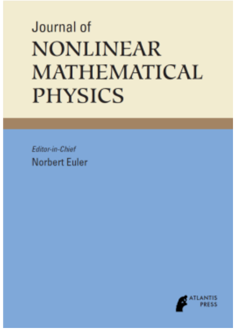

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DERIVATIONS OF THE 3-LIE ALGEBRA REALIZED BY $gl(n, \mathbb{C})$

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This paper studies structures of the 3-Lie algebra M realized by the general linear Lie algebra $gl(n, \mathbb{C})$. We show that M has only one nonzero proper ideal. We then give explicit expressions of both derivations and inner derivations of M. Finally, we investigate substructures of the (inner) derivation algebra of M.

Keywords: 3-Lie algebra; derivation algebra; inner derivation algebra; realization.

2010 Mathematics Subject Classification: 17B05, 17D99

1. Introduction

Derivations appear in many areas of mathematics. The derivation of an algebra is itself a significant object of study, a useful tool in constructing new algebraic structures and an important bridge relating algebras to geometries. For example, let $(A, \circ)$ be a commutative associative algebra and D a derivation of A. Then A defines a left-symmetric algebra $(A, *)$ by $x * y = x \circ D(y)$ and A defines a Lie algebra $(A, [,])$ in which the bracket operation $[x, y] = x \circ D(y) - y \circ D(x)$ for all $x, y \in A$ (see [1]). Also, from n commutative derivations $D_1, \ldots, D_n$ of $(A, \circ)$, we can obtain an n-Lie algebra by the n-ary operation $[x_1, \ldots, x_n] = det(c_{ij})$, where $x_1, \ldots, x_n \in A, c_{ij} = D_i(x_j), 1 \leq i, j \leq n$ (see [2]).

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An n-Lie algebra is a vector space endowed with an n-ary skew-symmetric multiplication satisfying the n-Jacobi identity (see [1] for more details):

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n]. \tag{1.1}$$

A lot of evidence shows that n-Lie algebras are useful in many fields in mathematics and mathematical physics. Indeed, motivated by some problems of quark dynamics, Nambu [3] introduces a ternary generalization of Hamiltonian dynamics by means of the ternary Poisson bracket

$$[f_1, f_2, f_3] = \det\left(\frac{\partial f_i}{\partial x_j}\right).$$

This identity satisfies (1.1). Takhtajan describes a theory of n-Poisson manifolds systematically [4]. From the work of Bagger and Lambert ([5–7]) and Gustavsson [8] one sees that the generalized Jacobi identity for 3-Lie algebras is essential in studying supersymmetry in superconformal fields. Their work stimulates the interest of researchers in mathematics and mathematical physics on n-Lie algebras [9–11].

There is a need, either from pure mathematics or physics point of view, to construct new n-Lie algebras and investigate their structures and derivations. However, it is difficult in general to deal with the n-ary ($n \geq 3$) multiplication in n-Lie algebras. So it is natural to construct n-Lie algebras from well-known existing algebras, which leads to the so-called "realization" theory. The authors of [12] give some realizations of 3-Lie algebras, showing that every m-dimensional 3-Lie algebra with $m \leq 5$ can be realized by existing algebras. They also investigate structures of the 3-Lie algebra $gl(n, \mathbb{C})_{tr}$, given in [13], realized by the general linear Lie algebra $gl(n, \mathbb{C})$, where $\mathbb{C}$ is the field of complex numbers. They conclude that every non-abelian realization $gl(n, \mathbb{C})_f$ for f in the dual space of $gl(n, \mathbb{C})$ is isomorphic to $gl(n, \mathbb{C})_{tr}$.

In the present paper we are interested in the 3-Lie algebra $gl(n, \mathbb{C})_{tr}$, which will be denoted by M for simplicity. We show some preliminary results about M in Sec. 2. We then give explicit expressions of inner derivations of M and describe the structure of the inner derivation algebra of M in Sec. 3. Finally, we investigate the structure of the derivation algebra of M in Sec. 4.

2. The Realization of a 3-Lie Algebra

The ternary operation of M is given by

$$[x, y, z] = \mathrm{Tr}(x)[y, z] + \mathrm{Tr}(y)[z, x] + \mathrm{Tr}(z)[x, y], \quad \text{for all } x, y, z \in M. \tag{2.1}$$

The subalgebra $[M, M, M]$ in M is called the derived algebra of M and will be denoted by M^1. Then

$$M^1 = \{x \in M \mid \mathrm{Tr}(x) = 0\}.$$

Choose a basis $\{e_{i,j}, e_{t,t} - e_{t+1,t+1}, \frac{1}{n}e | 1 \leq i \neq j \leq n, 1 \leq t \leq n-1\}$ of M, where $e_{i,j}$ are matrix units with 1 at the (i, j)-entry and 0 otherwise, and e is the unit matrix with 1 at the (i,i)-entry for $i=1,\dots,n$ and 0 elsewhere. It follows from (2.1) that

$$e_{i,j}=\frac{1}{n}[e,e_{i,k},e_{k,j}],\quad e_{i,i}-e_{j,j}=\frac{1}{n}[e,\,e_{i,j},\,e_{j,i}],\quad \text{for } 1\le i\ne j\ne k\ne i\le n.$$

It is routine to check that

$$M=M^1\oplus\mathbb{C}e\quad\text{(as a direct sum of vector spaces).}$$

To study the structure of M, we arrange the above basis elements in the following order.

$$e_{1,1}-e_{2,2},\, e_{1,2},\dots,e_{1,n},\, e_{2,1},e_{2,2}-e_{3,3},e_{2,3},\dots,e_{2,n},\dots,$$
$$e_{t,1},e_{t,2},\dots,e_{t,t-1},e_{t,t}-e_{t+1,t+1},e_{t,t+1},\dots,e_{t,n},\dots,$$
$$e_{n-1,1},e_{n-1,2},\dots,e_{n-1,n-2},e_{n-1,n-1}-e_{n,n},e_{n-1,n},e_{n,1},e_{n,2},\dots,e_{n,n-1},\frac{1}{n}e$$

for $1\le t\le n-1$. For simplicity, we denote them by e_i, respectively, where $i=1,\dots,n^2$. In other words, we write $e_1=e_{1,1}-e_{2,2}, e_2=e_{1,2},\dots,$ and $e_{n^2}=\frac{1}{n}e$. Then

$$\mathrm{Tr}(e_{n^2})=1,\quad \mathrm{Tr}(e_i)=0\quad\text{for } 1\le i\le n^2-1,\quad\text{and}$$
$$[e_{n^2},e_i,e_j]=[e_i,e_j]\quad\text{and}\quad [e_i,e_j,e_k]=0\quad\text{for } 1\le i,j,k\le n^2-1. \tag{2.2}$$

Therefore, the derived algebra $M^1=\sum_{i=1}^{n^2-1}\mathbb{C}e_i$. Clearly, the dimension of M^1 is n^2-1.

Theorem 2.1. *The derived algebra M^1 is the only nonzero proper ideal of M and the center of M is zero.*

Proof. If I is a nonzero proper ideal of M, then $[e_{n^2},I,M]=[I,gl(n,\mathbb{C})]\subseteq I$, that is I is a proper ideal of $gl(n,\mathbb{C})$. It follows that I equals the derived algebra of $gl(n,\mathbb{C})$, and hence I equals M^1 as vector spaces. Next, if z is in the center of M, then $[z,x,e_{n^2}]=0$ for all $x\in M$. We have $[z,x]=0$ for all $x\in gl(n,\mathbb{C})$, and hence $z=\alpha e_{n^2}$ for some $\alpha\in\mathbb{C}$. It follows from (2.2) that $z=0$. □

An ideal I of a 3-Lie algebra L is 2-solvable, if there is an integer $r\ge 0$ such that $I^{(r,2)}=0$, where $I^{(0,2)}=I$ and inductively $I^{(s,2)}=[I^{(s-1,2)},I^{(s-1,2)},I]$ for $s>0$. If L has no nonzero 2-solvable ideals, then L is called 2-semisimple. The 3-Lie algebra M is 2-semisimple. See [12] for more details.

3. Inner Derivation Algebra of M

We now study the inner derivation algebra of M. Let $x,y\in M$. The left multiplication operator $\mathrm{ad}(x,y)$ of M is defined by $\mathrm{ad}(x,y)(z)=[x,y,z]$ for all $z\in M$. Let $\mathrm{ad}(M)$ be the Lie algebra generated by all left multiplication operators $\mathrm{ad}(x,y)$ for $x,y\in M$. A simple calculation yields that

$$\mathrm{ad}(e_{n^2},e_i)(e_k)=[e_i,e_k],\qquad 1\le i,k\le n^2-1.$$

$$\mathrm{ad}(e_i,e_j)(e_k)=\begin{cases}0, & 1\le i,j,k\le n^2-1,\\ [e_i,e_j], & k=n^2.\end{cases}$$

We then have, for $1 \le i, j, k, l < n^2 - 1$,

$$[\mathrm{ad}(e_i, e_j), \mathrm{ad}(e_k, e_l)] = 0 \quad \text{and}$$
$$[\mathrm{ad}(e_{n^2}, e_i), \mathrm{ad}(e_k, e_l)] = \mathrm{ad}([e_i, e_k], e_l) + \mathrm{ad}(e_k, [e_i, e_l]). \tag{3.1}$$

Let $S(M)$ be the set of left multiplication operators of the form $\mathrm{ad}(e_{n^2}, x)$ for $x \in M$. Then $S(M)$ is a subalgebra of $\mathrm{ad}(M)$. We obtain the following result.

Theorem 3.1. *The Lie algebra $S(M)$ is isomorphic to the simple Lie algebra* $\mathrm{ad}(gl(n, \mathbb{C}))$.

Proof. Define $\sigma : S(M) \to \mathrm{ad}(gl(n, \mathbb{C}))$ by

$$\sigma(\mathrm{ad}(e_{n^2}, x)) = \mathrm{ad}(x) \quad \text{for all } x \in M,$$

where $\mathrm{ad}(x)$ is the left multiplication operator of $gl(n, \mathbb{C})$. Then $\sigma(\mathrm{ad}(e_{n^2}, x)) = 0$ if and only if x is in the center of the general linear Lie algebra $gl(n, \mathbb{C})$. It follows that σ is bijective. Since

$$[\mathrm{ad}(e_{n^2}, x), \mathrm{ad}(e_{n^2}, y)] = \mathrm{ad}(e_{n^2}, [e_{n^2}, x, y]) = \mathrm{ad}(e_{n^2}, [x, y]) \in S(M),$$

we have $\sigma([\mathrm{ad}(e_{n^2}, x), \mathrm{ad}(e_{n^2}, y)]) = \mathrm{ad}([x, y]) = [\sigma(\mathrm{ad}(e_{n^2}, x)), \sigma(\mathrm{ad}(e_{n^2}, y))]$. Therefore, σ is an isomorphism. □

Corollary 3.2. *The Lie algebra $S(M)$ is isomorphic to $sl(n, \mathbb{C})$ and* $\dim S(M) = n^2 - 1$.

Let $A(M)$ be the subalgebra of $\mathrm{ad}(M)$ generated by $\{\mathrm{ad}(e_i, e_j) | 1 \le i, j \le n^2 - 1\}$. Then we have

$$[\mathrm{ad}(e_{n^2}, x), \mathrm{ad}(e_i, e_j)] = \mathrm{ad}([e_{n^2}, x, e_i], e_j) + \mathrm{ad}(e_i, [e_{n^2}, x, e_j]) \in A(M), \tag{3.2}$$

and $[\mathrm{ad}(e_k, e_l), \mathrm{ad}(e_i, e_j)] = 0$ for $1 \le i, j, k, l \le n^2 - 1$. This leads to the following result.

Theorem 3.3. *The inner derivation algebra of M is a direct sum of $S(M)$ and $A(M)$ (as subalgebras, not ideals). Furthermore, $A(M)$ is an abelian ideal and*

$$[S(M), A(M)] = A(M).$$

Proof. The result follows from Theorem 3.1 and the identity (3.1). □

We investigate the structures of $S(M)$ and $A(M)$. To this end, we need explicit matric expressions of all inner derivations. From (2.1), the multiplication table of M with respect to the basis $e_1, \ldots, e_{n^2}$ is as follows:

$$[e_{n^2}, e_{j+n(i-1)}, e_{i+n(j-1)}] = e_{i+n(i-1)} + e_{i+1+ni} + \cdots + e_{j-1+n(j-2)}, \quad 1 \le i < j \le n;$$
$$[e_{n^2}, e_{j+n(i-1)}, e_{i+n(j-1)}] = -(e_{j+n(j-1)} + e_{j+1+nj} + \cdots + e_{i-1+n(i-2)}), \quad 1 \le j < i \le n;$$
$$[e_{n^2}, e_{j+n(i-1)}, e_{k+n(j-1)}] = e_{k+n(i-1)}, \quad 1 \le i \ne j \ne k \ne i \le n;$$
$$[e_{n^2}, e_{j+n(i-1)}, e_{i+n(s-1)}] = -e_{j+n(s-1)}, \quad 1 \le i \ne j \ne s \ne i \le n;$$
$$[e_{n^2}, e_{t+n(t-1)}, e_{k+n(t-1)}] = e_{k+n(t-1)}, \quad 1 \le t \le n-1,\ 1 \le k \le n,\ k \ne t,\ k \ne t+1;$$
$$[e_{n^2}, e_{t+n(t-1)}, e_{t+1+n(s-1)}] = e_{t+1+n(s-1)}, \quad 1 \le t \le n-1,\ 1 \le s \le n,\ s \ne t,\ s \ne t+1;$$
$$[e_{n^2}, e_{t+n(t-1)}, e_{t+1+n(t-1)}] = 2e_{t+1+n(t-1)}, \quad 1 \le t \le n-1;$$

$$
\begin{aligned}
[e_{n^2}, e_{t+n(t-1)}, e_{t+nt}] &= -2e_{t+nt}, \quad 1 \le t \le n-1;\\
[e_{n^2}, e_{t+n(t-1)}, e_{k+nt}] &= -e_{k+nt}, \quad 1 \le t \le n-1,\ 1 \le k \le n,\ k \ne t,\ k \ne t+1;\\
[e_{n^2}, e_{t+n(t-1)}, e_{t+n(s-1)}] &= -e_{t+n(s-1)}, \quad 1 \le t \le n-1,\ 1 \le s \le n,\ s \ne t,\ s \ne t+1;\\
[e_{n^2}, e_{j+n(i-1)}, e_{k+n(s-1)}] &= 0, \quad 1 \le i \ne j \le n,\ 1 \le s \ne k \le n,\ j \ne s,\ k \ne i;\\
[e_{n^2}, e_{t+n(t-1)}, e_{i+n(i-1)}] &= 0, \quad 1 \le t < i \le n-1.\\
[e_i, e_j, e_k] &= 0, \quad 1 \le i \ne j \ne k \ne i \le n^2-1.
\end{aligned}
$$

We compute the matrix forms, relative to the basis $e_1, \dots, e_{n^2}$, of the generators

$$
\begin{gathered}
\operatorname{ad}(e_{n^2}, e_{t+n(t-1)}), \quad \operatorname{ad}(e_{n^2}, e_{j+n(i-1)}), \quad \operatorname{ad}(e_{j+n(i-1)}, e_{t+n(t-1)}),\\
\operatorname{ad}(e_{j+n(i-1)}, e_{k+n(s-1)}), \quad \operatorname{ad}(e_{p+n(p-1)}, e_{q+n(q-1)}),
\end{gathered}
$$

where $1 \le i, j, s, k, \le n, i \ne j, s \ne k, 1 \le t \le n-1, 1 \le p \ne q \le n-1$. Suppose that the matrix form of $\operatorname{ad}(x,y)$, for every $x, y \in M$, relative to the same basis is

$$
B(x,y) = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n}\\ B_{21} & B_{22} & \cdots & B_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{pmatrix},
$$

where B_{ij} is an $n \times n$-matrix over $\mathbb{C}$. Denote by E_{ij} the matrix unit, of size n^2, whose (i,j)-entry is 1 and other entries are zero. We introduce

$$
\bar{\delta}_{i,j} = \begin{cases} 0 & \text{if } i = j;\\ 1 & \text{if } i \ne j \end{cases}
$$

to denote the dual Kronecker delta; it will be used below. We divide the entire argument into five cases and obtain the following identities using the above multiplication table.

Case 1: For $1 \le t \le n-1$, let

$$
\operatorname{ad}(e_{n^2}, e_{t+n(t-1)})(e_1, \dots, e_{n^2}) = (e_1, \dots, e_{n^2}) B(e_{n^2}, e_{t+n(t-1)}).
$$

Then

$$
B(e_{n^2}, e_{t+n(t-1)}) = \operatorname{diag}(B_{11}, \dots, B_{tt}, B_{t+1,t+1}, \dots, B_{nn}), \tag{3.3}
$$

where

$B_{tt} = \operatorname{diag}(1, \dots, 1, 0, 2, 1, \dots, 1)$ whose $(t+1)$-th position is 2,

$B_{t+1,t+1} = \operatorname{diag}(-1, \dots, -1, -2, 0, -1, \dots, -1)$ whose t-th position is -2,

$B_{ii} = \operatorname{diag}(0, \dots, 0, -1, 1, 0, \dots, 0)$ whose t-th position is -1, for $1 \le i \le n$ with $i \ne t, t+1$.

Thus the matrix form of $\operatorname{ad}(e_{n^2}, e_{t+n(t-1)})$ relative to the basis $e_1, \dots, e_{n^2}$ is

$$
\begin{aligned}
\Gamma_t = \sum_{j=1}^{n} (&E_{t+1+(j-1)n, t+1+(j-1)n} - E_{t+(j-1)n, t+(j-1)n}\\
&+ E_{j+n(t-1), j+n(t-1)} - E_{j+nt, j+nt}).
\end{aligned} \tag{3.4}
$$

Case 2: A similar discussion to the above shows that the matrix form of $\mathrm{ad}(e_{n^2}, e_{j+n(i-1)})$ for $1 \le i < j \le n$ under the basis $e_1, \ldots, e_{n^2}$ is

$$\Phi_{j,i} = E_{j+n(i-1),i-1+n(i-2)} - E_{j+n(i-1),j-1+n(j-2)} - \sum_{0 \le k \ne j-1}^{n-1} E_{j+nk,i+nk} + \sum_{k=1}^{n} \bar{\delta}_{n^2,\,k+n(j-1)} E_{k+n(i-1),k+n(j-1)} + \sum_{k=i+1}^{j-1} E_{k+n(k-1),i+n(j-1)}, \tag{3.5}$$

where we agree that $E_{j+n(i-1),i-1+n(i-2)} = 0$ if $i = 1$.

Similarly, for $1 \le j < i \le n$, the matrix form of $\mathrm{ad}(e_{n^2}, e_{j+n(i-1)})$ relative to the basis $e_1, \ldots, e_{n^2}$ is

$$\Psi_{j,i} = E_{j+n(i-1),i-1+n(i-2)} - E_{j+n(i-1),j-1+n(j-2)} - \sum_{0 \le k \ne j-1}^{n-1} \bar{\delta}_{n^2,\,i+nk} E_{j+nk,i+nk} + \sum_{1 \le k \ne i}^{n} E_{k+n(i-1),k+n(j-1)} - \sum_{k=j}^{i-1} E_{k+n(k-1),i+n(j-1)}, \tag{3.6}$$

where we agree that $E_{j+n(i-1),j-1+n(j-2)} = 0$ if $j = 1$.

Case 3: For $1 \le i \ne j \le n$ and $1 \le s \ne k \le n$, by (2.2) and (3.2) the matrix form of $\mathrm{ad}(e_{j+n(i-1)}, e_{k+n(s-1)})$ with respect to the basis $e_1, \ldots, e_{n^2}$ is

$$B(e_{j+n(i-1)}, e_{k+n(s-1)}) = \begin{cases} 0, & j \ne s, \ i \ne k; \\ E_{k+n(i-1),n^2}, & j = s, \ i \ne k; \\ -E_{j+n(s-1),n^2}, & j \ne s, \ i = k; \\ \sum_{r=i}^{j-1} E_{r+n(r-1),n^2}, & j = s > i = k; \\ -\sum_{p=j}^{i-1} E_{p+n(p-1),n^2}, & j = s < i = k. \end{cases} \tag{3.7}$$

Case 4: When $1 \le s \ne k \le n$ and $1 \le t \le n-1$, the matrix form of $\mathrm{ad}(e_{k+n(s-1)}, e_{t+n(t-1)})$ relative to the basis $e_1, \ldots, e_{n^2}$ is

$$B(e_{k+n(s-1)}, e_{t+n(t-1)}) = \begin{cases} 0, & t \ne s, \ s-1, k, k-1; \\ -E_{k+n(t-1),n^2}, & t = s, \ t \ne k-1; \\ -2E_{t+1+n(t-1),n^2}, & t = s, \ t = k-1; \\ -E_{t+1+n(s-1),n^2}, & t \ne s, \ t = k-1; \\ 2E_{t+nt,n^2}, & t = k, \ t = s-1; \\ E_{k+nt,n^2}, & t \ne k, \ t = s-1; \\ E_{t+n(s-1),n^2}, & t = k, \ t \ne s-1. \end{cases} \tag{3.8}$$

Case 5: If $1 \leq p \neq q \leq n-1, \mathrm{ad}(e_{p+n(p-1)}, e_{q+n(q-1)}) = 0$.

Summarizing above discussions, we are now in a position to state the following results about the structure of $S(M)$ and $A(M)$ in terms of elementary matrices of size n^2.

Theorem 3.4. *Let M be the 3-Lie algebra defined by* (2.1). *Then*

$$S(M) = \sum_{1 \leq i < j \leq n} \mathbb{C}\Phi_{j,i} + \sum_{1 \leq j < i \leq n} \mathbb{C}\Psi_{j,i} + \sum_{t=1}^{n-1} \mathbb{C}\Gamma_t.$$

Proof. By the multiplication table of M and the matrix forms above, the left multiplication operators $\mathrm{ad}(e_{n^2}, e_{j+n(i-1)})$ and $\mathrm{ad}(e_{n^2}, e_{t+n(t-1)})$ for $1 \leq t \leq n-1, 1 \leq i \neq j \leq n$ are linear independent. Furthermore, they form a basis of $S(M)$. In other words,

$$S(M) = \sum_{1 \leq i \neq j \leq n} \mathbb{C}\, \mathrm{ad}(e_{n^2}, e_{j+n(i-1)}) + \sum_{t=1}^{n-1} \mathbb{C}\, \mathrm{ad}(e_{n^2}, e_{t+n(t-1)}).$$

The result of the theorem follows from the identities (3.4), (3.5), and (3.6). □

Theorem 3.5. *Let M be the 3-Lie algebra defined by* (2.1). *Then*

$$A(M) = \sum_{i=1}^{n^2-1} \mathbb{C}E_{i,n^2}.$$

Proof. A direct (yet tedious) calculation yields that $\mathrm{ad}(e_{3+n}, e_{1+2n}), \mathrm{ad}(e_3, e_{2+2n})$, $\mathrm{ad}(e_{1+n(i-1)}, e_k)$ for $2 \leq i \neq k \leq n, \mathrm{ad}(e_2, e_{k+n})$ and $\mathrm{ad}(e_{2+n(k-1)}, e_{1+n})$ for $3 \leq k \leq n$, and $\mathrm{ad}(e_{i+1+(i-1)n}, e_{i+\mathrm{in}})$ for $1 \leq i \leq n-1$ form a basis of $A(M)$. We then have

$$A(M) = \mathbb{C}\, \mathrm{ad}(e_3, e_{2+2n}) + \mathbb{C}\, \mathrm{ad}(e_{3+n}, e_{1+2n}) + \sum_{1 \leq i \leq n-1} \mathbb{C}\, \mathrm{ad}(e_{i+1+(i-1)n}, e_{i+\mathrm{in}})$$

$$+ \sum_{2 \leq i \neq k \leq n} \mathbb{C}\, \mathrm{ad}(e_{1+n(i-1)}, e_k) + \sum_{3 \leq k \leq n} (\mathbb{C}\, \mathrm{ad}(e_2, e_{k+n}) + \mathbb{C}\, \mathrm{ad}(e_{2+n(k-1)}, e_{1+n})).$$

In view of the identities (3.7) and (3.8), the theorem holds. □

The following corollaries follow from Theorems 3.3, 3.4, and 3.5.

Corollary 3.6. *The inner derivation algebra of M is the non-essential extension of $S(M)$ by $A(M)$, and $A(M)$ is an irreducible $S(M)$-module in the regular representation.*

Corollary 3.7. *Use the notation above we obtain that $A(M)$ is an abelian ideal of* $\mathrm{ad}(M)$ *and* $\dim S(M) = \dim A(M)$ *and* $\dim \mathrm{ad}(M) = 2(n^2-1)$.

Proof. By Theorem 3.5 $A(M)$ is abelian. Theorem 3.4 indicates that $\dim S(M) = n^2 - 1$ and it can be seen from Theorem 3.5 that $\dim A(M) = n^2 - 1$. We then have that $\dim \mathrm{ad}(M) = 2(n^2-1)$ by Theorem 3.3. □

4. Derivation Algebra of M

In this section we determine explicit expressions of derivations of M and describe its derivation algebra $\mathrm{Der}\, M$. Let D be a derivation of M and

$$D(e_i) = \sum_{j=1}^{n^2} a_{j,i} e_j, \quad a_{i,j} \in \mathbb{C}, \quad 1 \le i,\ j \le n^2. \tag{4.1}$$

Then the matrix form of D under the basis $e_1, \dots, e_{n^2}$ is $D = \sum_{i,j=1}^{n^2} a_{i,j} E_{i,j}$. Note that $[e_i, e_j, e_k] = 0$ for $1 \le i, j, k \le n^2 - 1$. Also, there exist numbers $b_s^{ij} \in \mathbb{C}$ such that $[e_{n^2}, e_i, e_j] = \sum_{s=1}^{n^2-1} b_s^{ij} e_s$ for $1 \le s \le n^2 - 1$. Then for $1 \le i, j, k \le n^2 - 1$ we have

$$[D(e_i), e_j, e_k] + [e_i, D(e_j), e_k] + [e_i, e_j, D(e_k)] = 0,$$

$$D([e_{n^2}, e_i, e_j]) = [D(e_{n^2}), e_i, e_j] + [e_{n^2}, D(e_i), e_j] + [e_{n^2}, e_i, D(e_j)] = \sum_{s=1}^{n^2-1} b_s^{ij} D(e_s).$$

Therefore, for $1 \le i, j, k \le n^2 - 1$,

$$a_{n^2,i}[e_{n^2}, e_j, e_k] + a_{n^2,j}[e_i, e_{n^2}, e_k] + a_{n^2,k}[e_i, e_j, e_{n^2}] = 0,$$

$$a_{n^2,n^2}[e_{n^2}, e_i, e_j] + \sum_{p=1}^{n^2} a_{p,i}[e_{n^2}, e_p, e_j] + \sum_{p=1}^{n^2} a_{p,j}[e_{n^2}, e_i, e_p] = \sum_{s=1}^{n^2-1} \left(b_s^{ij} \sum_{p=1}^{n^2} a_{p,s} e_p \right).$$

A rigorous calculation shows that the constraints on the coefficients in the identity (4.1) are as follows. We omit its tedious details.

$$\begin{cases}
a_{j,i} = a_{j+nk,i+kn}, & 1 \le i < j \le n, \quad 1 \le k \ne j-1 \le n-1; \\
a_{j,i} = -a_{k+n(i-1),k+n(j-1)}, & 1 \le i < j \le n, 1 \le k \le n, \ \ k + n(j-1) \ne n^2; \\
a_{j,i} = -a_{k+n(k-1),i+n(j-1)}, & i+1 < j, \quad i+1 \le k \le j-1; \\
a_{j,i} = a_{j+nk,i+nk}, & 1 \le j < i \le n, \quad 1 \le k \ne j-1 \le n-1, \quad i + nk \ne n^2; \\
a_{j,i} = -a_{k+n(i-1),k+n(j-1)}, & 1 \le j < i \le n, 1 \le k \ne i \le n; \\
a_{j,i} = a_{k+n(k-1),i+n(j-1)}, & 1 \le j < i \le n, \quad j \le k \le i-1; \\
a_{j,i} = -a_{j+n(i-1),i-1+n(i-2)}, & 1 \le i \ne j \le n, \quad i \ne 1; \\
a_{j,i} = a_{j+n(i-1),j-1+n(j-2)}, & 1 \le i \ne j \le n, \quad j \ne 1; \\
a_{i,n^2} = k_i, & 1 \le i \le n^2 - 1, \quad k_i \in \mathbb{C}; \\
a_{n^2,n^2} = -a_{t+n(t-1),t+n(t-1)} & \\
\qquad\ \ = -a_{t+1+nt,t+1+nt}, & 1 \le t \le n-2; \\
a_{n^2,n^2} = -a_{i+n(j-1),i+n(j-1)} & \\
\qquad\quad -a_{j+n(i-1),j+n(i-1)} & \\
\qquad\quad +a_{t+n(t-1),t+n(t-1)}, & 1 \le i \ne j \le n, \quad 1 \le t \le n-1; \\
a_{i,j} = 0, & \text{otherwise.}
\end{cases} \tag{4.2}$$

For convenience we introduce the following notation for $1 \le i < j \le n$ and $j \ne i+1$,

$$\Upsilon_{i,j} = E_{j+n(i-1),j+n(i-1)} - E_{i+n(j-1),i+n(j-1)}, \tag{4.3}$$

$$\Theta = E_{n^2,n^2} - \sum_{i=1}^{n^2-1} E_{i,i}. \tag{4.4}$$

Theorem 4.1. *Every derivation D of M is of the matrix form below with respect the basis* $e_1, \ldots, e_{n^2}$,

$$\begin{aligned} D =& a_{n^2,n^2}\Theta + \sum_{1\le i<j\le n} a_{j,i}\Phi_{j,i} + \sum_{1\le j<i\le n} a_{j,i}\Psi_{j,i} + \sum_{t=1}^{n-1} a_{t+1+n(t-1),t+1+n(t-1)}\Gamma_t \\ &+ \sum_{1\le i<j\le n, j\ne i+1} a_{j+n(i-1),j+n(i-1)}\Upsilon_{i,j} + \sum_{i=1}^{n^2-1} a_{i,n^2}E_{i,n^2}. \end{aligned}$$

Proof. It follows from the multiplication table of M that $\Upsilon_{i,j}, \Theta, (1 \le i < j \le n)$ are derivations of M. Furthermore, from the identities (3.3), (3.4), (3.5), (3.6), (3.7) and (3.8), $\Phi_{j,i}(1 \le i < j \le n), \Psi_{j,i}(1 \le j < i \le n), \Gamma_t(1 \le t \le n-1)$ and $E_{i,n^2}(1 \le i \le n^2-1)$ are derivations of M. In view of the constraint in (4.2) on the coefficients of each derivation of M, we obtain that

$$\begin{aligned} &\Theta, \\ &\Phi_{j,i}, \quad 1 \le i < j \le n; \\ &\Psi_{j,i}, \quad 1 \le j < i \le n; \\ &\Gamma_t, \quad 1 \le t \le n-1; \\ &\Upsilon_{i,j}, \quad 1 \le i < j \le n, \quad j \ne i+1; \\ &E_{i,n^2}, \quad 1 \le i \le n^2-1, \end{aligned}$$

form a basis of $\operatorname{Der} M$. The completes the proof. □

Let

$$T(M) = \sum_{1\le i<j\le n, j\ne i+1} \mathbb{C}\Upsilon_{i,j} \oplus \mathbb{C}\Theta.$$

The following theorem describes the structure of the derivation algebra of M.

Theorem 4.2. *As a direct sum of subalgebras,*

$$\operatorname{Der}(M) = S(M) \oplus A(M) \oplus T(M),$$

where $[T(M), T(M)] = 0$ *and* $[T(M), A(M)] = A(M)$. *Moreover,* $\dim \operatorname{Der}(M) = \frac{5n^2-3n}{2}$.

Proof. From the identities (4.3) and (4.4), we have $[T(M), T(M)] = 0$. By Theorem 3.5 and the identities (4.3) and (4.4) we obtain $[T(M), A(M)] = A(M)$. The dimension of $Der(M)$ follows from Theorem 4.1. □

Corollary 4.3. *The derivation algebra of M is the non-essential extension of the abelian algebra $T(M)$ by the inner derivation algebra* $\mathrm{ad}(M)$.

Proof. This is the direct result of Theorem 4.2. $\square$

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