other infinite-dimensional Lie (super)algebras. Same applies to the deformations of the anti-bracket (tackled in [20]).

In this paper I describe the main Lie (super)algebras to be quantized (deformed) over the fields of characteristic 2.

Lie **super**algebras first appeared (under the incorrect name "graded Lie algebras": Lie superalgebras are not Lie algebras, graded or not) in 1930s, in topology, in cohomology theories — the language of topological field theories. Lie superalgebras appeared there over finite fields, and although the (co)homology were mainly considered in these questions over the 2-element field $\mathbb{Z}/2$ of residues modulo 2, no definition of Lie **super**algebras over $\mathbb{Z}/2$ was given until recently.

Although the Lie (super)algebras appearing in topology are solvable, it was recently discovered that at least some of them are subalgebras of certain simple Lie superalgebras, cf. [15] and references therein.

Lie (super)algebras over fields of positive characteristic, a.k.a. *modular Lie (super) algebras*, drew new attention with the advent of quantum groups $U_q(\mathfrak{g})$ (even over $\mathbb{C}$) whose representations resemble, if q is a primitive root of unity, representations of simple finite-dimensional modular Lie algebras.

The bilinear forms over fields of characteristic 2 were actively studied in 1930s by Albert and Arf but they were abandoned since then although such forms naturally appear recently in topological problems of the theory of real manifolds, for example, in singularity theory: As related to "symplectic analogs of Weyl groups" and related bilinear forms over $\mathbb{Z}/2$, cf. [9]. To consider corresponding analogs of the Calogero model, see [10, 11, 13], is a tempting open problem.

Symmetric bilinear forms over $\mathbb{Z}/2$ recently appeared in Lando's description of Vasiliev invariants [14].

It is therefore important, not only interesting, to investigate possible equivalences of bilinear forms, classify bilinear forms up to a reasonable equivalence (there are several non-obvious versions of such equivalences, and to select "reasonable" among them is one of the problems) and describe the Lie algebras that preserve the most interesting of such forms. These classifications over $\mathbb{Z}/2$ are complicated to perform, so we work here over algebraically closed fields.

Mathematicians tackled the classification problem of simple finite dimensional modular Lie algebras. The Kostrikin–Shafarevich conjecture (it describes the case of algebraically closed fields of characteristic $p > 7$), generalized to embrace $p > 3$, was recently proved [27, 2]. For a super version of the KSh-conjecture formulated together with a non-super version for $p = 3$ and 2, see [22].

We consider the two levels: linear algebra and differential geometry. On both levels we encounter surprising results. For details of the proofs, see [16].

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[a]Berezin [3] was the first to describe them for infinite dimensional orthogonal and symplectic Lie algebras; all discoveries of 1970s-80s on spinor and oscillator representations of Kac–Moody, Virasoro and other infinite dimensional Lie (super)algebras are based on Berezin's result; for details, see Neretin's works [25, 26].

1.2. *Main results*

It is shown that whereas all non-degenerate symmetric bilinear forms on any odd-dimensional space over a perfect field are equivalent, there are two (for $p = 2$) equivalence classes on even dimensional spaces; the Lie algebras that preserve these forms and the derived of these Lie algebras are non-isomorphic. Similarly, there are three types of ortho-orthogonal Lie superalgebras.

All these Lie (super)algebras have nontrivial Cartan prolongs, so we have four types of Hamiltonian Lie superalgebras — prolongs of $\mathfrak{oo}$, and four more types — prolongs of their first derived algebras $\mathfrak{oo}^{(1)}$; and one more type is prolong of the second derived algebras $\mathfrak{oo}^{(2)}$.

However, in presence of odd indeterminates, another stratification of the ortho-orthogonal Lie superalgebras is more reasonable: With regard of the traces on them; we should accordingly treat their Cartan prolongs.

Our $\mathfrak{h}_\Pi(2n; \underline{N}) := (\mathrm{id}, \mathfrak{o}_\Pi(2n))_{*,\underline{N}}$ and their derived algebras are not isomorphic to Lin's ones [24], at least, as graded Lie algebras, and hence, are "new": Ironically they are analogous to prolongs of split forms of $\mathfrak{o}$ for $p \neq 2$, but nobody noticed (at, least, nobody had published) that the split and non-split forms of $\mathfrak{o}$ are non-isomorphic (if $p = 2$), although at the level of (finite) groups this was known.

In the super setting, we introduce the divided powers of differential forms. This helps us to interpret several of the series of Hamiltonian Lie superalgebras as preserving an analog of symplectic form, but two of the series of Hamiltonian Lie (super)algebras defy such an interpretation; they cannot be realized as preserving an **exterior** 2-form; we realize them as preserving a tensor is a **non-exterior** 2-form.

The antibracket superalgebras and their quotients modulo center — analogs of Lie superalgebras Leites introduced in [17] — prolongs of $\mathfrak{pe}$; and prolongs of $\mathfrak{pe}^{(1)}$ (observe that, unlike the case where $p \neq 2$, we have $\mathfrak{pe}^{(1)} \not\simeq \mathfrak{spe}$ for $p = 2$) are also described.

Contact Lie superalgebras are described in terms of generating functions and as generalized Cartan prolongs. Lin's description of contact Lie algebras with many continuous parameters [23] is refuted.

For $\mathfrak{o}_I$ and $\mathfrak{oo}_{I\Pi}$, the prolongs of the trivial central extension of their derived algebras exist, but at the moment I cannot describe them lucidly and succinctly.

From our description of the contact Lie (super)algebras we see that, in characteristic 2, the non-degenerate (symplectic) 2-forms are sometimes replaced — quite unexpectedly — by degenerate ones. Moreover, the contact algebra is not the universal ambient Lie algebra with the given negative part, as is the case for $p \neq 2$.

1.3. *Notation*

We use the following notations for matrices, sometimes skipping the index:

$$\Pi_n = \begin{cases} \Pi_{2k} := \mathrm{antidiag}_2(1_k, 1_k) = \begin{pmatrix} 0 & 1_k \\ 1_k & 0 \end{pmatrix} & \text{if } n = 2k, \\ \Pi_{2k+1} := \mathrm{antidiag}_3(1_k, 1, 1_k) = \begin{pmatrix} 0 & 0 & 1_k \\ 0 & 1 & 0 \\ 1_k & 0 & 0 \end{pmatrix} & \text{if } n = 2k+1, \end{cases} \tag{1.1}$$

$$S_n = \mathrm{antidiag}_n(1, \ldots, 1), \quad Z_{2k} = \mathrm{diag}_k(\Pi_2, \ldots, \Pi_2).$$

Let parity be also denoted by Π since p denotes the characteristic of the ground field.

We identify a given bilinear form with its Gram matrix (in a fixed basis). Let $\mathfrak{o}_I(n), \mathfrak{o}_\Pi(n)$ and $\mathfrak{o}_S(n)$ be Lie algebras that preserve bilinear forms $1_n, \Pi_n$ and S_n, respectively.

Different normal forms of symmetric bilinear forms are used: In some problems, the form 1_n is used; in other problems (usually, mathematical ones) the forms Π_n and S_n are more preferable (so that the corresponding orthogonal Lie algebra has a Cartan subalgebra consisting of diagonal matrices).

Let $\Pi_{k\,|\,k} := \Pi_{2k}$, but considered as a supermatrix in the standard format $k\,|\,k$. Any square matrix is said to be *zero-diagonal* if it has only zeros on the main diagonal; let $ZD(n)$ be the space (Lie algebra if $p = 2$) of symmetric zero-diagonal $n \times n$-matrices. For any Lie algebra $\mathfrak{g}$ and $p \neq 2$, its *derived algebras* are defined to be

$$\mathfrak{g}^{(0)} := \mathfrak{g}, \quad \mathfrak{g}^{(1)} := [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}].$$

2. What the Lie Superalgebra in Characteristic 2 is

Let us give a naive definition of a Lie superalgebra for $p = 2$. (For a scientific one, as a Lie algebra in the category of supervarieties, needed, for example, for a rigorous study and interpretation of odd parameters of deformations, see [20, 21].) We define a Lie superalgebra as a superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that the even part $\mathfrak{g}_{\bar{0}}$ is a Lie algebra, the odd part $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module (made into the two-sided one by symmetry; more exactly, by *anti*-symmetry, but if $p = 2$, it is the same) and on $\mathfrak{g}_{\bar{1}}$ a *squaring* (roughly speaking, the halved bracket) is defined as a map

$$x \mapsto x^2 \quad \text{such that } (ax)^2 = a^2 x^2 \quad \text{for any } x \in \mathfrak{g}_{\bar{1}} \text{ and } a \in \mathbb{K}, \text{ and}$$
$$(x + y)^2 - x^2 - y^2 \text{ is a bilinear form on } \mathfrak{g}_{\bar{1}} \text{ with values in } \mathfrak{g}_{\bar{0}}. \tag{2.1}$$

(We use a minus sign, so the definition also works for $p \neq 2$.) The origin of this operation is as follows: If char $\mathbb{K} \neq 2$, then for any Lie superalgebra $\mathfrak{g}$ and any odd element $x \in \mathfrak{g}_{\bar{1}}$, the universal enveloping algebra $U(\mathfrak{g})$ contains the element x^2, which is equal to the even element $\frac{1}{2}[x, x] \in \mathfrak{g}_{\bar{0}}$. It is desirable to keep this operation for the case of $p = 2$, but, since it cannot be defined in the same way, we define it separately, and then define the bracket of odd elements to be (this equation is valid for $p \neq 2$ as well):

$$[x, y] := (x + y)^2 - x^2 - y^2. \tag{2.2}$$

We also assume, as usual, that

if $x, y \in \mathfrak{g}_{\bar{0}}$, then $[x, y]$ is the bracket on the Lie algebra;
if $x \in \mathfrak{g}_{\bar{0}}$ and $y \in \mathfrak{g}_{\bar{1}}$, then $[x, y] := l_x(y) = -[y, x] = -r_x(y)$, where l and r are the left and right $\mathfrak{g}_{\bar{0}}$-actions on $\mathfrak{g}_{\bar{1}}$, respectively.

The Jacobi identity involving two or three odd elements has now the following form:

$$[x^2, y] = [x, [x, y]] \quad \text{for any } x \in \mathfrak{g}_{\bar{1}}, \ y \in \mathfrak{g}. \tag{2.3}$$

If $\mathbb{K} \neq \mathbb{Z}/2$, we can replace the condition (2.3) on two odd elements by a simpler one:

$$[x, x^2] = 0 \quad \text{for any } x \in \mathfrak{g}_{\bar{1}}. \tag{2.4}$$

Because of the squaring, the definition of derived Lie superalgebras should be modified. For any Lie superalgebra $\mathfrak{g}$, set $\mathfrak{g}^{(0)} := \mathfrak{g}$ and (for $i \geq 0$)

$$\mathfrak{g}^{(i+1)} := [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] + \mathrm{Span}\{g^2 \,|\, g \in \mathfrak{g}_{\bar{1}}^{(i)}\}. \tag{2.5}$$

An even linear map $r : \mathfrak{g} \to \mathfrak{gl}(V)$ is said to be a *representation of the Lie superalgebra* $\mathfrak{g}$ in the module V if

$$
\begin{aligned}
r([x, y]) &= [r(x), r(y)] & &\text{for any } x, y \in \mathfrak{g}; \\
r(x^2) &= (r(x))^2 & &\text{for any } x \in \mathfrak{g}_{\bar{1}}.
\end{aligned}
\tag{2.6}
$$

2.1. *Examples: Lie superalgebras preserving non-degenerate forms*

We say ([16]) that two bilinear forms B and B' on a superspace V are *equivalent* if there is an even non-degenerate linear map $M : V \to V$ such that

$$B'(x, y) = B(Mx, My) \quad \text{for all } x, y \in V. \tag{2.7}$$

We fix some basis in V and identify a bilinear form with its *Gram matrix* in this basis; let us also identify any linear operator on V with its matrix. Then two bilinear forms (rather supermatrices) are equivalent if there is an even invertible matrix M such that

$$B' = MBM^T, \quad \text{where } T \text{ is for transposition.} \tag{2.8}$$

A bilinear form B on V is said to be *symmetric* if $B(v, w) = B(w, v)$ for any $v, w \in V$; it is *anti-symmetric* if $B(v, v) = 0$ for any $v \in V$.

A linear map F is said to preserve a bilinear form B, if

$$B(Fx, y) + (-1)^{\Pi(x)\Pi(F)} B(x, Fy) = 0 \quad \text{for all } x, y \in V.$$

All linear maps preserving a given bilinear form constitute a Lie sub(super)algebra $\mathfrak{aut}_B(V)$ of $\mathfrak{gl}(V)$ (denoted $\mathfrak{aut}_B(n) \subset \mathfrak{gl}(n)$ in matrix realization).

Let us consider the case of purely even space V of dimension n over a field of characteristic $p \neq 2$. If $p \neq 2$, every nonzero form B can be uniquely represented as the sum of a symmetric and an anti-symmetric form and it is possible to consider automorphisms and equivalence classes of each summand separately. If the ground field is perfect (i.e., such that every element of $\mathbb{K}$ has a square root[b]), then there is just one equivalence class of non-degenerate symmetric even forms, and the corresponding Lie algebra $\mathfrak{aut}_B(V)$ is called *orthogonal* and denoted $\mathfrak{o}_B(n)$ (or just $\mathfrak{o}(n)$). Non-degenerate anti-symmetric forms over V exist only if n is even; in this case, there is also just one equivalence class of non-degenerate anti-symmetric even forms; the corresponding Lie algebra $\mathfrak{aut}_B(n)$ is called *symplectic* and denoted $\mathfrak{sp}_B(2k)$ (or just $\mathfrak{sp}(2k)$). Both algebras $\mathfrak{o}(n)$ and $\mathfrak{sp}(2k)$ are simple.

If $p = 2$, then, instead of a unique representation of a given form as a sum of an anti-symmetric and symmetric forms, we have a subspace of symmetric forms and the quotient space of non-symmetric forms; in particular, it is not immediately clear how to classify non-symmetric forms. Instead of orthogonal and symplectic Lie algebras we have two different

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[b]Since $a^2 - b^2 = (a - b)^2$ if $p = 2$, it follows that no element can have two distinct square roots.

types of orthogonal Lie algebras (see Theorem 2.1). Either the derived algebras of these algebras or their quotient modulo center are simple if n is large enough, so the canonical expressions of the forms B are needed as a step towards classification of simple Lie algebras in characteristic 2 which is an open problem, and as a step towards a version of this problem for Lie superalgebras, even more open.

In [16], I showed that with respect to the above natural equivalence of forms (2.8), every **even** symmetric non-degenerate form on a superspace of dimension $n_{\bar{0}} \,|\, n_{\bar{1}}$ over a perfect field of characteristic 2 is equivalent to a form of the shape (here: $i = \bar{0}$ or $\bar{1}$ and each n_i may equal to 0)

$$B = \begin{pmatrix} B_{\bar{0}} & 0 \\ 0 & B_{\bar{1}} \end{pmatrix}, \quad \text{where } B_i = \begin{cases} 1_{n_i} & \text{if } n_i \text{ is odd;} \\ \text{either } 1_{n_i} \text{ or } \Pi_{n_i} & \text{if } n_i \text{ is even.} \end{cases}$$

In other words, the bilinear forms with matrices 1_n and Π_n are equivalent if n is odd and non-equivalent if n is even. The Lie superalgebra preserving B — by analogy with the orthosymplectic Lie superalgebras $\mathfrak{osp}$ in characteristic 0 we call it *ortho-orthogonal* and denote $\mathfrak{oo}_B(n_{\bar{0}} \,|\, n_{\bar{1}})$ — is spanned by the supermatrices which in the standard format are of the form

$$\begin{pmatrix} A_{\bar{0}} & B_{\bar{0}} C^T B_{\bar{1}}^{-1} \\ C & A_{\bar{1}} \end{pmatrix}, \quad \begin{array}{l} \text{where } A_{\bar{0}} \in \mathfrak{o}_{B_{\bar{0}}}(n_{\bar{0}}), \quad A_{\bar{1}} \in \mathfrak{o}_{B_{\bar{1}}}(n_{\bar{1}}), \quad \text{and} \\ C \text{ is arbitrary } n_{\bar{1}} \times n_{\bar{0}} \text{ matrix.} \end{array}$$

Since, as is easy to see,

$$\mathfrak{oo}_{\Pi I}(n_{\bar{0}} \,|\, n_{\bar{1}}) \simeq \mathfrak{oo}_{I\Pi}(n_{\bar{1}} \,|\, n_{\bar{0}}),$$

we do not have to consider the Lie superalgebra $\mathfrak{oo}_{\Pi I}(n_{\bar{0}} \,|\, n_{\bar{1}})$ separately unless we study Cartan prolongations where the difference between these two incarnations of one algebra is vital: For a linear superspace V of superdimension $n_{\bar{0}} \,|\, 2k_{\bar{1}}$, the prolong $(V, \mathfrak{oo}_{I\Pi}^{(1)}(V))_*$ is finite-dimensional, but the prolong $(\Pi(V), \mathfrak{oo}_{\Pi I}^{(1)}(V))_* \simeq (\Pi(V), \mathfrak{oo}_{\Pi I}^{(1)}(\Pi(V)))_*$ is infinite-dimensional. (Structurally, the latter prolong is the closest $p = 2$ analog of the Lie super-algebra of Hamiltonian vector fields (there is a one-to-one correspondence between their bases and almost identical presentations) while the former one, $(V, \mathfrak{oo}_{I\Pi}^{(1)}(V))_*$, is analogous to the automorphism algebra of the straightforward (but physically less meaningful than super Minkowski spaces) super analog of the Riemann geometry.) The difference between $(V, \mathfrak{oo}_{I\Pi}(V))_*$ and $(\Pi(V), \mathfrak{oo}_{I\Pi}(V))_* \simeq (\Pi(V), \mathfrak{oo}_{\Pi I}(\Pi(V)))_*$ is not so drastic.

For an **odd** symmetric bilinear form B on a superspace of dimension $(n_{\bar{0}} \,|\, n_{\bar{1}})$ over a field of characteristic 2 to be non-degenerate, we need $n_{\bar{0}} = n_{\bar{1}}$, and every such form B is equivalent to $\Pi_{k \,|\, k}$, where $k = n_{\bar{0}} = n_{\bar{1}}$. This form is preserved by linear transformations with supermatrices in the standard format of the shape

$$\begin{pmatrix} A & C \\ D & A^T \end{pmatrix}, \quad \text{where } A \in \mathfrak{gl}(k), \quad C \text{ and } D \text{ are symmetric } k \times k \text{ matrices.} \tag{2.9}$$

As over $\mathbb{C}$ or $\mathbb{R}$, the Lie superalgebra of linear maps preserving B will be referred to as *periplectic*, as A. Weil suggested; it is denoted $\mathfrak{pe}_B(k)$ or just $\mathfrak{pe}(k)$. Note, though, that even

the superdimensions of $\mathfrak{aut}_B(k)$ for $p = 2$ and $p = 0$ differ both in the case of even and odd form B.

> **The fact that two bilinear forms are inequivalent does not, generally, imply that the Lie (super)algebras that preserve them are not isomorphic.** (2.10)

In [16], I proved that for the *non-degenerate symmetric* forms, this implication (2.10) is, however, true (except for $\mathfrak{oo}_{I\Pi} \simeq \mathfrak{oo}_{\Pi I}$), and therefore we have several types of non-isomorphic Lie (super)algebras (except for occasional isomorphisms intermixing the types, e.g., $\mathfrak{oo}_{\Pi\Pi}^{(1)}(6 \,|\, 2) \simeq \mathfrak{pe}^{(1)}(4)$).

The problem of describing preserved bilinear forms has two levels: we can consider *linear transformations* (Linear Algebra) and *arbitrary coordinate changes* (Differential Geometry).

In the literature, both levels are completely investigated, except for the case where $p = 2$. More precisely, the fact that the non-split and split forms of the Lie algebras that preserve the symmetric forms are not always isomorphic was never mentioned. A similar fact is known on the Chevalley group level preserving quadratic forms, cf. [30]; although similar, the classification of non-degenerate quadratic form and groups preserving them has no relation — if $p = 2$ — with two types of non-degenerate symmetric forms and Lie algebras preserving them.

2.2. *Known facts: The case $p = 2$*

(1) Arf has discovered *the Arf invariant* — an important invariant of non-degenerate quadratic forms in characteristic 2; for an exposition, see [7]. Two such forms are equivalent if and only if their Arf invariants are equal.

With any symmetric bilinear form B a quadratic form $Q(x) := B(x, x)$ is associated. The other way round, given a quadratic form Q, we define a symmetric bilinear form, called *the polar form* of Q, by setting

$$B_Q(x, y) = Q(x + y) - Q(x) - Q(y).$$

If $p = 2$, the correspondence $Q \leftrightarrow B_Q$ is not one-to-one. More precisely:

- the Arf invariant cannot be used for classification of symmetric bilinear forms because one symmetric bilinear form can serve as the polar form for two non-equivalent (and having different Arf invariants) quadratic forms;
- not every symmetric bilinear form can be represented as a polar form.

(2) Albert [1] classified symmetric bilinear forms over a field of characteristic 2 and proved that

(1) two anti-symmetric forms of equal ranks are equivalent;
(2) every non-anti-symmetric form has a matrix which is equivalent to a diagonal matrix;
(3) if $\mathbb{K}$ is perfect, then every two non-anti-symmetric forms of equal ranks are equivalent.

(3) Albert also obtained some results on the classification of quadratic forms over a field $\mathbb{K}$ of characteristic 2 (considered as elements of the quotient space of all bilinear forms modulo

the space of anti-symmetric forms). In particular, he showed that if $\mathbb{K}$ is algebraically closed, then every quadratic form is equivalent to exactly one of the forms

$$x_1 x_{r+1} + \cdots + x_r x_{2r} \quad \text{or} \quad x_1 x_{r+1} + \cdots + x_r x_{2r} + x_{2r+1}^2, \tag{2.11}$$

where $2r$ is the rank of the form.

Theorem 2.1 ([1, 16]). *Let $\mathbb{K}$ be a perfect field of characteristic 2. Let V be an n-dimensional space over $\mathbb{K}$.*

(1) *For n odd, there is only one equivalence class of non-degenerate symmetric bilinear forms on V.*

(2) *For n even, there are two equivalence classes of non-degenerate symmetric bilinear forms, one contains 1_n and the other one (none of the Gram matrices of this class has a nonzero entry on the main diagonal) contains S_n and Π_n.*

Remark 2.2. For the purposes of representation theory, it is desirable to have the Cartan subalgebra consisting of matrices with nonzero entries only on the main diagonal (or as close as possible). So it may be preferable to replace 1_{n_i} in the above bilinear forms by an equivalent form

$$\begin{aligned}
\operatorname{diag}(1_2, \Pi_{2k-2}) &\sim \operatorname{diag}(1_2, S_{2k-2}) &&\text{if } n_i = 2k, \\
\operatorname{diag}(1, \Pi_{2k}) &\sim \Pi_{2k+1} \sim S_{2k+1} &&\text{if } n_i = 2k+1.
\end{aligned} \tag{2.12}$$

2.3. New results

In view of (2.10) the next Lemma is nontrivial.

Lemma 2.3. (1) *The Lie algebras $\mathfrak{o}_I(2k)$ and $\mathfrak{o}_\Pi(2k)$ are not isomorphic (though are of the same dimension); the same applies to their derived algebras:*

(2) $\mathfrak{o}_I^{(1)}(2k) \not\simeq \mathfrak{o}_\Pi^{(1)}(2k)$, *though* $\dim \mathfrak{o}_I^{(1)}(2k) = \dim \mathfrak{o}_\Pi^{(1)}(2k)$;

(3) $\mathfrak{o}_I^{(2)}(2k) \not\simeq \mathfrak{o}_\Pi^{(2)}(2k)$ *unless $k = 1$.*

Based on these results, I describe the two types of analogs of the Poisson bracket in the purely even case, four types of analogs of the Poisson bracket in the super case, and (just one) contact bracket, cf. [21]. Similar results for the odd bilinear form yield a description of the anti-bracket (a.k.a. Buttin bracket), and the (peri)contact bracket, cf. [21].

If $p \neq 2$, the quotient of the Poisson $\mathfrak{po}(2n \,|\, m)$ (resp. the Buttin $\mathfrak{b}(n)$) Lie (super)algebra modulo the center coincides with Cartan prolongs of orthogonal/orthosymplectic (resp. periplectic) Lie (super)algebra, called the Lie (super)algebra of Hamiltonian vector fields $\mathfrak{h}(2n; \underline{N} \,|\, m)$, and $\mathfrak{le}(n; \underline{N})$, respectively. If $p = 2$, these prolongs contain elements corresponding to non-existing functions — for example, squares of odd variables.

Altogether there are 5 types of analogs of Hamiltonian Lie algebras:

(1) There are two types of the "full" Hamiltonian Lie algebras — $\mathfrak{h}_I(n; \underline{N})$ and $\mathfrak{h}_\Pi(2n; \underline{N})$ — the Cartan prolongs of the respective Lie algebras $\mathfrak{o}_I(n)$ and $\mathfrak{o}_\Pi(n)$.

(2) There are two types of Cartan prolongs of the derived orthogonal Lie algebras $\mathfrak{o}_B^{(1)}$. These prolongs "little" Hamiltonian Lie algebras — $\mathfrak{lh}_I(n; \underline{N})$ and $\mathfrak{lh}_\Pi(2k; \underline{N})$ — consist of elements (vector fields) $A = \sum_{1 \leq i \leq n} A_i \partial_i$ of the "full" Lie algebras $\mathfrak{h}_I(n; \underline{N})$ and

$\mathfrak{h}_{\Pi}(2k; \underline{N})$ satisfying the following conditions:

$$
\begin{aligned}
&\text{for } \mathfrak{o}_I^{(1)}(n)\colon \ \partial_i A_i = 0 && \text{for all } i = 1, \ldots, n; \\
&\text{for } \mathfrak{o}_{\Pi}^{(1)}(2k)\colon \ \partial_i A_{k+i} = \partial_{k+i} A_i = 0 && \text{for all } i = 1, \ldots, k.
\end{aligned}
\tag{2.13}
$$

(3) There is $\mathfrak{slh}_{\Pi}(2k; \underline{N})$, the Cartan prolong of the second derived Lie algebra $\mathfrak{o}_{\Pi}^{(2)}$ consisting of divergence-free elements of $\mathfrak{lh}_{\Pi}(2k; \underline{N})$.

Altogether there are 9 types of analogs of Hamiltonian Lie superalgebras:

(1) There are four types of the "full" Hamiltonian superalgebras — $\mathfrak{h}_{II}(n; \underline{N} \mid m)$, $\mathfrak{h}_{I\Pi}(n; \underline{N} \mid 2m)\mathfrak{h}_{\Pi I}(2n; \underline{N} \mid m)$, $\mathfrak{h}_{\Pi\Pi}(2n; \underline{N} \mid 2m)$ — the Cartan prolongs of the respective Lie superalgebras $\mathfrak{oo}_{II}(n \mid m)$, $\mathfrak{oo}_{I\Pi}(n \mid 2m)$, $\mathfrak{oo}_{\Pi I}(2n \mid m)$, $\mathfrak{oo}_{\Pi\Pi}(2n \mid 2m)$.

(2) There are four types of Cartan prolongs of the derived Lie superalgebras $\mathfrak{oo}_B^{(1)}$, the "little" Hamiltonian superalgebras — $\mathfrak{lh}_{II}(n_{\bar{0}}; \underline{N} \mid n_{\bar{1}})$, $\mathfrak{lh}_{I\Pi}(n_{\bar{0}}; \underline{N} \mid 2k_{\bar{1}})$ $\mathfrak{lh}_{\Pi I}(2k_{\bar{0}}; \underline{N} \mid n_{\bar{1}})$, $\mathfrak{lh}_{\Pi\Pi}(2k_{\bar{0}}; \underline{N} \mid 2k_{\bar{1}})$. These prolongs consist of elements $A = \sum_{1 \le i \le n_{\bar{0}} + n_{\bar{1}}} A_i \partial_i$ of the "full" superalgebras satisfying the following conditions, where $n_{\bar{0}} = 2k_{\bar{0}}$ and $n_{\bar{1}} = 2k_{\bar{1}}$ if $n_{\bar{0}}$ or $n_{\bar{1}}$ is even:

$$
\begin{aligned}
&\text{for } \mathfrak{oo}_{II}^{(1)}(n_{\bar{0}} \mid n_{\bar{1}})\colon && \operatorname{div} A = 0; \\
&\text{for } \mathfrak{oo}_{I\Pi}^{(1)}(n_{\bar{0}} \mid 2k_{\bar{1}})\colon && \partial_i A_i = 0 \quad \text{for all } i = 1, \ldots, n_{\bar{0}}; \\
&\text{for } \mathfrak{oo}_{\Pi I}^{(1)}(2k_{\bar{0}} \mid n_{\bar{1}})\colon && \partial_i A_i = 0 \quad \text{for all } i = 2k_{\bar{0}} + 1, \ldots, 2k_{\bar{0}} + n_{\bar{1}}; \\
&\text{for } \mathfrak{oo}_{\Pi\Pi}^{(1)}(2k_{\bar{0}} \mid 2k_{\bar{1}})\colon && \partial_i A_{k_{\bar{0}}+i} = \partial_{k_{\bar{0}}+i} A_i = 0 \quad \text{for all } i = 1, \ldots, k_{\bar{0}}; \quad \text{and} \\
&&& \partial_{2k_{\bar{0}}+i} A_{2k_{\bar{0}}+k_{\bar{1}}+i} = \partial_{2k_{\bar{0}}+k_{\bar{1}}+i} A_{2k_{\bar{0}}+i} = 0 \quad \text{for all } i = 1, \ldots, k_{\bar{1}}.
\end{aligned}
\tag{2.14}
$$

(3) There is $\mathfrak{slh}_{\Pi\Pi}(2k_{\bar{0}}; \underline{N} \mid 2k_{\bar{1}})$, the Cartan prolong of $\mathfrak{oo}_{\Pi\Pi}^{(2)}(2k_{\bar{0}} \mid 2k_{\bar{1}})$, consisting of divergent-free elements of the prolong of $\mathfrak{lh}_{\Pi\Pi}(2k_{\bar{0}}; \underline{N} \mid 2k_{\bar{1}})$.

There are three types of analogs of the anti-bracket (Buttin) Lie superalgebras and their quotients modulo center:

(1) There is "full" Lie superalgebra $\mathfrak{le}(k; \underline{N}) := \mathfrak{le}(k; \underline{N} \mid k)$, the Cartan prolong of $\mathfrak{pe}(k \mid k)$;
(2) There is "little" Lie superalgebra $\mathfrak{lle}(k; \underline{N}) := \mathfrak{le}(k; \underline{N} \mid k)$, the Cartan prolongs of $\mathfrak{pe}^{(1)}(k \mid k)$, consisting of the elements $A = \sum_{1 \le i \le n} A_i \partial_i$ of the "full" superalgebra satisfying the condition

$$
\partial_i A_{k+i} = \partial_{k+i} A_i = 0 \quad \text{for all } i = 1, \ldots, k.
\tag{2.15}
$$

(3) There is $\mathfrak{slle}(k; \underline{N})$, the Cartan prolong of $\mathfrak{pe}^{(2)}(k \mid k)$, consisting of divergence-free elements of $\mathfrak{lle}(k; \underline{N})$.

We offer several analogs of the notion "differential form": The "usual" one, the divided power ones, and the tensor one. Note that the usual, valid for $p \ne 2$, interpretation of the Hamiltonian Lie superalgebra $\mathfrak{h}$ as the one preserving a non-degenerate closed differential 2-form (neither "usual" nor divided power ones) is **not** applicable to some of the analogs of $\mathfrak{h}$ introduced below.

Particular cases of Hamiltonian Lie (super)algebras were partly investigated in [24, 22].

3. Functions, Vector Fields, and Differential Forms for $p > 0$

3.1. *Divided powers*

Let us consider the supercommutative superalgebra $\mathbb{C}[x]$ of polynomials in a indeterminates $x = (x_1, \ldots, x_a)$, for convenience ordered in a "standard format", i.e., so that the first m indeterminates are even and the rest n ones are odd ($m + n = a$). Among the integer bases of $\mathbb{C}[x]$ (i.e., the bases, in which the structure constants are integers), there are two canonical ones, — the usual, monomial, one and the basis of *divided powers*, which is constructed in the following way.

For any multi-index $\underline{r} = (r_1, \ldots, r_a)$, where $r_1, \ldots, r_m$ are non-negative integers, and $r_{m+1}, \ldots, r_n$ are 0 or 1, we set

$$u_i^{(r_i)} := \frac{x_i^{r_i}}{r_i!} \quad \text{and} \quad u^{(\underline{r})} := \prod_{i=1}^{a} u_i^{(r_i)}.$$

These $u^{(\underline{r})}$ form an integer basis of $\mathbb{C}[x]$. Clearly, their multiplication relations are

$$u^{(\underline{r})} \cdot u^{(\underline{s})} = \prod_{i=m+1}^{n} \min(1, 2 - r_i - s_i) \cdot (-1)^{\sum_{m < i < j \le a} r_j s_i} \cdot \binom{\underline{r} + \underline{s}}{\underline{r}} u^{(\underline{r} + \underline{s})},$$

$$\text{where} \quad \binom{\underline{r} + \underline{s}}{\underline{r}} := \prod_{i=1}^{m} \binom{r_i + s_i}{r_i}. \tag{3.1}$$

In what follows, for clarity, we will write exponents of divided powers in parentheses, as above, especially if the usual exponents might be encountered as well.

Now, for an arbitrary field $\mathbb{K}$ of characteristic $p > 0$, we may consider the supercommutative superalgebra $\mathbb{K}[u]$ spanned by elements $u^{(\underline{r})}$ with multiplication relations (3.1). For any m-tuple $\underline{N} = (N_1, \ldots, N_m)$, where N_i are either positive integers or infinity, denote

$$\mathcal{O}(m; \underline{N}) := \mathbb{K}[u; \underline{N}] := \mathrm{Span}_{\mathbb{K}}(u^{(\underline{r})} \mid r_i < p^{N_i} \text{ for } i \le m \text{ and } r_i = 0 \text{ or } 1 \text{ for } i > m) \tag{3.2}$$

(we assume that $p^{\infty} = \infty$). As is clear from (3.1), $\mathbb{K}[u; \underline{N}]$ is a subalgebra of $\mathbb{K}[u]$. The algebra $\mathbb{K}[u]$ and its subalgebras $\mathbb{K}[u; \underline{N}]$ are called the *algebras of divided powers*; they can be considered as analogs of the polynomial algebra.

In what follows we will drop the parameter $\underline{N}$ and write just $\mathcal{O}(m)$ or even $\mathcal{O}(x)$ (having in mind $\mathcal{O}(u)$), when the exact value of $\underline{N}$ is not important.

Only one of these numerous algebras of divided powers $\mathcal{O}(n; \underline{N})$ are indeed generated by the indeterminates declared: If $N_i = 1$ for all i. Otherwise, in addition to the u_i, we have to add $u_i^{(p^{k_i})}$ for all $i \le m$ and all k_i such that $1 < k_i < N_i$ to the list of generators. Since any derivation D of a given algebra is determined by the values of D on the generators, we see that $\mathfrak{der}(\mathcal{O}(m; \underline{N}))$ has more than m functional parameters (coefficients of the analogs of partial derivatives) if $N_i \neq 1$ for at least one i. Define *distinguished partial derivatives* by setting

$$\partial_i(u_j^{(k)}) = \delta_{ij} u_j^{(k-1)} \quad \text{for all } k < p^{N_j}. \tag{3.3}$$

The simple vectorial Lie algebras over $\mathbb{C}$ have only one parameter: the number of indeterminates. If char $\mathbb{K} = p > 0$, the vectorial Lie algebras acquire one more parameter: $\underline{N}$.

For Lie superalgebras, $\underline{N}$ only concerns the even indeterminates. In what follows, by abuse of notation, we often denote the divided power indeterminates by x^r, not $u^{(r)}$.

The Lie (super)algebra of all derivations $\mathfrak{der}(\mathcal{O}(m; \underline{N}))$ turns out to be not so interesting as its *Lie subsuperalgebra of distinguished derivations*: The *general vectorial Lie algebra of distinguished derivations* is denoted by (W is in honor of Witt who in 1930s considered the characteristic p version of the "centerless Virasoro algebra")

$$\mathfrak{vect}(m; \underline{N} \mid n) \quad \text{a.k.a } W(m; \underline{N} \mid n) \text{ a.k.a}$$
$$\mathfrak{der}_{\text{dist}} \, \mathbb{K}[u; \underline{N}] = \text{Span}_{\mathbb{K}}(u^{\underline{(r)}} \partial_k \mid r_i < p^{N_i} \quad \text{for } i \leq m \tag{3.4}$$
$$\text{and } r_i = 0 \text{ or } 1 \text{ for } i > m; 1 \leq k \leq a).$$

3.2. CTS-prolongations in the modular case

Let DS^k be the operation of rising to the kth divided symmetric power and $DS^{\cdot} := \oplus_k DS^k$; we set

$$i: DS^{k+1}(\mathfrak{g}_-)^* \otimes \mathfrak{g}_- \to DS^k(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-^* \otimes \mathfrak{g}_-;$$
$$j: DS^k(\mathfrak{g}_-)^* \otimes \mathfrak{g}_0 \to DS^k(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-^* \otimes \mathfrak{g}_- \tag{3.5}$$

be the natural maps. Let the $(i, \underline{N})$-th prolong of the pair $(\mathfrak{g}_-, \mathfrak{g}_0)$ be:

$$\mathfrak{g}_{k, \underline{N}} = (j(DS^{\cdot}(\mathfrak{g}_-)^* \otimes \mathfrak{g}_0) \cap i(DS^{\cdot}(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-))_{k, \underline{N}}, \tag{3.6}$$

where the subscript k in the right hand side singles out the component of degree k. Together with $\mathcal{O}(n; \underline{N})$ all prolongs acquire one more — shearing — parameter: $\underline{N}$.

Superization of the prolongation construction is immediate.

Set $(\mathfrak{g}_-, \mathfrak{g}_0)_{*, \underline{N}} = \oplus_{i \geq -d} \mathfrak{g}_{i, \underline{N}}$; then, as is easy to verify, $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ is a Lie (super)algebra. Provided $\mathfrak{g}_0$ acts on $\mathfrak{g}_{-1}$ without kernel, $(\mathfrak{g}_-, \mathfrak{g}_0)_{*, \underline{N}}$ is a subalgebra of $\mathfrak{vect}(m; \underline{N} \mid n)$ for $m \mid n = \text{sdim } \mathfrak{g}_-$ and some $\underline{N}$.

3.2.1. Superizations of the Cartan prolongs and its Tanaka–Shchepochkina generalization

A necessary condition for a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ of finite depth to be simple is $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$; so being interested in simple algebras, we note, that if $[\mathfrak{g}_{-1}, \mathfrak{g}_1] \neq \mathfrak{g}_0$ in $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*, \underline{N}}$, we can — if $p \neq 2$ — replace $\mathfrak{g}_0$ by $[\mathfrak{g}_{-1}, \mathfrak{g}_1]$, and the resulting space is still a Lie algebra.

If $p = 2$, then we cannot, in general, replace in $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ the Lie **super**algebra $\mathfrak{g}_0$ by $[\mathfrak{g}_1, \mathfrak{g}_{-1}]$, since $[\mathfrak{g}_1, \mathfrak{g}_{-1}]$ may be not closed under squaring. So, if we want to replace $\mathfrak{g}_0$ by the minimal possible space containing $[\mathfrak{g}_1, \mathfrak{g}_{-1}]$ and closed relative to the bracket and squaring, we should take

$$\overline{[\mathfrak{g}_1, \mathfrak{g}_{-1}]} := [\mathfrak{g}_1, \mathfrak{g}_{-1}] + \text{Span}\{g^2 \mid g \in [\mathfrak{g}_1, \mathfrak{g}_{-1}]_{\bar{1}}\}. \tag{3.7}$$

Example 3.1. Now, recall that the 1-form α on a superdomain M is said to be *contact* if it singles out a non-integrable distribution in the tangent bundle TM and $d\alpha$ is non-degenerate on the fibers of this distribution; for details, see [19]. The Lie superalgebra that preserves the distribution singled out by a contact form is said to consist of contact vector fields.

What is the Lie algebra of contact vector fields on $\mathbb{K}^n$ in these terms? Denote by $\mathfrak{hei}(2n)$ the Heisenberg Lie algebra: its space is $W \oplus \mathbb{K} \cdot z$, where W is a $2n$-dimensional space endowed with a non-degenerate anti-symmetric bilinear form B and the bracket in $\mathfrak{hei}(2n)$ is given by the following relations:

$$z \text{ is in the center and } [v, w] = B(v, w) \cdot z \quad \text{for any } v, w \in W. \tag{3.8}$$

Clearly, for $p \neq 2$, we have the following realization of the Lie algebra of contact vector fields (here: $\mathfrak{c}(\mathfrak{g}) := \mathfrak{g} \oplus \mathbb{K}z$):

$$\mathfrak{k}(2n + 1) \cong (\mathfrak{hei}(2n), \mathfrak{c}(\mathfrak{sp}(2n)))_{*,\underline{N}}. \tag{3.9}$$

3.3. *On vectorial Lie superalgebras, there are TWO analogs of trace*

More precisely, there are *traces* and their Cartan prolongs, called *divergencies*. On any Lie (super)algebra $\mathfrak{g}$ over a field $\mathbb{K}$, a *trace* is any map $\mathrm{tr} : \mathfrak{g} \to \mathbb{K}$ such that

$$\mathrm{tr}([\mathfrak{g}, \mathfrak{g}]) = 0. \tag{3.10}$$

The straightforward analogs of the trace are, therefore, the linear functionals that vanish on $\mathfrak{g}^{(1)} := [\mathfrak{g}, \mathfrak{g}]$ (the first derived is often briefly denoted by $\mathfrak{g}'$ if $p = 0$); the number of linearly independent traces is equal to $\dim \mathfrak{g}/\mathfrak{g}^{(1)}$; if $\mathfrak{g}$ is a Lie superalgebra, these traces are called supertraces and they can be even or odd. Each trace is defined up to a nonzero scalar factor selected *ad lib*.

Let now $\mathfrak{g}$ be a $\mathbb{Z}$-graded vectorial Lie superalgebra with $\mathfrak{g}_- := \oplus_{i<0}\, \mathfrak{g}_i$ generated by $\mathfrak{g}_{-1}$, and let tr be a (super)trace on $\mathfrak{g}_0$. The *divergence* $\mathrm{div} : \mathfrak{g} \to \mathcal{F}$ is an $\mathrm{ad}_{\mathfrak{g}_{-1}}$-invariant prolongation of the trace satisfying the following conditions:

$$\mathrm{div} : \mathfrak{g} \to \mathcal{F} \quad \text{preserves the degree, i.e., } \deg \mathrm{div} = 0;$$
$$X_i(\mathrm{div} D) = \mathrm{div}[X_i, D] \quad \text{for all elements } X_i \text{ that span } \mathfrak{g}_{-1};$$
$$\mathrm{div}\,|_{\mathfrak{g}_0} = \mathrm{tr};$$
$$\mathrm{div}\,|_{\mathfrak{g}_-} = 0.$$

By construction, the Lie (super)algebra $\mathfrak{sg} := \mathrm{Ker}\,\mathrm{div}\,|_{\mathfrak{g}}$ of divergence-free elements of $\mathfrak{g}$ is the complete prolong of $(\mathfrak{g}_-, \mathrm{Ker}\,\mathrm{tr}\,|_{\mathfrak{g}_0})$. This fact explains why we say that div is the prolongation of the trace.

Strictly speaking, divergences are not traces (they do not satisfy (3.10)) but for vectorial Lie (super)algebras they embody the idea of the trace (understood as property (3.10)) better than the traces. We denote the *special* (*divergence free*) subalgebra of a vectorial algebra $\mathfrak{g}$ by $\mathfrak{sg}$, e.g., $\mathfrak{svect}(n\,|\,m)$. If there are several traces on $\mathfrak{g}_0$, there are several types of special subalgebras of $\mathfrak{g}$ and we need a different name for each.

3.4. *Symmetric and exterior differential forms*

In what follows, as is customary in supergeometry, we use the anti-symmetric $\wedge$ product for the analogs of the *exterior differential* forms, and the symmetric $\circ$ product for the *symmetric differential* forms, e.g., analogs of the metrics. We can also consider the divided power versions of the exterior and symmetric forms. Usually we suppress the $\wedge$ or $\circ$ signs, since all is clear from the context, unless both multiplications are needed simultaneously.

Considering *exterior* differential forms, we use divided powers $dx_i^{(\wedge k)}$ with multiplication relations (3.1), where the indeterminates are now the dx_i of parity $\Pi(x_i) + \bar{1}$, and the Lie derivative along the vector field X given by the formula

$$L_X(dx_i^{(\wedge k)}) = (L_X dx_i) \wedge dx_i^{(\wedge k-1)}.$$

Note that if we consider divided power differential forms in characteristic 2, then, for x_i odd, we have $dx_i \wedge dx_i = 2(dx_i^{(\wedge 2)}) = 0$.

Considering divided powers of chains and cochains of Lie superalgebras affects the formula for the (co)chain differentials. For cochains of a given Lie superalgebra $\mathfrak{g}$, this only means that a divided power of an odd element must be differentiated as a whole:

$$d(\varphi^{(\wedge k)}) = d\varphi \wedge \varphi^{(\wedge(k-1))} \quad \text{for any } \varphi \in (\mathfrak{g}^*)_{\bar{1}}.$$

For chains, the modification is a little more involved: Let $g_1, \ldots, g_n$ be a basis of $\mathfrak{g}$. Then for chains of $\mathfrak{g}$ with coefficients in a right module A, and $a \in A$, we have

$$
\begin{aligned}
d\left(a \otimes \bigwedge_{i=1}^n g_i^{(\wedge r_i)}\right) &= \sum_{\Pi(g_k)=\bar{1}, r_k \geq 2} a \otimes \bigwedge_{i<k} g_i^{(\wedge r_i)} \wedge g_k^2 \wedge g_k^{(\wedge(r_k-2))} \wedge \bigwedge_{i>k} g_i^{(\wedge r_i)} \\
&+ \sum_{1 \leq k < l \leq n, r_k, r_l \geq 1} (-1)^{\sum_{k<m<l} r_m \Pi(g_m)} a \\
&\otimes \bigwedge_{i<k} g_i^{(\wedge r_i)} \wedge [g_k, g_l] \wedge g_k^{(\wedge(r_k-1))} \wedge \bigwedge_{k<i<l} g_i^{(\wedge r_i)} \wedge g_l^{(\wedge(r_l-1))} \wedge \bigwedge_{i>l} g_i^{(\wedge r_i)} \\
&+ \sum_{r_k \geq 1} (-1)^{\Pi(g_k) \sum_{m<k} r_m \Pi(g_m)} (ag_k) \otimes \bigwedge_{i<k} g_i^{(\wedge r_i)} \wedge g_k^{(\wedge(r_k-1))} \wedge \bigwedge_{i>k} g_i^{(\wedge r_i)}.
\end{aligned}
$$

Denote the divided power cohomology by $DPH^{i,\underline{N}}(\mathfrak{g}; M)$ and divided power homology by $DPH_{i,\underline{N}}(\mathfrak{g}; M)$. Note that if $\mathfrak{g}$ is a Lie **super**algebra, we cannot interpret its generating relations in terms of homology, as we do for $p = 0$, instead we need $H_2(\mathfrak{g}) := H_2(\mathfrak{g}; \mathbb{K})$ if $p \neq 2$ (cf. [12]) and $DPH_{2,\underline{N}}(\mathfrak{g}) := DPH_{2,\underline{N}}(\mathfrak{g}; \mathbb{K})$ if $p = 2$: We **must** use divided powers (with $\underline{N}$ such that $N_i \geq 2$ for all i) since otherwise we won't be able to take into account the relations of the form $x^2 = 0$.

4. Relation with 1-Forms (Differential Geometry)

In this section, $p = 2$; so the space of bilinear forms is filtered, with symmetric forms forming a subspace invariant with respect to the changes of bases.

4.1. *A factor-class approach to bilinear forms*

For reasons given in [16], equivalences (2.7) and (2.8) are inadequate for classification of non-symmetric forms. Instead of considering non-symmetric forms individually, we can consider the quotient space $NB(n)$ of the space of all forms modulo the space of symmetric forms. We will denote the element of this quotient space with representative B, by $\{B\}$. We say that $\{B\}$ and $\{C\}$ are *equivalent* (and denote it $\{B\} \sim \{C\}$), if there exists an invertible

matrix M such that

$$\{MBM^T\} = \{C\}, \quad \text{i.e., if } MBM^T - C \text{ is symmetric} \tag{4.1}$$

(this definition does not depend on the choice of representatives B and C).

Any $\{B\}$ has both degenerate and non-degenerate representatives: the representative with nonzero elements only above the diagonal (such representative is unique and characterizes $\{B\}$) is degenerate, and if we add the unit matrix to it, we get a non-degenerate representative of $\{B\}$.

Note that $\{B\}$ can be also characterized by the symmetric zero-diagonal matrix $B + B^T$. The rank of $B + B^T$ is said to be *the rank of* $\{B\}$. According to a Lemma [16], it is always even and is equal to doubled minimal rank of representatives of $\{B\}$. We say that the class $\{B\}$ is non-degenerate, if so is the matrix $B + B^T$.

Theorem 4.1. *The classes $\{B\}$ and $\{C\}$ are equivalent if and only if they have equal ranks.*

4.2. *Matrices and 1-forms*

Let B and B' be the matrices of bilinear forms on an n-dimensional space V over a field $\mathbb{K}$ of characteristic 2. Let $x_0, x_1, \ldots, x_n$ be independent indeterminates; set

$$\deg x_0 = 2, \ \ \deg x_1 = \cdots = \deg x_n = 1.$$

We say that B and B' are 1-*form-equivalent* if there exists a degree preserving transformation, i.e., a set of independent variables $x'_0, x'_1, \ldots, x'_n$ such that

$$\deg x'_0 = 2, \quad \deg x'_1 = \cdots = \deg x'_n = 1, \tag{4.2}$$

which are polynomials in $x_0, x_1, \ldots, x_n$ in divided powers with shearing parameter

$$\underline{N} = (N_0, \ldots, N_n) \quad \text{such that } N_i > 1 \ \ \text{for every } i \text{ from 1 to } n, \tag{4.3}$$

and such that

$$dx_0 + \sum_{i,j=1}^{n} B_{ij} x_i dx_j = dx'_0 + \sum_{i,j=1}^{n} B'_{ij} x'_i dx'_j. \tag{4.4}$$

Lemma 4.2. *B and B' are 1-form-equivalent if and only if $\{B\} \sim \{B'\}$.*

4.3. *The case of odd indeterminates*

Let us modify the definition of 1-form-equivalence for the super case where $x_1, \ldots, x_n$ are all *odd*. In this case, we can only use divided powers with $\underline{N} = (N_0, 1, \ldots, 1)$.

We say that B and B' are 1-*superform-equivalent* if there exists a set of indeterminates $x'_0, x'_1, \ldots, x'_n$, which are polynomials in $x_0, x_1, \ldots, x_n$, such that

$$\Pi(x'_0) = \bar{0}, \quad \Pi(x'_1) = \cdots = \Pi(x'_n) = \bar{1}, \quad \deg x'_0 = 2, \quad \deg x'_1 = \cdots = \deg x'_n = 1 \tag{4.5}$$

and

$$dx_0 + \sum_{i,j=1}^{n} B_{ij}x_i dx_j = dx'_0 + \sum_{i,j=1}^{n} B'_{ij}x'_i dx'_j. \tag{4.6}$$

Lemma 4.3. *The matrices B and B' are 1-superform-equivalent if and only if exist an invertible matrix M and a symmetric zero-diagonal matrix A such that*

$$B = MB'M^T + A. \tag{4.7}$$

4.3.1. *Relation with quadratic forms*

In pre-super era, Albert [1] considered the equivalence (4.7) as an equivalence of (matrices of) quadratic forms. In particular, he proved the following

Statement 4.4. *If $\mathbb{K}$ is algebraically closed, every matrix B is equivalent in the sense (4.7) to exactly one of the matrices*

$$Y(n,r) = \begin{pmatrix} 0_r & 1_r & 0 \\ 0_r & 0_r & 0 \\ 0 & 0 & 0_{n-2r} \end{pmatrix} \quad or \quad \widetilde{Y}(n,r) = \begin{pmatrix} 0_r & 1_r & 0 & 0 \\ 0_r & 0_r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0_{n-2r-1} \end{pmatrix},$$

where $2r = rank(B + B^T)$. The corresponding quadratic form is non-degenerate if and only if either (a) $n = 2r$ and the matrix is equivalent to $Y(n,r)$, or (b) $n = 2r+1$ and the matrix is equivalent to $\widetilde{Y}(n,r)$.

If, in 1-form-equivalence, we consider divided powers with shearing parameter $\underline{N} = (N_0, 1, \dots, 1)$, it is the same as to consider 1-superform-equivalence.

Lemma 4.5. *Let $x_0, \dots, x_n$ be indeterminates,*

$$\Pi(x_0) = \bar{0}, \quad \Pi(x_1) = \dots = \Pi(x_n) = \Pi.$$

Then the 1-form on the $(n+1\,|\,0)$-dimensional (if $\Pi = \bar{0}$) or $(1\,|\,n)$-dimensional (if $\Pi = \bar{1}$) superspace

$$\alpha = dx_0 + \sum_{i,j=1}^{n} B_{ij}x_i dx_j \tag{4.8}$$

is contact (see Subsec. 3.1) if and only if one of the following conditions holds:

(1) $\Pi = \bar{0}$, *and $\{B\}$ is non-degenerate. i.e., $n = rank(B + B^T)$ (this rank is always even);*
(2) $\Pi = \bar{1}$, *and the quadratic form corresponding to B is non-degenerate.*

From this, we get the following

Theorem 4.6. *The following are the canonical expressions of the odd contact forms if the indeterminates $x_1, \dots, x_n$ are of the same parity (for the general case, see*

Theorem 4.8):

$$\alpha = dx_0 + \sum_{i=1}^{k} x_i dx_{k+i} \begin{cases} & \text{for } n = 2k \text{ and } x_1, \ldots, x_n \\ & \qquad \text{all even or all odd;} \\ + x_{2k+1} dx_{2k+1} & \text{for } n = 2k+1 \text{ and } x_1, \ldots, x_n \text{ odd.} \end{cases} \qquad (4.9)$$

Remark 4.7. (1) If $n > 1$ and $x_1, \ldots, x_n$ are odd, the 1-form $\alpha = dx_0 + \sum_{i=1}^{n} x_i dx_i$ is not contact since (recall that $p = 2$)

$$\alpha = d\left(x_0 + \sum_{i<j} x_i x_j\right) + \left(\sum_{i=1}^{n} x_i\right) d\left(\sum_{i=1}^{n} x_i\right), \quad \text{hence}$$

$$\operatorname{rk} d\alpha = \operatorname{rk} d\left(\sum_{i=1}^{n} x_i\right) \wedge d\left(\sum_{i=1}^{n} x_i\right) = 1.$$

(2) Let $p = 2$. Since there are two types of orthogonal Lie algebras if n is even, and orthogonal algebras coincide, in a sense, with symplectic ones, it seems natural to expect that there are also two types of the Lie algebras of Hamiltonian vector fields (preserving I or S (or Π), respectively).

Are there two types of contact Lie algebras corresponding to these cases? The (somewhat unexpected) answer is NO:

The classes of 1-(super)form-equivalence of bilinear forms which correspond to contact forms have nothing to do with classes of classical equivalence of symmetric bilinear forms. The 1-forms, corresponding to symmetric bilinear forms are exact if $x_1, \ldots, x_n$ are even, and are of rank ≤ 2 if $x_1, \ldots, x_n$ are odd.

(3) Lin [23] considered an n-parameter family of simple Lie algebras for $p = 2$ preserving in dimension $2n + 1$ the distribution given by the contact form

$$\alpha = dt + \sum_{i=1}^{n} ((1 - a_i) p_i dq_i + a_i q_i dp_i), \quad \text{where } a_i \in \mathbb{K}.$$

Obviously, the linear change

$$t' = t + \sum a_i p_i q_i \quad \text{and identical on other indeterminates} \qquad (4.10)$$

reduces α to the canonical form $dt + \sum_{i=1}^{n} p_i dq_i$. So the parameters a_i can be eliminated. Although Lin mentioned the change (4.10) on p. 21 of [23], its consequence was not formulated and, seven years after, Brown [6] reproduced Lin's misleading n-parameter description of $\mathfrak{k}(2n + 1)$.

4.4. *The general case of an odd 1-form*

Let

$$\Pi(x_0) = \Pi(x_1) = \cdots = \Pi(x_{n_{\bar{0}}}) = \bar{0}, \quad \Pi(x_{n_{\bar{0}}+1}) = \cdots = \Pi(x_n) = \bar{1}.$$

This corresponds to the following equivalence (we call it 1-*superform-equivalence* again) of *even* bilinear forms on a superspace V of superdimension $(n_{\bar{0}} \,|\, n_{\bar{1}})$, where $n_{\bar{1}} = n - n_{\bar{0}}$: Two

such forms B and B' are said to be 1-*superform-equivalent* if, for their supermatrices, we have (4.7), where $M \in GL(n_{\bar{0}} \,|\, n_{\bar{1}})$ and A is a symmetric even supermatrix such that the restriction of the bilinear form corresponding to it onto the odd subspace $V_{\bar{1}}$ is anti-symmetric. This means that, in the standard format of supermatrices,

$$B = \begin{pmatrix} B_{\bar{0}} & 0 \\ 0 & B_{\bar{1}} \end{pmatrix} \quad \text{and} \quad B' = \begin{pmatrix} B'_{\bar{0}} & 0 \\ 0 & B'_{\bar{1}} \end{pmatrix}$$

are 1-superform-equivalent if and only if

(1) $B_{\bar{0}}$ and $B'_{\bar{0}}$ are 1-form-equivalent, and
(2) $B_{\bar{1}}$ and $B'_{\bar{1}}$ are 1-superform-equivalent. Then, from (4.9) we get the following.

Theorem 4.8. *The following are the canonical expressions for an odd contact form on a superspace*:

$$dt + \sum_{i=1}^{k} p_i dq_i + \sum_{j=1}^{l} \xi_i d\eta_i \begin{cases} & \text{for } n_{\bar{0}} = 2k \ \text{ and } \ n_{\bar{1}} = 2l, \\ +\theta d\theta & \text{for } n_{\bar{0}} = 2k \ \text{ and } \ n_{\bar{1}} = 2l + 1, \end{cases}$$

where $t = x_0$, *and* $p_i = x_i, q_i = x_{k+i}$ *for* $1 \leq i \leq k$ *are the even indeterminates*; $\xi_i = x_{n_{\bar{0}}+i}, \eta_i = x_{n_{\bar{0}}+l+i}$ *for* $1 \leq i \leq l$, *and* $\theta = x_n$ *for* $n_{\bar{1}} = 2l + 1$ *are the odd indeterminates*.

This follows from the fact proved in [16] that the 1-form

$$dx_0 + \sum_{i,j=1}^{n_{\bar{0}}} A_{ij} x_i dx_j + \sum_{i,j=1}^{n_{\bar{1}}} B_{ij} x_{n_{\bar{0}}+i} dx_{n_{\bar{0}}+j}$$

is contact if and only if the forms

$$dx_0 + \sum_{i,j=1}^{n_{\bar{0}}} A_{ij} x_i dx_j \quad \text{and} \quad dx_0 + \sum_{i,j=1}^{n_{\bar{1}}} B_{ij} x_{n_{\bar{0}}+i} dx_{n_{\bar{0}}+j}$$

are contact on the superspaces of superdimension $(n_{\bar{0}} + 1 \,|\, 0)$ and $(1 \,|\, n_{\bar{1}})$, respectively.

4.5. *Generating functions*

Recall that the contact Lie superalgebra consists of the vector fields D that preserve the contact structure (non-integrable distribution given by a contact form α) on the supervariety $M = \mathbb{K}^{n_{\bar{0}}+1 \,|\, n_{\bar{1}}}$. Such fields satisfy

$$L_D(\alpha) = F_D \alpha \quad \text{for some } F_D \in \mathcal{F}, \text{ where } \mathcal{F} \text{ is the space of functions on } M. \quad (4.11)$$

Let us consider the form

$$\alpha = dx_0 + \sum_{i=1}^{k} x_i dx_{k+i} \begin{cases} & \text{if } n = n_{\bar{0}} + n_{\bar{1}} = 2k \\ +x_{2k+1} dx_{2k+1} & \text{if } n = 2k + 1, \end{cases}$$

such that

$$\Pi(x_0) = \cdots = \Pi(x_{k_{\bar{0}}}) = \Pi(x_{k+1}) = \cdots = \Pi(x_{k+k_{\bar{0}}}) = \bar{0},$$

$$\Pi(x_{k_{\bar{0}}+1}) = \cdots = \Pi(x_k) = \Pi(x_{k+k_{\bar{0}}+1}) = \cdots = \Pi(x_{2k+1}) = \bar{1}$$

(here $n_{\bar{0}} = 2k_{\bar{0}}$; if $n_{\bar{0}}$ is odd, no contact form exists).

The vector fields D that satisfy (4.11) for some function F_D look differently for different characteristics. For $p \neq 2$, and also if $p = 2$ and $n = 2k$, they have the following form (for any $f \in \mathcal{F}$):

$$K_f = (1 - E')(f)\frac{\partial}{\partial x_0} + \frac{\partial f}{\partial x_0}E' + \sum_{i=1}^{k_{\bar{0}}}\left(\frac{\partial f}{\partial x_{k+i}}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_{k+i}}\right)$$

$$- (-1)^{\Pi(f)}\left(\sum_{i=k_{\bar{0}}+1}^{k}\left(\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_{k+i}} + \frac{\partial f}{\partial x_{k+i}}\frac{\partial}{\partial x_i}\right)\right.$$

$$+ \left.\begin{cases} & \text{if } n = 2k \\ \frac{1}{2}\frac{\partial f}{\partial x_{2k+1}}\frac{\partial}{\partial x_{2k+1}} & \text{if } n = 2k+1 \end{cases}\right), \tag{4.12}$$

where $E' = \sum_{i=1}^{k}x_i\frac{\partial}{\partial x_i} + \begin{cases} & \text{if } n = 2k \\ \frac{1}{2}x_{2k+1}\frac{\partial}{\partial x_{2k+1}} & \text{if } n = 2k+1 \end{cases}$.

If $p = 2$ and $n = 2k + 1$, we cannot use this formula for K_f anymore (at least, not for arbitrary f) since it contains $\frac{1}{2}$. In this case, the elements of the contact algebra have the following forms:

(a) for any f such that $\frac{\partial f}{\partial x_0} = \frac{\partial f}{\partial x_{2k+1}} = 0$, i.e., for any $f \in \mathcal{O}(x_1, \ldots, x_{2k}; \underline{N})$, we have

$$K_f = (1 - E')(f)\frac{\partial}{\partial x_0} + \sum_{i=1}^{k_{\bar{0}}}\left(\frac{\partial f}{\partial x_{k+i}}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_{k+i}}\right)$$

$$- (-1)^{\Pi(f)}\left(\sum_{i=k_{\bar{0}}+1}^{k}\left(\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_{k+i}} + \frac{\partial f}{\partial x_{k+i}}\frac{\partial}{\partial x_i}\right)\right), \tag{4.13}$$

where $E' = \sum_{i=1}^{k}x_i\frac{\partial}{\partial x_i}$. (Note, that if $\mathcal{F}$ consists of polynomials (or series), instead of divided powers, then we can use $f \in \mathbb{K}[x_0^p, x_1, \ldots, x_{2k}]$ for any characteristic $p > 0$.)

(b) For any $g \in \mathcal{O}(x_0, \ldots, x_{2k}; \underline{N})$, we have

$$\text{(b1) } A_g := g\left(x_{2k+1}\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_{2k+1}}\right)$$

$$\text{(b2) } B_g := gx_{2k+1}\frac{\partial}{\partial x_{2k+1}}. \tag{4.14}$$

Remark 4.9. We do not unite these two formulas into one although they both can be obtained from (4.12) by "multiplication by 2" for the following reason. Indeed, unification gives

$$x_{2k+1}\frac{\partial f}{\partial x_{2k+1}}\frac{\partial}{\partial x_0} + \left(\frac{\partial f}{\partial x_{2k+1}} + x_{2k+1}\frac{\partial f}{\partial x_0}\right)\frac{\partial}{\partial x_{2k+1}}. \tag{4.15}$$

If $f = gx_{2k+1}$, where $g \in \mathcal{O}(x_0, \ldots, x_{2k}; \underline{N})$, then (4.15) is equal to (b1) from (4.14), all right. But if $f \in \mathcal{O}(x_0, \ldots, x_{2k})$, then (4.15) is equal to

$$\frac{\partial f}{\partial x_0}x_{2k+1}\frac{\partial}{\partial x_{2k+1}};$$

but more general (since the endomorphism $\frac{\partial}{\partial x_0}$ is not onto on $\mathcal{O}(x_0, \ldots, x_{2k})$) vector fields (b2) from (4.14) also preserve the form α (not only the distribution given by it).

The brackets and squarings of contact vector fields, and the corresponding contact brackets of generating functions are as follows:

$$
\begin{aligned}
[K_f, K_{f'}] &= K_{[f,f']_{K.b.}}; \quad [M_f, M_{f'}] = M_{[f,f']_{K.b.}}; \\
(K_f)^2 &= K_{\sum_{i=1}^{k} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_{k+i}}}; \quad (M_f)^2 = M_{\sum_{i=1}^{k} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_{k+i}}}; \\
[K_f, A_g] &= A_{[f,g]_{K.b.}}; \quad (A_g)^2 = B_{g \frac{\partial g}{\partial x_0}}; \\
[K_f, B_g] &= B_{[f,g]_{K.b.}}; \quad [A_g, B_{g'}] = A_{gg'}; \\
[A_g, A_{g'}] &= B_{\frac{\partial}{\partial x_0}(gg')}, \quad [B_g, B_{g'}] = (B_g)^2 = 0,
\end{aligned}
\tag{4.16}
$$

where the contact bracket is of the form

$$
\{f, g\}_{k.b.} = \frac{\partial f}{\partial x_0}(1 - E')(g) + (1 - E')(f)\frac{\partial g}{\partial x_0} + \sum_{i=1}^{k} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_{k+i}} + \frac{\partial f}{\partial x_{k+i}} \frac{\partial g}{\partial x_i} \right) \tag{4.17}
$$

and where

$$
x_0 := \begin{cases} \tau & \text{for } \mathfrak{m}(n; \underline{N} \,|\, n+1) \\ t & \text{for } \mathfrak{k}(2k+1; \underline{N} \,|\, n_{\bar{1}}). \end{cases}
$$

For $\mathfrak{k}(2k+1; \underline{N} \,|\, 2l)$ and $\mathfrak{m}(n; \underline{N} \,|\, n+1)$, the bracket of contact vector fields reduces to k.b.

4.5.1. *Contact algebra as a CTS prolong*

If one tries to build the contact algebra $\mathfrak{g}$ by means of a non-degenerate symmetric bilinear form B on the space V by setting (like it is done in characteristic 0) $\mathfrak{g}$ to be the CTS prolong $(\mathfrak{g}_-, \mathfrak{g}_0)_*$, where the non-positive terms of $\mathfrak{g}$ are:

$$
\mathfrak{g}_i = \begin{cases} 0 & \text{if } i \leq -3; \\ \mathbb{K} \cdot K_1 & \text{if } i = -2 \\ V = \mathrm{Span}_{\mathbb{K}}(K_{x_1}, \ldots, K_{x_n}) & \text{if } i = -1 \\ \mathfrak{o}_B(V) \oplus \mathbb{K}K_t \simeq \mathrm{Span}_{\mathbb{K}}(K_{x_i x_j} \,|\, i, j = 1, \ldots, n) \oplus \mathbb{K}K_t & \text{if } i = 0 \end{cases}
\tag{4.18}
$$

and where the multiplication is given by the formulas

$$
\begin{aligned}
&[X, Y] = B(X, Y)K_1 \quad \text{for any } X, Y \in \mathfrak{g}_{-1}; \\
&\mathfrak{o}_B(V) \text{ acts on } V \text{ via the standard action;} \\
&[\mathfrak{g}_0, \mathfrak{g}_{-2}] = 0; \\
&K_t \text{ acts as id on } \mathfrak{g}_{-1}, \\
&[K_t, \mathfrak{o}_B(V)] = 0,
\end{aligned}
$$

then the form B must be zero-diagonal one (because $0 = [X, X] = B(X, X)K_1$ for $X \in \mathfrak{g}_{-1}$).

One can also try to construct a Lie superalgebra in a similar way by letting $\mathfrak{g}_{-1}$ be purely odd and

$$X^2 = B(X, X)K_1 \quad \text{for any } X \in \mathfrak{g}_{-1}. \tag{4.19}$$

Let us realize this Lie superalgebra by vector fields on a superspace of superdimension $(1 \,|\, n)$ with basis $x_0, \ldots, x_n$ such that

$$\Pi(x_0) = \bar{0}; \quad \Pi(x_i) = \bar{1} \quad \text{for } 1 \leq i \leq n.$$

If $e_1, \ldots, e_n$ is a basis of V and we set (here $\partial_i = \frac{\partial}{\partial x_i}$ for $i = 0, \ldots, n$):

$$K_1 = \partial_0; \quad e_i = \partial_i + \sum_{j=1}^{n} A_{ij} x_j \partial_0 \quad \text{for } i = 1, \ldots, n,$$

then, to satisfy relations (4.19), we need the following (here the Gram matrix B is taken in the basis $e_1, \ldots, e_n$):

$$A_{ii} = B_{ii} \quad \text{for } 1 \leq i \leq n;$$

$$A_{ij} + A_{ji} = B_{ij} + B_{ji} \quad \text{for } 1 \leq i < j \leq n$$

i.e., $A \in \{B\}$, where the equivalence class is taken modulo zero-diagonal symmetric matrices. (For a discussion of various possible equivalences of non-symmetric forms, see [16].)

These vector fields preserve the 1-form

$$\alpha = dx_0 + \sum_{i,j=1}^{n} A_{ij} x_i dx_j.$$

So, to get a contact Lie superalgebra in **this** way, one needs B to be non-symmetric with non-degenerate class $\{B\}$ (i.e., such that $B + B^T$ is non-degenerate).

4.6. *The case of an even* **1**-*form*

Let $\Pi(x_0) = \bar{1}$. This corresponds to the following equivalence of *odd* bilinear forms on a superspace V of superdimension $(n_{\bar{0}} \,|\, n_{\bar{1}})$: two such forms B and B' are said to be 1-*superform-equivalent* if for their (super)matrices we have (4.7), where $M \in GL(n_{\bar{0}} \,|\, n_{\bar{1}})$ and A is a symmetric odd supermatrix. Then, since

$$\left(\begin{array}{c|c} 1_{n_{\bar{0}}} & 0 \\ \hline 0 & M \end{array} \right) \left(B + \left(\begin{array}{c|c} 0 & C \\ \hline C^T & 0 \end{array} \right) \right) \left(\begin{array}{c|c} 1_{n_{\bar{0}}} & 0 \\ \hline 0 & M^T \end{array} \right) = \left(\begin{array}{c|c} 0 & 0 \\ \hline X(D + C^T) & 0 \end{array} \right)$$

$$\text{for } B = \left(\begin{array}{c|c} 0 & C \\ \hline D & 0 \end{array} \right),$$

any such B is equivalent to a form with a supermatrix of the shape (the indices above and to the left of the supermatrix are the sizes of the blocks)

$$\begin{array}{c} \\ n_{\bar{0}} \\ r \\ n_{\bar{1}} - r \end{array} \begin{array}{c} r \quad n_{\bar{0}} - r \quad n_{\bar{1}} \\ \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1_r & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \end{array}$$

where $r = \text{rank}(D + C^T)$. The corresponding form is contact if and only if $r = n_{\bar{0}} = n_{\bar{1}}$. Hence, we get the following somewhat unexpected result:

Theorem 4.10. *The following expressions for the canonical form of an even (peri) contact 1-form on a superspace of dimension $(k \,|\, k+1)$ are equivalent for all l (0 to k):*

$$(1)\ d\tau + \sum_{i=1}^{k} \xi_i dq_i, \quad (2)\ d\tau + \sum_{i=1}^{k} q_i d\xi_i, \quad (3)\ d\tau + \sum_{i=1}^{l} \xi_i dq_i + \sum_{i=l+1}^{k} q_i d\xi_i, \qquad (4.20)$$

where $\tau = x_0$, and $\xi_i = x_{k+i}, q_i = x_i$ for $1 \leq i \leq k$.

The pericontact vector fields that preserve the contact structure with the form (4.20.3) are of the shape

$$M_f = (1 - E')(f)\frac{\partial}{\partial \tau} + \sum_{i=1}^{l} \left(\frac{\partial f}{\partial q_i}\frac{\partial}{\partial \xi_i} + (-1)^{\Pi(f)}\frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial q_i} \right)$$

$$- \sum_{i=l+1}^{k} \left((-1)^{\Pi(f)}\frac{\partial f}{\partial q_i}\frac{\partial}{\partial \xi_i} + \frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial q_i} \right) + (-1)^{\Pi(f)}\frac{\partial f}{\partial \tau}E', \qquad (4.21)$$

where

$$E' = \sum_{i=1}^{l} \xi_i \frac{\partial}{\partial \xi_i} + \sum_{i=l+1}^{k} q_i \frac{\partial}{\partial q_i}$$

(we usually select one of the extreme cases $l = 0$ or $l = k$, i.e., either $E' = \sum \xi_i \frac{\partial}{\partial \xi_i}$ or $E' = \sum q_i \frac{\partial}{\partial q_i}$).

5. The Hamiltonian Lie Superalgebras

Let $B = (B_{ij})$ be an even symmetric non-degenerate bilinear form on a superspace V of dimension $n_{\bar{0}} \,|\, n_{\bar{1}}$ with a basis $\{x_1, \ldots, x_n\}$, where $n = n_{\bar{0}} + n_{\bar{1}}$, such that

$$\Pi(x_1) = \cdots = \Pi(x_{n_{\bar{0}}}) = \bar{0}, \quad \Pi(x_{n_{\bar{0}}+1}) = \cdots = \Pi(x_n) = \bar{1}.$$

Then the Cartan prolong of the Lie superalgebra $\mathfrak{oo}_B(n_{\bar{0}} \,|\, n_{\bar{1}})$ is analogous to the Hamiltonian Lie superalgebra.

5.1. *Relations with differential 2-forms*

Let W be a free module over a supercommutative superalgebra A. The module

$$T^{\cdot}(W) = \bigoplus_{n=0}^{\infty} T^n(W), \quad \text{where } T^n(W) = W \otimes_A \cdots \otimes_A W \ (n \text{ factors})$$

possesses a natural algebra structure. It is called the *tensor algebra* of W over A.

The two quotients of $T^\bullet(W)$ are of special interest: the symmetric algebra $S^\bullet(W)$ and the external algebra $\wedge^\bullet(W)$. The *symmetric algebra* is defined as the quotient modulo the ideal generated by elements of the forms

$$x \otimes y - y \otimes x, \quad \text{where } x, y \in W_{\bar{0}} \quad \text{or } x \in W_{\bar{0}}, \ y \in W_{\bar{1}};$$
$$x \otimes x, \quad\quad\quad\quad \text{where } x \in W_{\bar{1}}.$$

The *exterior algebra* is defined as the quotient modulo the ideal generated by elements of the forms

$$x \otimes x, \quad\quad\quad\quad \text{where } x \in W_{\bar{0}};$$
$$x \otimes y - y \otimes x, \quad \text{where } x, y \in W_{\bar{1}} \quad \text{or } x \in W_{\bar{0}}, \ y \in W_{\bar{1}}.$$

(These two algebras can also be considered as subalgebras in $T^\bullet(W)$. Note that if we define them so, then for $p = 2$ and purely even W, we have $\wedge^\bullet(W) \subset S^\bullet(W)$.)

For a given vector superspace V, one can construct in this way the superalgebra $D^\bullet(V)$ — the *tensor algebra of the free module of differential 1-forms* over the supercommutative superalgebra of polynomials in V. This algebra was seldom if ever considered in geometry whereas the algebras of exterior and symmetric differential forms are indispensable in Riemannian and differential geometries, respectively.

In the case of characteristic $p \neq 2$, the Hamiltonian Lie superalgebra can be represented as the Lie superalgebra of vector fields preserving a given exterior differential 2-form of maximal rank with constant coefficient. In the case of characteristic 2, the Cartan prolong of the Lie superalgebra $\mathfrak{oo}_B(n_{\bar{0}} \,|\, n_{\bar{1}})$ can be represented as a Lie superalgebra of vector fields preserving a given exterior differential 2-form only if B is equivalent to $B_{\Pi\Pi}$ or $B_{\Pi I}$; the corresponding 2-form is

$$\omega_B = \sum_{1 \leq i \leq j \leq n} B_{ij} dx_i \wedge dx_j, \tag{5.1}$$

where $dx_i \wedge dx_j$ denotes the equivalence class with a representative $dx_i \otimes dx_j$. However, this prolong can be represented as a Lie superalgebra of vector fields preserving a given **tensor, not exterior**, differential 2-form for any equivalence class of B; the corresponding 2-form is

$$\omega_B = \sum_{1 \leq i \leq j \leq n} B_{ij} dx_i \otimes dx_j. \tag{5.2}$$

5.2. *The structure of the prolongs*

As a linear space, the above Cartan prolong can be represented as

$$\mathrm{Reg}_B \oplus \mathrm{Irreg}^1_B \oplus \mathrm{Irreg}^2_B, \tag{5.3}$$

where

$$\mathrm{Reg}_B = \left\{ \sum_{i,j=1}^n (B^{-1})_{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} \,\middle|\, f \in \mathcal{O}(x_1, \ldots, x_n; \underline{N}) \right\},$$

$$\mathrm{Irreg}_B^1 = \mathrm{Span}\left(\sum_{j=1}^n (B^{-1})_{ij} x_i \frac{\partial}{\partial x_j} \mid n_{\bar{0}} < i \leq n\right)$$

$$\mathrm{Irreg}_B^2 = \mathrm{Span}\left(\sum_{j=1}^n (B^{-1})_{ij} x_i^{(2^{\underline{N}_i} - 1)} \frac{\partial}{\partial x_j} \mid 1 \leq i \leq n_{\bar{0}} \text{ such that } \underline{N}_i < \infty\right). \qquad (5.4)$$

Note that sdim $\mathrm{Irreg}_B^1 = n_{\bar{1}} \mid 0$, and this space is spanned by elements "generated" by nonexisting "Hamiltonians" $x_i^{(2)}$, where $n_{\bar{0}} < i \leq n$; the space Irreg_B^2 is spanned by elements "generated" by nonexisting "Hamiltonians" $x_i^{(2^{\underline{N}_i})}$, where $1 \leq i \leq n_{\bar{0}}$.

This description implies, in particular, that the superdimensions of the prolongs do not depend on the type of the superalgebra (i.e., is it $\mathfrak{oo}_{II}, \mathfrak{oo}_{I\Pi}, \mathfrak{oo}_{\Pi I}$ or $\mathfrak{oo}_{\Pi\Pi}$) — they only depend on the superdimension $n_{\bar{0}} \mid n_{\bar{1}}$, the number of the prolong and the shearing parameter $\underline{N}$.

For $\underline{N} = \underline{N}_\infty$, the dimension of the kth prolong is equal to a coefficient of the supercharacter of $\mathcal{O}(x_1, \ldots, x_{n_{\bar{0}}}, \xi_1, \ldots, \xi_{n_{\bar{1}}}; \underline{N})$, i.e.,

$$\sum_{i=0}^{n_{\bar{1}}} \binom{n_{\bar{0}}}{k+2-i} \binom{n_{\bar{1}}}{i}$$

$$= \text{the coefficient of } x^{k+2} \text{ in the Taylor series expansion of } \frac{(1+x)^{n_{\bar{1}}}}{(1-x)^{n_{\bar{0}}}} \text{ at } x = 0.$$

For $\underline{N} = \underline{N}_s$, the dimension of the kth prolong is equal to $\binom{n}{k+2}$, and the dimension of the complete prolong is equal to $2^n + n - 1$. The general formula for the superdimensions seems to be complicated (I was unable to derive it).

5.3. *Cartan prolongs of the derived algebras*

If $n_{\bar{0}}, n_{\bar{1}} > 0$, then the Cartan prolong of $\mathfrak{oo}_{II}^{(1)}(n_{\bar{0}} \mid n_{\bar{1}})$ can be represented as

$$\mathrm{Reg}'_{II} \oplus \mathrm{Irreg}'^1_{II},$$

where

$$\mathrm{Reg}'_{II} = \left\{ H_{II,f} \mid f \in \mathcal{O}(x_1, \ldots, x_n), \ \sum_{j=1}^{n_{\bar{0}}} \frac{\partial^2 f}{\partial x_i^2} = 0 \right\},$$

$$\mathrm{Irreg}'^1_{II} = \mathrm{Span}\left(x_{i-1} \frac{\partial}{\partial x_{i-1}} + x_i \frac{\partial}{\partial x_i} \mid n_{\bar{0}} < i \leq n \right).$$

If $n_{\bar{0}} > 1$ and $\underline{N}_1 = \cdots = \underline{N}_{n_{\bar{0}}} = \infty$, then the dimension of the $(k, \underline{N})$-th prolong of $\mathfrak{oo}_{II}^{(1)}(n_{\bar{0}} \mid n_{\bar{1}})$ is equal to the dimension of the kth prolong of $\mathfrak{oo}_{II}(n_{\bar{0}} - 1 \mid n_{\bar{1}} + 1)$.

Let $n_{\bar{0}} = 1$ and $\underline{N}_1 = \infty$. Then the dimension of the $(k, \underline{N})$-th prolong of $\mathfrak{oo}_{II}^{(1)}(n_{\bar{0}} \mid n_{\bar{1}})$ is equal to $\binom{n_{\bar{1}}+1}{k+2}$.

If $n_{\bar{0}}, n_{\bar{1}} > 0$, and $n_{\bar{1}} = 2k_{\bar{1}}$, then the Cartan prolong of $\mathfrak{oo}_{I\Pi}^{(1)}(n_{\bar{0}} \,|\, n_{\bar{1}})$ can be represented as

$$\mathrm{Reg}'_{I\Pi} \oplus \mathrm{Irreg}^1_{I\Pi},$$

where

$$\mathrm{Reg}'_{I\Pi} = \{H_{I\Pi,f} \,|\, f \in \mathcal{O}(x_1, \ldots, x_n), \ f \text{ has degree} \leq 1 \text{ w.r.t. any } x_i, 1 \leq i \leq n\}.$$

So, independently of $\underline{N}$, the dimension of the kth prolong of $\mathfrak{oo}_{I\Pi}^{(1)}(n_{\bar{0}} \,|\, n_{\bar{1}})$ is equal to $\binom{n}{k+2}$.

If $n_{\bar{0}}, n_{\bar{1}} > 0$, and $n_{\bar{0}} = 2k_{\bar{0}}$, then the Cartan prolong of $\mathfrak{oo}_{\Pi I}^{(1)}(n_{\bar{0}} \,|\, n_{\bar{1}})$ is equal to

$$\mathrm{Reg}_{\Pi I} \oplus \mathrm{Irreg}^2_{\Pi I}.$$

Thus, the superdimension of the kth prolong of $\mathfrak{oo}_{\Pi I}^{(1)}(n_{\bar{0}} \,|\, n_{\bar{1}})$ is equal to the superdimension of the kth prolong of $\mathfrak{oo}_{\Pi I}(n_{\bar{0}} \,|\, n_{\bar{1}})$.

If $n_{\bar{0}}, n_{\bar{1}} > 0$, and $n_{\bar{0}} = 2k_{\bar{0}}, n_{\bar{1}} = 2k_{\bar{1}}$, then the Cartan prolong of $\mathfrak{oo}_{\Pi\Pi}^{(1)}(n_{\bar{0}} \,|\, n_{\bar{1}})$ is equal to

$$\{H_{\Pi\Pi,f} \,|\, f \in \mathcal{O}(x_1, \ldots, x_n), \ f \text{ has degree} \leq 1 \text{ w.r.t. any } x_i, 1 \leq i \leq n\}.$$

So, independently of $\underline{N}$, the dimension of the kth prolong of $\mathfrak{oo}_{\Pi\Pi}^{(1)}(n_{\bar{0}} \,|\, n_{\bar{1}})$ is equal to $\binom{n}{k+2}$.

5.3.1. *The prolong of the second derived*

The Cartan prolong of $\mathfrak{oo}_{\Pi\Pi}^{(2)}(n_{\bar{0}} \,|\, n_{\bar{1}})$ consists of elements of the Cartan prolong of $\mathfrak{oo}_{\Pi\Pi}^{(1)}$, generated by functions f such that

$$\sum_{i=0}^{k_{\bar{0}}} \frac{\partial^2 f}{\partial p_i \partial q_i} + \sum_{i=0}^{k_{\bar{1}}} \frac{\partial^2 f}{\partial \xi_i \partial \eta_i} = 0.$$

The formula for the dimension of the kth prolong of $\mathfrak{oo}_{\Pi\Pi}^{(2)}(n_{\bar{0}} \,|\, n_{\bar{1}})$ seems to be rather complicated. One can show, though, that this dimension depends only on n and k — not on $n_{\bar{0}}$ and $n_{\bar{1}}$.

6. The Poisson Lie Superalgebras

As it was said above, the space Reg_B consists of vector fields of the form

$$H_{B,f} = \sum_{i,j=1}^{n} (B^{-1})_{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}, \quad \text{where } f \in \mathcal{O}(x_1, \ldots, x_n).$$

Here are the precise forms of these fields for the above bilinear forms:

$$H_{II,f} := \sum_{i=1}^{n_{\bar{0}}} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i} + \sum_{i=1}^{n_{\bar{1}}} \frac{\partial f}{\partial \theta_i} \frac{\partial}{\partial \theta_i}$$

$$H_{I\Pi,f} := \sum_{i=1}^{n_{\bar{0}}} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i} + \sum_{i=1}^{k_{\bar{1}}} \left(\frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i} \frac{\partial}{\partial \xi_i} \right)$$

$$H_{\Pi I,f} := \sum_{i=1}^{k_{\bar{0}}} \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} + \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} \right) + \sum_{i=1}^{n_{\bar{1}}} \frac{\partial f}{\partial \theta_i} \frac{\partial}{\partial \theta_i}$$

$$H_{\Pi\Pi,f} := \sum_{i=1}^{k_{\bar{0}}} \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} + \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} \right) + \sum_{i=1}^{k_{\bar{1}}} \left(\frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i} \frac{\partial}{\partial \xi_i} \right), \tag{6.1}$$

where

$$
\begin{aligned}
p_i &= x_i, \quad q_i = x_{k_{\bar{0}}+i} && \text{for } n_{\bar{0}} = 2k_{\bar{0}} \text{ and } 1 \le i \le k_{\bar{0}}; \\
\theta_i &= x_{n_{\bar{0}}+i} && \text{for } 1 \le i \le n_{\bar{1}}; \\
\xi_i &= x_{n_{\bar{0}}+i}, \quad \eta_i = x_{n_{\bar{0}}+k_{\bar{1}}+i} && \text{for } n_{\bar{1}} = 2k_{\bar{1}} \text{ and } 1 \le i \le k_{\bar{1}}.
\end{aligned}
$$

The space Reg_B is closed under the Lie bracket (but may be not closed under squaring). The corresponding *Poisson bracket* of the nonexisting "generating functions" is

$$\{f,g\}_B = \sum_{i,j=1}^{n} (B^{-1})_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}. \tag{6.2}$$

In particular, the Poisson brackets corresponding to the above bilinear forms B are

$$
\begin{aligned}
\{f,g\}_{II} &:= \sum_{i=1}^{n_{\bar{0}}} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i} + \sum_{i=1}^{n_{\bar{1}}} \frac{\partial f}{\partial \theta_i} \frac{\partial g}{\partial \theta_i} \\
\{f,g\}_{I\Pi} &:= \sum_{i=1}^{n_{\bar{0}}} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i} + \sum_{i=1}^{k_{\bar{1}}} \left(\frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i} \frac{\partial g}{\partial \xi_i} \right) \\
\{f,g\}_{\Pi I} &:= \sum_{i=1}^{k_{\bar{0}}} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} + \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) + \sum_{i=1}^{n_{\bar{1}}} \frac{\partial f}{\partial \theta_i} \frac{\partial g}{\partial \theta_i} \\
\{f,g\}_{\Pi\Pi} &:= \sum_{i=1}^{k_{\bar{0}}} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} + \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) + \sum_{i=1}^{k_{\bar{1}}} \left(\frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i} \frac{\partial g}{\partial \xi_i} \right).
\end{aligned}
\tag{6.3}
$$

In the cases ΠI and $\Pi\Pi$, if $\underline{N}$ is such that $N_i \ge 2$ for all i, then the space Reg_B is closed under squaring (i.e., it is a Lie superalgebra), and so $(H_f)^2 = H_{f^{[2]}}$, where the respective expressions of $f^{[2]}$ are

$$
f^{[2]} := \begin{cases}
\displaystyle\sum_{i=1}^{k_{\bar{0}}} \frac{\partial f}{\partial p_i} \frac{\partial f}{\partial q_i} + \sum_{i=1}^{n_{\bar{1}}} \left(\frac{\partial f}{\partial \theta_i} \right)^{(2)} & \text{for } \Pi I \\[4mm]
\displaystyle\sum_{i=1}^{k_{\bar{0}}} \frac{\partial f}{\partial p_i} \frac{\partial f}{\partial q_i} + \sum_{i=1}^{k_{\bar{1}}} \frac{\partial f}{\partial \xi_i} \frac{\partial f}{\partial \eta_i} & \text{for } \Pi\Pi.
\end{cases}
\tag{6.4}
$$

Remark 6.1. In Eq. (6.4) we use divided square for arbitrary polynomials, not only for the indeterminates x_i, where $i \le n_{\bar{0}}$. We mean that $X^{(2)} = 0$ for any monomial X not

proportional to x_i, where $i \leq n_{\bar{0}}$, and that the following relation holds:

$$(a + \lambda b)^{(2)} = a^{(2)} + \lambda^2 b^{(2)} + \lambda ab \quad \text{for any } a, b \in \mathcal{O}(x_1, \ldots, x_n), \ \lambda \in \mathbb{K}.$$

7. Deformations of the Buttin Superalgebra Over $\mathbb{K}$

7.1. *The antibracket (Buttin) Lie superalgebras*

Schouten discovered what is called in Differential Geometry is called Schouten bracket (over $\mathbb{R}$ or $\mathbb{C}$), C. Buttin proved that the Schouten bracket satisfies the super Jacobi identity, that was the reason for Leites [17] to call the Lie superalgebra of functions with the Schouten bracket the *Buttin superalgebra* $\mathfrak{b}(n)$ (he also called the bracket the *Buttin bracket*). Later, Batalin and Vilkovysky rediscovered this bracket and dubbed it the *antibracket* (independently this was done by Zinn–Justin). Leites [17] interpreted the quotient Lie superalgebra, later called $\mathfrak{le}(n) = \mathfrak{b}(n)/\mathfrak{z}$, of the antibracket (a.k.a. Buttin) Lie superalgebra modulo its center as an analog of Lie algebra of Hamiltonian vector fields, i.e., as preserving a non-degenerate closed differential 2-form, an odd one. The Lie superalgebra $\mathfrak{le}(n)$ is the Cartan prolong of Lie superalgebra $\mathfrak{pe}_B$. It also allows the description (5.3) and (5.4), so the dimensions of the prolongs are the same as of the prolongs of ortho-orthogonal superalgebras. The space Reg_B is closed under the Lie (super)bracket and under squaring. In particular, if $B = \Pi_{m \,|\, m}$, then Reg_B consists of vector fields of the form

$$Le_f := \sum_{i=1}^{m} \left(\frac{\partial f}{\partial x_i} \frac{\partial}{\partial \theta_i} + \frac{\partial f}{\partial \theta_i} \frac{\partial}{\partial x_i} \right).$$

The corresponding *antibracket* and squaring of the generating functions are, respectively:

$$\{f, g\}_{a.b.} := \sum_{i=1}^{m} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \theta_i} + \frac{\partial f}{\partial \theta_i} \frac{\partial g}{\partial x_i} \right); \quad f^{[2]} := \sum_{i=1}^{m} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial \theta_i}.$$

The Cartan prolong $\mathfrak{lle}(m; \underline{N})$ of $\mathfrak{pe}_{\Pi}^{(1)}(m)$ consists of

$$\{ Le_f \mid f \in \mathcal{O}(x_1, \ldots, x_{2m}), \ \deg_{x_i} f \leq 1 \text{ for any } i, \text{ where } 1 \leq i \leq 2m \}.$$

So, independently of $\underline{N}$, the dimension of the kth prolong of $\mathfrak{pe}_{\Pi}^{(1)}(m)$ is equal to $\binom{2m}{k+2}$.

The Cartan prolong $\mathfrak{slle}(m; \underline{N})$ of $\mathfrak{pe}_{\Pi}^{(2)}(m)$ consists of

$$\left\{ Le_f \in \mathfrak{lle}(m; \underline{N}) \,\middle|\, Le_f \in \sum_{i=0}^{m} \frac{\partial^2 f}{\partial x_i \partial \theta_i} = 0 \right\}.$$

7.2. *Deformations of the Buttin superalgebra over $\mathbb{C}$*

As is clear from the definition of the antibracket, there is a regrading (namely, $\mathfrak{b}(n; n)$ given by $\deg \xi_i = 0, \deg q_i = 1$ for all i) under which $\mathfrak{b}(n)$, initially of depth 2, takes the form $\mathfrak{g} = \oplus_{i \geq -1} \mathfrak{g}_i$ with $\mathfrak{g}_0 = \mathfrak{vect}(0 \,|\, n)$ and $\mathfrak{g}_{-1} \cong \Pi(\mathbb{C}[\xi])$. Replace now the $\mathfrak{vect}(0 \,|\, n)$-module $\mathfrak{g}_{-1}$ of functions (with inverted parity) by the module of λ-densities, i.e., set $\mathfrak{g}_{-1} \cong \Pi(\mathrm{Vol}(0 \,|\, n)^{\lambda})$, where the Lie derivative L_D along the vector field $D \in \mathfrak{vect}(0 \,|\, n)$ is given by the

formula

$$L_D(f(\xi)\text{vol}_\xi^\lambda) = (D(f) + (-1)^{p(f)p(D)}\lambda f \, \text{div}D)\text{vol}_\xi^\lambda \quad \text{and} \quad p(\text{vol}_\xi^\lambda) = \bar{1}. \qquad (7.1)$$

Define $\mathfrak{b}_\lambda(n;n)$ to be the Cartan prolong

$$(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = (\Pi(\text{Vol}(0\,|\,n)^\lambda), \mathfrak{vect}(0\,|\,n))_*. \qquad (7.2)$$

Clearly, this is a deform of $\mathfrak{b}(n;n)$. The collection of these $\mathfrak{b}_\lambda(n;n)$ for all λ's is called the *main deformation*, the other deformations, defined in what follows, will be called *singular*.

The deform $\mathfrak{b}_\lambda(n)$ of $\mathfrak{b}(n)$ is a regrading of $\mathfrak{b}_\lambda(n;n)$ described as follows. Set

$$\mathfrak{b}_{a,b}(n) = \left\{ M_f \in \mathfrak{m}(n) \,\middle|\, a \, \text{div}M_f = (-1)^{p(f)}2(a - bn)\frac{\partial f}{\partial \tau} \right\}. \qquad (7.3)$$

For future use, we will denote the operator that singles out $\mathfrak{b}_\lambda(n)$ in $\mathfrak{m}(n)$ as follows:

$$\text{div}_\lambda = (bn - aE)\frac{\partial}{\partial \tau} - a\Delta, \quad \text{where } \lambda = \frac{2a}{n(a - b)} \quad \text{and} \quad \Delta = \sum_{i \leq n}\frac{\partial^2}{\partial q_i \partial \xi_i}. \qquad (7.4)$$

Taking into account the explicit form of the divergence of M_f we get

$$\mathfrak{b}_{a,b}(n) = \left\{ M_f \in \mathfrak{m}(n) \,\middle|\, (bn - aE)\frac{\partial f}{\partial \tau} = a\Delta f \right\}$$
$$= \{D \in \mathfrak{vect}(n\,|\,n+1) \,|\, L_D(\text{vol}_{q,\xi,\tau}^a \alpha_0^{a-bn}) = 0\}. \qquad (7.5)$$

It is subject to a direct verification that $\mathfrak{b}_{a,b}(n) \simeq \mathfrak{b}_\lambda(n)$ for $\lambda = \frac{2a}{n(a-b)}$. This isomorphism shows that λ actually runs over $\mathbb{C}P^1$, not $\mathbb{C}$. Obviously, the Lie superalgebra $\mathfrak{b}_\infty(n)$ differs from other members of the parametric family and should be considered separately.

As follows from the description of $\mathfrak{vect}(m\,|\,n)$-modules ([4]) and the criteria for simplicity of $\mathbb{Z}$-graded Lie superalgebras ([13]), the Lie superalgebras $\mathfrak{b}_\lambda(n)$ are simple for $n > 1$ and $\lambda \neq 0, 1, \infty$. It is also clear that the Lie superalgebras $\mathfrak{b}_\lambda(n)$ are non-isomorphic for distinct λ's, bar occasional isomorphisms in small dimensions.

The Lie superalgebra $\mathfrak{b}(n) = \mathfrak{b}_0(n)$ is not simple: it has an ε-dimensional, i.e., $(0\,|\,1)$-dimensional, center. At $\lambda = 1$ and ∞ the Lie superalgebras $\mathfrak{b}_\lambda(n)$ are not simple either: they has an ideal of codimension ε^n and ε^{n+1}, respectively. The corresponding exact sequences are

$$0 \to \mathbb{C}M_1 \to \mathfrak{b}(n) \to \mathfrak{le}(n) \to 0,$$
$$0 \to \mathfrak{b}_1'(n) \to \mathfrak{b}_1(n) \to \mathbb{C} \cdot M_{\xi_1 \cdots \xi_n} \to 0, \qquad (7.6)$$
$$0 \to \mathfrak{b}_\infty'(n) \to \mathfrak{b}_\infty(n) \to \mathbb{C} \cdot M_{\tau\xi_1 \cdots \xi_n} \to 0.$$

Clearly, at the exceptional values of λ, i.e., 0, 1, and ∞, the deformations of $\mathfrak{b}_\lambda(n)$ should be investigated extra carefully; for the complete description of deformation of $\mathfrak{b}_\lambda(n)$, see [20].

7.3. *The Lie (super)algebras preserving symmetric bilinear forms*

If $p = 2$, the analogs of symplectic and periplectic Lie (super)algebras accrue additional elements: If the matrix of the bilinear form $\mathcal{B}$ is Π_{2n} (resp. $\Pi_{n\,|\,n}$), then $\mathfrak{aut}(\mathcal{B})$ consists of the (super)matrices of the form

$$\begin{pmatrix} A & B \\ C & A^t \end{pmatrix}, \tag{7.7}$$

where B and C are symmetric. Denote these *general* Lie (super)algebras $\mathfrak{aut}_{\mathrm{gen}}(\mathcal{B})$; for $\mathcal{B} = \Pi_{2n}$ (resp. $\Pi_{n\,|\,n}$) the notation is $\mathfrak{o}_{\mathrm{gen}}(2n)$ (resp. $\mathfrak{pe}_{\mathrm{gen}}(n)$).

Let ZD denote the space of symmetric matrices with zeros on their main diagonals. The derived Lie (super)algebra $\mathfrak{aut}^{(1)}(\mathcal{B})$ consists of the (super)matrices of the form (7.7), where $B, C \in ZD$. In other words, these Lie (super)algebras resemble the orthogonal Lie algebras. On these Lie (super)algebras $\mathfrak{aut}^{(1)}(\mathcal{B})$ the following (super)trace (*half-trace*) is defined:

$$\mathrm{htr} : \begin{pmatrix} A & B \\ C & A^t \end{pmatrix} \to \mathrm{tr}\ A. \tag{7.8}$$

The traceless Lie sub(super)algebra of $\mathfrak{aut}^{(1)}(\mathcal{B})$ is isomorphic to $\mathfrak{aut}^{(2)}(\mathcal{B})$.

There is, however, an algebra $\widetilde{\mathfrak{aut}}(\mathcal{B})$, such that $\mathfrak{aut}^{(1)}(\mathcal{B}) \subset \widetilde{\mathfrak{aut}}(\mathcal{B}) \subset \mathfrak{aut}(\mathcal{B})$, consisting of (super)matrices of the form (7.7), where $B \in ZD$s (or isomorphic to it version with $C \in ZD$). Shchepochkina suggests to denote this $\widetilde{\mathfrak{aut}}(\mathcal{B})$ by $\mathfrak{op}$ if $\mathcal{B}$ is even and $\mathfrak{pe}$ if $\mathcal{B}$ is odd. Consider now these cases separately.

7.4. *Generalized Cartan prolongations of the Lie (super)algebras preserving symmetric bilinear forms*

7.4.1. *Let $p \neq 2$ and $\mathfrak{g}_0 = \mathfrak{pe}_B(n)$*

If the form $\mathcal{B}$ is in canonical shape $\mathcal{B} = \Pi_{n\,|\,n}$, then $\mathfrak{g}_0$ consists of the supermatrices of the form

$$X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \quad \text{where } B \text{ is symmetric and } C \text{ antisymmetric.} \tag{7.9}$$

Clearly, $\mathrm{str}\ X = 2\,\mathrm{tr}\ A$. We also have $\mathfrak{g}^{(1)} = \mathfrak{spe}(n)$, i.e., is of codimension 1 and singled out by the condition $\mathrm{str}\ X = 0$, which is equivalent to $\mathrm{tr}A = 0$.

The Lie superalgebra $\mathfrak{le}(n; \underline{N}\,|\,n)$ is, by definition, the Cartan prolong $(\mathrm{id}, \mathfrak{pe}(n))_{*,\underline{N}}$.

Over $\mathbb{C}$, there is no shearing parameter, and $\mathfrak{le}(n) := \mathfrak{le}(n\,|\,n)$ is spanned by the elements Le_f, where $f \in \mathbb{C}[q, \xi]$.

Let $p > 2$. If $\underline{N}_i = \infty$ for all coordinates, the generating functions are $f \in \mathcal{O}[q; \underline{N}\,|\,\xi]$, whereas if $N_i < \infty$ for some i, the generating functions are $f \in \mathcal{O}(q; \underline{N}\,|\,\xi) \cup \mathrm{Span}(q_i^{p^{N_i}})$.

The prolong $(\mathrm{id}, \mathfrak{spe}(n))_{*,\underline{N}}$ is singled out by the condition

$$\mathrm{div}\ Le_f = 0 \Leftrightarrow \Delta f = 0, \quad \text{where } \Delta = \sum_{i \leq n} \frac{\partial^2}{\partial q_i \partial \xi_i}.$$

The operator Δ is, therefore, the Cartan prolong of the supertrace expressed as an operator acting on the space of generating functions.

What modifications should be performed in the above description if $p = 2$?

The Lie superalgebra $\mathfrak{pe}(n)_{\mathrm{gen}}$ is larger than $\mathfrak{pe}(n)$: both B and C are symmetric, see (7.7). Observe that $\mathfrak{pe}(n)_{\mathrm{gen}} \subset \mathfrak{sl}(n \,|\, n)$.

The Cartan prolong $(\mathrm{id}, \mathfrak{pe}(n)_{\mathrm{gen}})_{*,\underline{N}}$ if $\underline{N} = \underline{N}_\infty$ consists of the regular part Reg and an additional part Irreg

$$\mathrm{Reg} = \mathrm{Span}(Le_f \,|\, f \in \mathcal{O}[q; \underline{N} \,|\, \xi]), \quad \mathrm{Irreg} = \mathrm{Span}(\xi_i \partial_{u_i})_{i=1}^n.$$

The part Irreg corresponds to the non-existing generating functions ξ_i^2.

The the additional part Irreg does not change while the regular part is of the form looking alike for any $p > 2$:

$$\mathrm{Reg} = \mathrm{Span}(Le_f \,|\, f \in \mathcal{O}(q; \underline{N} \,|\, \xi) \cup \mathrm{Span}(q_i^{2^{N_i}})), \quad \mathrm{Irreg} = \mathrm{Span}(\xi_i \partial_{u_i})_{i=1}^n.$$

We denote this Cartan prolong $\mathfrak{le}(n; \underline{N} \,|\, n)_{\mathrm{gen}} := (\mathrm{id}, \mathfrak{pe}(n)_{\mathrm{gen}})_{*,\underline{N}}$. Clearly, it is contained in $\mathfrak{svect}(n; \underline{N} \,|\, n)$, and therefore coincides with $\mathfrak{sle}(n; \underline{N} \,|\, n)_{\mathrm{gen}}$.

For $\mathfrak{g} = \mathfrak{pe}(n)_{\mathrm{gen}}$, their derived $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$ and the Cartan prolongs of these derived are already considered. We saw that $\mathfrak{g}^{(1)}$ consists of supermatrices of the form (7.7) with zero-diagonal matrices B and C, and $\mathfrak{g}^{(2)}$ is singled out of $\mathfrak{g}^{(1)}$ by the condition $\mathrm{htr} = 0$. The Cartan prolongs of each of these Lie superalgebras only have the regular part:

$$\begin{aligned} (\mathrm{id}, \mathfrak{g}^{(1)})_{*,\underline{N}} &= \mathrm{Span}(Le_f \,|\, f \in \mathcal{O}(q; \underline{N}_s \,|\, \xi)); \\ (\mathrm{id}, \mathfrak{g}^{(2)})_{*,\underline{N}} &= \mathrm{Span}(Le_f \,|\, f \in \mathcal{O}(q; \underline{N}_s \,|\, \xi) \text{ and } \Delta f = 0). \end{aligned} \tag{7.10}$$

Consider now the direct analog of the complex superalgebra $\mathfrak{pe}(n)$, i.e., the Lie superalgebra consisting of the supermatrices of the form (7.7) with $B \in ZD$. It is this Lie superalgebra which is natural to designate by $\mathfrak{pe}(n)$. Its commutant $\mathfrak{pe}(n)^{(1)}$ is of codimension 1 and singled out in $\mathfrak{pe}(n)$ by the condition $\mathrm{htr} = 0$.

Thus, htr plays the role of supertrace on $\mathfrak{g} = \mathfrak{pe}(n)$.

The Cartan prolong $(\mathrm{id}, \mathfrak{pe}(n))_{*,\underline{N}}$ consists of the regular part only, and therefore looks the same for any $p > 0$. The Cartan prolong $(\mathrm{id}, (\mathfrak{pe}(n))^{(1)})_{*,\underline{N}}$ is singled out in $\mathfrak{le}(n; \underline{N} \,|\, n)$ by the following condition in terms of generating functions: $\Delta(f) = 0$.

Thus, the "correct" direct analogs of the complex Lie superalgebras $\mathfrak{sle}(n)$ and $\mathfrak{spe}(n)$ are $(\mathrm{id}, (\mathfrak{pe}(n))^{(1)})_{*,\underline{N}}$ and $\mathfrak{pe}(n)^{(1)}$, respectively.

Remark 7.1. For $\underline{N}$ with $N_i < \infty$ for all i, the Lie superalgebra $\mathfrak{le}(n; \underline{N} \,|\, n)^{(1)}$ is spanned, for *any* $p > 0$, by the elements $f \in \mathcal{O}(q; \underline{N} \,|\, \xi)$, whereas the "virtual" generating functions belonging to $\cup_i \mathrm{Span}(q_i^{p^{N_i}})$ determine outer derivations of $\mathfrak{le}(n; \underline{N} \,|\, n)^{(1)}$. (Indeed: The brackets and squarings are given in terms of generating functions and there is no way to obtain $q_i^{p^{N_i}}$ from lesser powers.) In other words, to obtain a simple Lie superalgebra, we have to take generating functions from the space $\mathcal{O}(q; \underline{N} \,|\, \xi)$.

If $p > 0$, the element of the highest degree does not belong to $\mathfrak{le}(n; \underline{N} \,|\, n)^{(1)}$:

$$f = q_1^{p^{N_1}-1} \cdots q_n^{p^{N_n}-1} \xi_1 \cdots \xi_n \notin \mathfrak{le}(n; \underline{N} \,|\, n)^{(1)}.$$

7.5. *On $\mathfrak{m}$ and $\mathfrak{b}$*

First of all, observe that $\mathfrak{le}_{\text{gen}}$ has no non-trivial central extension. Only $\mathfrak{le}$ has it; *this* central extension is a correct direct analog of the complex Buttin superalgebra $\mathfrak{b}$.

To pass from $\mathfrak{b}(n; \underline{N} \,|\, n)$ to $\mathfrak{m}(n; \underline{N} \,|\, n+1)$, we have to add the central element to $\mathfrak{b}(n)_0 = \mathfrak{pe}(n)$; this element will serve as a grading operator of the prolong. We see that $\mathfrak{m}$ is the generalized Cartan prolong of $(\mathfrak{b}(n)_-, \mathfrak{cb}(n)_0)$.

The commutant of $\mathfrak{m}(n; \underline{N} \,|\, n+1)_0$ is the same as that of $\mathfrak{b}(n)_0 = \mathfrak{pe}(n)$, so is of codimension 2. Hence there are two traces on $\mathfrak{m}(n; \underline{N} \,|\, n+1)_0$, and therefore there are two divergences on $\mathfrak{m}$. One of them is

$$\partial_\tau, \quad \text{more precisely } D_\tau := \partial_\tau \circ \text{sign}, \tag{7.11}$$

i.e., the operator such that

$$D_\tau(f) = (-1)^{p(f)} \partial_\tau(f) \quad \text{for any } f \in \mathcal{O}(q; \underline{N} \,|\, \xi) \tag{7.12}$$

since (see [29]) this should be the map *commuting, not **super**commuting* with $\mathfrak{m}_-$. The condition $D_\tau(f) = 0$ singles out precisely $\mathfrak{b}(n)$.

7.5.1. $\mathfrak{sb}(n; \underline{N})$

The definition of $\mathfrak{sb}(n; \underline{N})$ is the same for any characteristic p (in terms of generating "functions" from the appropriate space $\mathcal{F}$):

$$\mathfrak{sb}(n; \underline{N}) = \text{Span}(f \in \mathcal{F} \,|\, \Delta(f) = 0). \tag{7.13}$$

7.5.2. $\mathfrak{b}_{a,b}(n; \underline{N} \,|\, n+1)$ *for $p = 2$*

The other trace on $\mathfrak{m}_0(n; \underline{N} \,|\, n+1)$ is htr. On $\mathfrak{le}$, the Cartan prolong of this trace was the operator Δ. But Δ does not commute with the whole $\mathfrak{m}_-$. To obtain the $\mathfrak{m}_-$-invariant prolong of this trace on $\mathfrak{m}_0$, we have to express htr in terms of the operators commuting with $\mathfrak{m}_-$ (*Y-type vectors* in terms of [29]). Taking $\mathfrak{m}_-$ spanned by the elements

$$\mathfrak{m}_{-2} = \mathbb{K} \cdot \partial_\tau, \quad \mathfrak{m}_{-1} = \text{Span}(\partial_{q_i} + \xi_i \partial_\tau, \partial_{\xi_i})_{i=1}^n,$$

we see that the operators commuting with $\mathfrak{m}_-$ are spanned by

$$\partial_\tau, \quad \partial_{q_i}, \quad \partial_{\xi_i} + q_i \partial_\tau.$$

In terms of these operators the vector field M_f takes the form:

$$M_f = f \partial_\tau + \sum_i (\partial_{q_i}(f)(\partial_{\xi_i} + q_i \partial_\tau) + (\partial_{\xi_i} + q_i \partial_\tau)(f) \partial_{q_i}) \tag{7.14}$$

and the invariant prolong of htr takes the form:

$$\Delta^{\mathfrak{m}}(f) = \sum_i ((\partial_{\xi_i} + q_i \partial_\tau) \partial_{q_i}(f) = \Delta(f) + E_q \partial_\tau(f), \quad \text{where } E_q = \sum_i q_i \partial_{q_i}. \tag{7.15}$$

The condition $\Delta^{\mathfrak{m}}(f) = 0$ singles out the $p = 2$ analog of $\mathfrak{sm}$, whereas the condition

$$a\partial_\tau(f) + b\Delta^{\mathfrak{m}}(f) = 0 \tag{7.16}$$

singles out the $p = 2$ analog of $\mathfrak{b}_{a,b}(n; \underline{N} \,|\, n + 1)$.

Having applied to the above described constructions the functor F of forgetting the superstructure we obtain new subalgebras in the Lie algebras of Hamiltonian and contact vector fields; some of them — $\mathfrak{b}_{a,b}(n; \underline{N} \,|\, n + 1)$ — have no analogs for $p \neq 2$.

8. The Contact Brackets. Contact Lie Superalgebras as CTS-Prolongs

All the minuses in what follows are used in order to make expressions look like their analogs in characteristic $p \neq 2$ (if this analogs exist).

8.1. *The odd (*contact*) form*

8.1.1. *Notation*

The superdimension of the superspace on which the contact structure is considered is equal to either $2k_{\bar{0}} + 1 \,|\, 2k_{\bar{1}}$ or $2k_{\bar{0}} + 1 \,|\, 2k_{\bar{1}} + 1$. Set $k = k_{\bar{0}} + k_{\bar{1}}$.

The indeterminates are denoted by t, p_i, q_i, θ, where $i = 1, \ldots, k$ and θ is present only if the superdimension is equal to $2k_{\bar{0}} + 1 \,|\, 2k_{\bar{1}} + 1$. The parities of the indeterminates are:

$$\Pi(t) = \bar{0}; \quad \Pi(\theta) = \bar{1}; \quad \Pi(p_i) = \Pi(q_i) = \begin{cases} \bar{0} & \text{if } i \leq k_{\bar{0}}; \\ \bar{1} & \text{if } i > k_{\bar{0}}. \end{cases}$$

The contact form is of the shape

$$\alpha = dt + \sum_i p_i dq_i (+\theta d\theta).$$

8.1.2. *Basis*

The basis elements of the zeroth part $\mathfrak{g}_0$ of the contact Lie superalgebra $\mathfrak{g}$ in its standard $\mathbb{Z}$-grading are as follows (since some of these elements have no analogs in characteristic $p \neq 2$, we use $+$ sign everywhere):

	Element	Conditions on existence
1	$t\partial_t + \sum p_i\partial_{p_i}$	$\text{sdim} = 2k_{\bar{0}} + 1 \,|\, 2k_{\bar{1}}$
2	$p_i\partial_{p_j} + q_j\partial_{q_i}$	$1 \leq i, j \leq k$
3	$p_ip_j\partial_t + p_i\partial_{q_j} + p_j\partial_{q_i}$	$1 \leq i \neq j \leq k$
4	$q_iq_j\partial_t + q_i\partial_{p_j} + q_j\partial_{p_i}$	$1 \leq i \neq j \leq k$
5	$p_i^{(2)}\partial_t + p_i\partial_{q_i}$	$1 \leq i \leq k_{\bar{0}}$ (i.e., the p_i and q_i are even)
6	$q_i^{(2)}\partial_t + q_i\partial_{p_i}$	$1 \leq i \leq k_{\bar{0}}$ (i.e., the p_i and q_i are even)
7	$p_i\theta\partial_t + p_i\partial_\theta$	$\text{sdim} = 2k_{\bar{0}} + 1 \,|\, 2k_{\bar{1}} + 1, \, 1 \leq i \leq k$
8	$q_i\theta\partial_t + q_i\partial_\theta$	$\text{sdim} = 2k_{\bar{0}} + 1 \,|\, 2k_{\bar{1}} + 1, \, 1 \leq i \leq k$
9	$\theta\partial_\theta$	$\text{sdim} = 2k_{\bar{0}} + 1 \,|\, 2k_{\bar{1}} + 1$

Remark 8.1. Clearly, the elements with θ (Cases 7–9) exist only if there is an odd number of odd variables; it is remarkable, though, that the element of case 1 has no analog in dimension $2k_{\bar{0}} + 1 \,|\, 2k_{\bar{1}} + 1$.

8.1.3. *Realization of $\mathfrak{g}_0$ in terms of ortho-orthogonal Lie superalgebras*

If sdim $= 2k_{\bar{0}} + 1 \mid 2k_{\bar{1}}$, then this algebra is the subalgebra of $\mathfrak{oo}'_{\Pi\Pi}(2k_{\bar{0}} \mid 2k_{\bar{1}})$ spanned by the grading operator $I_0 = \mathrm{diag}(1_{k_{\bar{0}} \mid k_{\bar{1}}}, 0_{k_{\bar{0}} \mid k_{\bar{1}}})$ and the supermatrices of format $k_{\bar{0}} \mid k_{\bar{1}} \mid k_{\bar{0}} \mid k_{\bar{1}}$ and having the form

$$
\begin{pmatrix} A & C \\ D & A^T \end{pmatrix}
\quad
\begin{array}{l}
\text{where } A \in \mathfrak{gl}(k_{\bar{0}} \mid k_{\bar{1}}), \\
C, D \text{ are symmetric,} \\
C_{ii} = D_{ii} = 0 \quad \text{for all } k_{\bar{0}} < i \le k.
\end{array}
$$

If sdim $= 2k_{\bar{0}} + 1 \mid 2k_{\bar{1}} + 1$, then $\mathfrak{g}_0$ is NOT a subalgebra of $\mathfrak{oo}_{\Pi\Pi}(2k_{\bar{0}} \mid 2k_{\bar{1}} + 1)$. It is a subalgebra of the algebra of supermatrices preserving the *degenerate* form

$$
\mathrm{antidiag}(1_{k_{\bar{0}} \mid k_{\bar{1}}}, 0, 1_{k_{\bar{0}} \mid k_{\bar{1}}}),
$$

and it is spanned by supermatrices of format $k_{\bar{0}} \mid k_{\bar{1}} + 1 \mid k_{\bar{0}} \mid k_{\bar{1}}$ and having the form

$$
\begin{pmatrix} A & X & C \\ 0 & z & 0 \\ D & Y & A^T \end{pmatrix}
\quad
\begin{array}{l}
\text{where } A \in \mathfrak{gl}(k_{\bar{0}} \mid k_{\bar{1}}), \\
C, D \text{ are symmetric,} \\
C_{ii} = D_{ii} = 0 \quad \text{for all } k_{\bar{0}} < i \le k_{\bar{0}} + k_{\bar{1}}, \\
X, Y \text{ are arbitrary } k_{\bar{0}} \mid k_{\bar{1}}\text{-vectors,} \\
z \in \mathbb{K}.
\end{array}
$$

8.2. The even (pericontact) form

In this case the superdimension of the superspace is equal to $2k \mid 2k + 1$, the coordinates are τ, q_i, ξ_i, where

$$
\Pi(p_i) = \bar{0}; \quad \Pi(\tau) = \Pi(\xi_i) = \bar{1}.
$$

Let the pericontact form be of the simplest form (4.20.1).

8.2.1. *Basis*

The basis elements of the zeroth part $\mathfrak{g}_0$ of the pericontact Lie superalgebra $\mathfrak{g}$ in its standard $\mathbb{Z}$-grading are as follows:

	Element	Conditions on existence
1	$\tau \partial_\tau + \sum q_i \partial_{q_i}$	—
2	$q_i \partial_{q_j} - \xi_j \partial_{\xi_i}$	$1 \le i, j \le k$
3	$q_i q_j \partial_\tau - q_i \partial_{\xi_j} - q_j \partial_{\xi_i}$	$1 \le i < j \le k$
4	$\xi_i \xi_j \partial_\tau - \xi_i \partial_{q_j} + \xi_j \partial_{q_i}$	$1 \le i < j \le k$
5	$q_i^{(2)} \partial_t - q_i \partial_{\xi_i}$	$1 \le i \le k$

(The situation here is analogous to the case of $p \ne 2$, so we keep minus signs.)

8.2.2. *Realization of $\mathfrak{g}_0$ in terms of $\mathfrak{pe}(k)$*

The Lie superalgebra $\mathfrak{g}_0$ is the subalgebra of $\mathfrak{pe}(k)$ consisting of supermatrices of format $k \,|\, k$ and of the form

$$\begin{pmatrix} A & C \\ D & A^T \end{pmatrix} \quad \begin{array}{l} \text{where } A \in \mathfrak{gl}(k), \\ C \text{ is symmetric}, \\ D \in ZD(k). \end{array}$$

9. Divergence-Free Subalgebras

9.1. *Contact vector fields*

We have

$$\mathrm{div}(K_f) = \begin{cases} (k+1)\dfrac{\partial f}{\partial x_0} & \text{if } n = 2k; \\ 0 & \text{if } n = 2k+1; \\ x_{2k+1}\dfrac{\partial g}{\partial x_0} & \text{if } n = 2k+1, \text{ case (4.14), b1}; \\ g & \text{if } n = 2k+1, \text{ case (4.14), b2}. \end{cases} \tag{9.1}$$

$$\mathrm{div}(M_f) = (k+1)\dfrac{\partial f}{\partial x_0}. \tag{9.2}$$

9.2. *Hamiltonian vector fields*

We have

$$\mathrm{div}(H_{I,f}) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial_{x_i}^2}; \;\; \mathrm{div}(H_{II,f}) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial_{x_i}^2}; \;\; \mathrm{div}(H_{\Pi\Pi,f}) = 0;$$

$$\mathrm{div}(H_{\Pi,f}) = 0; \qquad \mathrm{div}(H_{I\Pi,f}) = \sum_{i=1}^{n_{\bar 0}} \frac{\partial^2 f}{\partial_{x_i}^2}; \;\; \mathrm{div}(H_{\Pi I,f}) = \sum_{i=n_{\bar 0}+1}^{n} \frac{\partial^2 f}{\partial_{x_i}^2}. \tag{9.3}$$

$$\mathrm{div}(Le_f) = 0. \tag{9.4}$$

Acknowledgments

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