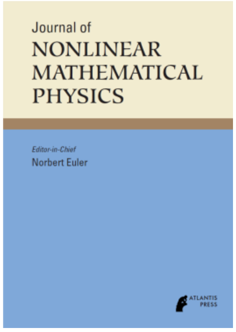

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 ISSN (Print): 1776-0852
Journal Home Page: https://www.tandfonline.com/loi/tnmp20

Integrability of Lie Systems through Riccati Equations

José F. Cariñena, Javier de Lucas

To cite this article: José F. Cariñena, Javier de Lucas (2011) Integrability of Lie Systems through Riccati Equations, Journal of Nonlinear Mathematical Physics 18:1, 29–54, DOI: https://doi.org/10.1142/S1402925111001131

To link to this article: https://doi.org/10.1142/S1402925111001131

Published online: 04 January 2021

Article

Journal of Nonlinear Mathematical Physics, Vol. 18, No. 1 (2011) 29–54

DOI: 10.1142/S1402925111001131

INTEGRABILITY OF LIE SYSTEMS THROUGH RICCATI EQUATIONS

JOSÉ F. CARIÑENA

Departamento de Física Teórica
Universidad de Zaragoza, c. Pedro Cerbuna 12
Zaragoza, 50.009, Spain
jfc@unizar.es

JAVIER DE LUCAS

Institute of Mathematics
Polish Academy of Sciences, ul. Śniadeckich 8
Warszawa, P.O. Box 21, 00-596, Poland
delucas@impan.pl

Received 27 January 2010
Accepted 13 May 2010

Integrability conditions for Lie systems are related to reduction or transformation processes. We here analyze a geometric method to construct integrability conditions for Riccati equations following these approaches. Our procedure provides us with a unified geometrical viewpoint that allows us to analyze some previous works on the topic and explain new properties. Moreover, this new approach can be straightforwardly generalized to describe integrability conditions for any Lie system. Finally we show the usefulness of our treatment in order to study the problem of the linearizability of Riccati equations.

Keywords: Lie–Scheffers systems; Riccati equations; integrability conditions.

2000 Mathematics Subject Classification: 17B80, 34A05, 34A26, 34A34

1. Introduction

The Riccati equation

$$\frac{dy}{dt} = b_0(t) + b_1(t)y + b_2(t)y^2, \tag{1.1}$$

is the simplest nonlinear differential equation [5, 8] and it appears in many different fields of Mathematics and Physics [3, 7, 9, 14, 30, 35, 40, 41, 44]. It is essentially the only first-order ordinary differential equation on the real line admitting a nonlinear superposition principle [27, 44] and in spite of its apparent simplicity, its general solution cannot be described by means of quadratures except in some very particular cases [1, 10, 21, 22, 24, 25, 28, 29, 31–34, 36–38, 42].

In this paper we review the geometric approach to Riccati equations according to the results of the works [5, 10] with the aim of proving that integrability conditions of Riccati equations can be understood in a very general way from the point of view of the theory of Lie systems [11, 16, 18–20, 27, 43, 44]. Furthermore, we recover various known results as particular cases of our approach and the method here derived can be applied to any other Lie system, e.g. [3, 7].

Each Lie system is associated with a Lie algebra of vector fields, the so-called Vessiot–Guldberg Lie algebra [7, 15, 16, 18, 43]. This Lie algebra can be used to classify those Lie systems that can be integrated by quadratures [8]. For instance, it is a known fact that Lie systems related to solvables Vessiot–Guldberg Lie algebras, e.g. affine homogeneous systems or linear homogeneous systems, can be integrated by quadratures [8, 22, 23]. Nevertheless, the general solution of Lie systems related to non-solvables Lie algebras, e.g. Riccati equations, cannot be completely determined and it frequently relies on the knowledge of certain special functions [22], the solution of other equations [32, 33, 41], etc.

The method developed here allows us to determine integrable cases of Lie systems related to non-solvables Vessiot–Gulberg Lie algebras. Such a procedure is detailed for Riccati equations, which are associated with a non-solvable Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, but it can also be applied to other Lie systems related to Vessiot–Guldberg Lie algebras isomorphic to this one, for example, the Lie systems connected to Milne–Pinney equations [4], Ermakov systems [6], harmonic oscillators [7], etc.

We finally analyze the linearization of Riccati equation by means of our new approach and recover a characterization previously proved by Ibragimov [21]. Furthermore, we detail a, as far as we know, new result about the properties of linearization of Riccati equations.

The paper is organized as follows. For the sake of completeness, we report some known facts on the integrability of Riccati equations in Sec. 2 and we review the geometric interpretation of the general Riccati equation as a t-dependent vector field on the one-point compactification of the real line in Sec. 3. As a consequence of the latter, Riccati equations can be studied through equations on $SL(2,\mathbb{R})$. Section 4 is devoted to reporting some known results on the action of the group of curves in $SL(2,\mathbb{R})$ on the set of Riccati equations and how this action can be seen in terms of transformations of the corresponding equations on $SL(2,\mathbb{R})$, see [12]. In Sec. 5 we build up a Lie system describing the transformation process of Riccati equations through the action of curves of $SL(2,\mathbb{R})$ and then, in Sec. 6, we analyze the general characteristics of our approach into integrability conditions and how the transformation processes described by the previous Lie system can be used to give a unified approach to the results of [5, 8]. In Sec. 7 we develop a particular case of the procedures of Sec. 6 in order to recover some results found in the literature [1, 25, 34, 36–38]. Section 8 is devoted to analyzing the theory of integrability through reduction from our new viewpoint. Finally, in Sec. 9 we describe how our Lie system for studying integrability conditions enables us to explain when certain linear fractional transformations allow us to linearize Riccati equations. As a particular instance we obtain a result given in [21, 38].

2. Integrability of Riccati Equations

In order to provide a first insight into the study of integrability conditions for Riccati equations, and for any Lie system in general, we review in this section some known results about the integrability of Riccati equations.

As a first particular example, Riccati equations (1.1) are integrable by quadratures when $b_2 = 0$. Indeed, in such a case these equations reduce to an inhomogeneous linear equation and two quadratures allow us to find the general solution.

Additionally, under the change of variable $w = -1/y$ the Riccati equation (1.1) reads

$$\frac{dw}{dt} = b_0(t)\, w^2 - b_1(t)\, w + b_2(t)$$

and if we suppose $b_0 = 0$ in Eq. (1.1), then the mentioned change of variable transforms the given equation into an integrable linear one.

Another very well-known property on integrability of Riccati equations is that given a particular solution $y_1(t)$ of Eq. (1.1), then the change of variable $y = y_1(t) + z$ leads to a new Riccati equation for which the coefficient of the term independent of z is zero, i.e.

$$\frac{dz}{dt} = [2\, b_2(t)\, y_1(t) + b_1(t)]z + b_2(t)\, z^2, \tag{2.1}$$

and, as we pointed out before, it can be reduced to an inhomogeneous linear equation with the change $z = -1/u$. Therefore, given one particular solution, the general solution can be found by means of two quadratures.

If not only one but two particular solutions, $y_1(t)$ and $y_2(t)$, of Eq. (1.1) are known, the general solution can be found by means of only one quadrature. In fact, the change of variable $z = (y - y_1(t))/(y - y_2(t))$ transforms the original equation into a homogeneous first-order linear differential equation in the new variable z and therefore the general solution can immediately be found.

Finally, giving three particular solutions, $y_1(1), y_2(t), y_3(t)$, the general solution can be written, without making use of any quadrature, in the following way

$$y(t) = \frac{y_1(t)(y_3(t) - y_2(t)) - k y_2(t)(y_1(t) - y_3(t))}{(y_3(t) - y_2(t)) - k(y_1(t) - y_3(t))}.$$

This is a nonlinear superposition rule studied in [9] from a group theoretical perspective.

The simplest case of Eq. (1.1), when it is an autonomous equation (b_0, b_1 and b_2 constants), has been fully studied (see e.g. [13] and references therein) and it is integrable by quadratures. This result can be considered as a consequence of the existence of a constant (maybe complex) solution enabling us to reduce the Riccati equation into an inhomogeneous linear one. Moreover, the separable Riccati equations of the form

$$\frac{dy}{dt} = \varphi(t)(c_0 + c_1\, y + c_2\, y^2),$$

with $\varphi(t)$ a non-vanishing function on a certain open interval $I \subset \mathbb{R}$ and c_0, c_1, c_2 real numbers, are integrable because a new time function $\tau = \tau(t)$ such that $d\tau/dt = \varphi(t)$ reduces the above equation into an autonomous one. Furthermore, the above Riccati equations are also integrable as they accept, in similarity to the autonomous case, a constant (maybe complex) solution.

3. Geometric Approach to Riccati Equations

Let us report in this section some known results about the geometrical approach to the Riccati equation [5]. Such a point of view is used in next sections to investigate integrability conditions for these equations and, in general, for any Lie system.

From the geometric viewpoint, the Riccati equation (1.1) can be considered as a differential equation determining the integral curves for the t-dependent vector field [2]

$$X(t,y) = [b_0(t) + b_1(t)y + b_2(t)y^2]\frac{\partial}{\partial y}. \tag{3.1}$$

This t-dependent vector field is a linear combination with t-dependent coefficients $b_0(t)$, $b_1(t)$ and $b_2(t)$ of the three vector fields

$$L_0 = \frac{\partial}{\partial y}, \quad L_1 = y\frac{\partial}{\partial y}, \quad L_2 = y^2\frac{\partial}{\partial y}, \tag{3.2}$$

with defining relations

$$[L_0, L_1] = L_0, \quad [L_0, L_2] = 2L_1, \quad [L_1, L_2] = L_2, \tag{3.3}$$

and therefore spanning a three-dimensional Lie algebra of vector fields V. Consequently, Riccati equations are Lie systems [27] and the Lie algebra V, the so-called Vessiot–Guldberg Lie algebra [16, 43], is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ being here considered as made up by traceless 2×2 matrices. A particular basis for $\mathfrak{sl}(2,\mathbb{R})$ is given by

$$M_0 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad M_1 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.4}$$

Moreover, it can be checked that the linear map $\rho\colon \mathfrak{sl}(2,\mathbb{R}) \to V$ obeying $\rho(M_j) = L_j$, with $j = 0, 1, 2$, is a Lie algebra isomorphism.

Note that L_2 is not a complete vector field on $\mathbb{R}$. However we can do the one-point compactification $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$ and then L_0, L_1 and L_2 are complete vector fields on $\overline{\mathbb{R}}$. Consequently, these vector fields are fundamental vector fields corresponding to the action $\Phi\colon (A, y) \in SL(2,\mathbb{R}) \times \overline{\mathbb{R}} \mapsto \Phi(A, y) \in \overline{\mathbb{R}}$ given by

$$\Phi(A,y) = \begin{cases} \dfrac{\alpha\, y + \beta}{\gamma\, y + \delta} & y \neq -\dfrac{\delta}{\gamma}, \quad y \neq \infty, \\ \dfrac{\alpha}{\gamma} & y = \infty, \\ \infty & y = -\dfrac{\delta}{\gamma}, \end{cases} \qquad \text{with} \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{R}). \tag{3.5}$$

Denote by $X^{\mathtt{R}}_j$ and $X^{\mathtt{L}}_j$, $j = 1, 2, 3$, the right- and left-invariant vector fields on $SL(2,\mathbb{R})$ such that $X^{\mathtt{R}}_j(I) = X^{\mathtt{L}}_j(I) = M_j$. Moreover, these vector fields satisfy that $X^{\mathtt{R}}_j(A) = M_j \cdot A$ and $X^{\mathtt{L}}_j(A) = A \cdot M_j$, with "$\cdot$" the usual matrix multiplication.

A remarkable property is that if $A(t)$ is the integral curve for the t-dependent vector field

$$X(t) = -\sum_{j=0}^{2} b_j(t)\, X^{\mathtt{R}}_j,$$

starting from the neutral element in $SL(2,\mathbb{R})$, i.e. $A(0) = I$, then $A(t)$ satisfies the equation

$$\dot{A}(t)A^{-1}(t) = -\sum_{j=0}^{2} b_j(t) M_j \equiv a(t), \tag{3.6}$$

and the solution of Riccati equation (1.1) with initial condition $y(0) = y_0$ is given by $y(t) = \Phi(A(t), y_0)$ [11].

Note that the r.h.s. in Eq. (3.6) is a curve in $T_I SL(2,\mathbb{R})$ that can be identified to a curve in the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of left-invariant vector fields on $SL(2,\mathbb{R})$ through the usual isomorphism: we relate each left-invariant vector field X^{L} to the element $X^{\mathrm{L}}(I) \in T_I SL(2,\mathbb{R})$. From now on, we do not distinguish explicitly elements in $T_I SL(2,\mathbb{R})$ and its corresponding ones in $\mathfrak{sl}(2,\mathbb{R})$.

In summary, the general solution of Riccati equations (1.1) can be obtained through solutions of an equation like (3.6) starting from I. Consequently, we have reduced the problem of finding the general solution of Riccati equations to determining the solution of Eq. (3.6) beginning at the neutral element of $SL(2,\mathbb{R})$. Note that, in a similar way, this procedure can be applied to any Lie system [3].

4. Transformation Laws of Riccati Equations

In this section we briefly describe an important property of Lie systems, in the particular case of Riccati equations, which plays a very relevant rôle for establishing, as indicated in [10], integrability criteria: *The group $\mathcal{G}$ of curves in a Lie group G associated with a Lie system*, *here $SL(2,\mathbb{R})$*, *acts on the set of these Lie systems*, *here Riccati equations*.

More explicitly, fixed a basis of vector fields on $\overline{\mathbb{R}}$, for instance $\{L_j \,|\, j = 0,1,2\}$, which spans a Vessiot–Guldberg Lie algebra of vector fields isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, each Riccati equation (1.1) can be considered as a curve $(b_0(t), b_1(t), b_2(t))$ in $\mathbb{R}^3$. The point now is that each element of the group of smooth curves in $SL(2,\mathbb{R})$, i.e. $\bar{A} \in \mathcal{G} \equiv \mathrm{Map}(\mathbb{R}, SL(2,\mathbb{R}))$, transforms every curve $y(t)$ in $\overline{\mathbb{R}}$ into a new curve $y'(t)$ in $\overline{\mathbb{R}}$ given by $y'(t) = \Phi(\bar{A}(t), y(t))$. Moreover, the t-dependent change of variables $y'(t) = \Phi(\bar{A}(t), y(t))$ transforms the Riccati equation (1.1) into a new Riccati equation with new t-dependent coefficients, b'_0, b'_1, b'_2 given by

$$\begin{cases} b'_2 = \delta^2\, b_2 - \delta\gamma\, b_1 + \gamma^2\, b_0 + \gamma\dot{\delta} - \delta\dot{\gamma}, \\ b'_1 = -2\,\beta\delta\, b_2 + (\alpha\delta + \beta\gamma)\, b_1 - 2\,\alpha\gamma\, b_0 + \delta\dot{\alpha} - \alpha\dot{\delta} + \beta\dot{\gamma} - \gamma\dot{\beta}, \\ b'_0 = \beta^2\, b_2 - \alpha\beta\, b_1 + \alpha^2\, b_0 + \alpha\dot{\beta} - \beta\dot{\alpha}, \end{cases} \tag{4.1}$$

with

$$\bar{A}(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix}.$$

The above transformation defines an affine action (see e.g. [26] for the general definition of this concept) of the group $\mathcal{G}$ on the set of Riccati equations, see [10].

The group $\mathcal{G}$ also acts on the set of equations of the form (3.6) on $SL(2,\mathbb{R})$. In order to show this, note first that $\mathcal{G}$ acts on the left on the set of curves in $SL(2,\mathbb{R})$ by left translations, i.e. given two curves $A(t)$ and $\bar{A}(t)$ in $SL(2,\mathbb{R})$, the curve $\bar{A}(t)$ transforms the curve $A(t)$ into a new one $A'(t) = \bar{A}(t)A(t)$. Moreover, if $A(t)$ is a solution of Eq. (3.6), then the new curve $A'(t)$ satisfies a new equation like (3.6) but with a different right-hand side $a'(t)$. Differentiating the relation $A'(t) = \bar{A}(t)A(t)$ in terms of time and taking into account the form of (3.6), we get that the relation between the curves $a(t)$ and $a'(t)$ in $\mathfrak{sl}(2,\mathbb{R})$ is

$$a'(t) = \bar{A}(t)a(t)\bar{A}^{-1}(t) + \dot{\bar{A}}(t)\bar{A}^{-1}(t) = -\sum_{j=0}^{2} b'_j(t)M_j \tag{4.2}$$

and such a relation implies the expressions (4.1). Conversely, if $A'(t) = \bar{A}(t)A(t)$ is the solution for the equation corresponding to the curve $a'(t)$ given by the transformation rule (4.2), then $A(t)$ is the solution of Eq. (3.6).

To sum up, we have shown that it is possible to associate each Riccati equation with an equation on the Lie group $SL(2,\mathbb{R})$ and to define an infinite-dimensional group of transformations acting on the set of Riccati equations. Additionally, this process can be easily derived in a similar way for any Lie system. In such a case, we must consider an equation on a Lie group G associated with the corresponding Lie system and the group $\mathcal{G}$ of curves in G acting on the set of curves in G in the form $A'(t) = L_{\bar{A}(t)}A(t)$ instead of $A'(t) = \bar{A}(t)A(t)$. This action induces other action of $\mathcal{G}$ on the set of equations of the form (3.6) but on the Lie group G. More explicitly, a curve $\bar{A}(t) \in \mathcal{G}$ transforms an equation on G of the form (3.6) determined by a curve $a(t) \subset T_IG$ into a new one determined by the new curve $a'(t) \subset T_IG$ given by

$$a'(t) = \mathrm{Ad}_{\bar{A}(t)}a(t) + R_{\bar{A}^{-1}(t)*\bar{A}(t)}\dot{\bar{A}}(t). \tag{4.3}$$

5. Lie Structure of an Equation of Transformation of Lie Systems

Our aim in this section is to construct a Lie system describing the curves in $SL(2,\mathbb{R})$ relating two Riccati equations associated with a pair of equations in $SL(2,\mathbb{R})$ characterized by two curves $a(t), a'(t) \subset \mathfrak{sl}(2,\mathbb{R})$. By means of this Lie system we are going to explain in next sections the developments of [5, 8] and other works from a unified viewpoint.

Let us multiply Eq. (4.2) on the right by $\bar{A}(t)$ to get

$$\dot{\bar{A}}(t) = a'(t)\bar{A}(t) - \bar{A}(t)a(t). \tag{5.1}$$

If we consider Eq. (5.1) as a first-order differential equation in the coefficients of the curve $\bar{A}(t)$ in $SL(2,\mathbb{R})$, with

$$\bar{A}(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix}, \quad \alpha(t)\delta(t) - \beta(t)\gamma(t) = 1,$$

then system (5.1) reads

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \\ \dot{\delta} \end{pmatrix} = \begin{pmatrix} \dfrac{b'_1 - b_1}{2} & b_2 & b'_0 & 0 \\ -b_0 & \dfrac{b'_1 + b_1}{2} & 0 & b'_0 \\ -b'_2 & 0 & -\dfrac{b'_1 + b_1}{2} & b_2 \\ 0 & -b'_2 & -b_0 & -\dfrac{b'_1 - b_1}{2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}. \tag{5.2}$$

In order to determine the solutions $x(t) = (\alpha(t), \beta(t), \gamma(t), \delta(t))$ of the above system relating two different Riccati equations, we should check that actually the matrices $\bar{A}(t)$, whose elements are the corresponding components of $x(t)$, are related to matrices in $SL(2,\mathbb{R})$, i.e. we have to verify that at any time $\alpha\delta - \beta\gamma = 1$. Nevertheless, we can drop such a restriction because it can be automatically implemented by a restraint on the initial conditions for the solutions and hence we can deal with the variables $\alpha, \beta, \gamma, \delta$ in the system (5.2) as being independent. Consider now the vector fields

$$
\begin{aligned}
N_0 &= -\alpha\frac{\partial}{\partial\beta} - \gamma\frac{\partial}{\partial\delta}, & N_0' &= \gamma\frac{\partial}{\partial\alpha} + \delta\frac{\partial}{\partial\beta},\\
N_1 &= \frac{1}{2}\left(\beta\frac{\partial}{\partial\beta} + \delta\frac{\partial}{\partial\delta} - \alpha\frac{\partial}{\partial\alpha} - \gamma\frac{\partial}{\partial\gamma}\right), & N_1' &= \frac{1}{2}\left(\alpha\frac{\partial}{\partial\alpha} + \beta\frac{\partial}{\partial\beta} - \gamma\frac{\partial}{\partial\gamma} - \delta\frac{\partial}{\partial\delta}\right),\\
N_2 &= \beta\frac{\partial}{\partial\alpha} + \delta\frac{\partial}{\partial\gamma}, & N_2' &= -\alpha\frac{\partial}{\partial\gamma} - \beta\frac{\partial}{\partial\delta},
\end{aligned}
$$

satisfying the non-null commutation relations

$$
\begin{aligned}
&[N_0, N_1] = N_0, \quad [N_0, N_2] = 2N_1, \quad [N_1, N_2] = N_2,\\
&[N_0', N_1'] = N_0', \quad [N_0', N_2'] = 2N_1', \quad [N_1', N_2'] = N_2'.
\end{aligned}
$$

Note that as $[N_i, N_j'] = 0$, for $i, j = 0, 1, 2$, the linear system of differential equation (5.2) is a Lie system on $\mathbb{R}^4$ associated with a Lie algebra of vector fields isomorphic to $\mathfrak{g} \equiv \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$. This Lie algebra decomposes into a direct sum of two Lie algebras of vector fields isomorphic to $\mathfrak{sl}(2,\mathbb{R})$: the first one is spanned by $\{N_0, N_1, N_2\}$ and the second one by $\{N_0', N_1', N_2'\}$.

If we denote $x \equiv (\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4$, the system (5.2) is a differential equations on $\mathbb{R}^4$

$$
\frac{dx}{dt} = N(t, x),
$$

with N being the t-dependent vector field

$$
N(t, x) = \sum_{j=0}^{2} \big(b_\alpha(t)N_\alpha(x) + b_\alpha'(t)N_\alpha'(x)\big).
$$

The vector fields $\{N_0, N_1, N_2, N_0', N_1', N_2'\}$ span a regular involutive distribution $\mathcal{D}$ with rank three in almost any point of $\mathbb{R}^4$ and thus there exists, at least locally, a first-integral. We can check that the function

$$
I : x \equiv (\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4 \to I(x) \equiv \det x \equiv \alpha\delta - \beta\gamma \in \mathbb{R}
$$

is a first-integral for the vector fields in the distribution $\mathcal{D}$. Moreover, such a first-integral is related to the determinant of the matrix $\bar{A}$ with coefficients given by the components of $x = (\alpha, \beta, \gamma, \delta)$. Therefore, if we have a solution of the system (5.2) with an initial condition $\det x(0) = \alpha(0)\delta(0) - \beta(0)\gamma(0) = 1$, then $\det x(t) = 1$ at any time t and the solution can be understood as a curve in $SL(2,\mathbb{R})$.

In summary, we have proved that:

Theorem 5.1. *The curves in* $SL(2,\mathbb{R})$ *transforming Eq.* (3.6) *into a new equation of the same form but characterized by a new curve* $a'(t) = -\sum_{j=0}^{2} b'_j(t)M_j$ *are described through the solutions of the Lie system*

$$\frac{dx}{dt} = N(t,x) \equiv \sum_{j=0}^{2}(b_j(t)N_j(x) + b'_j(t)N'_j(x)) \tag{5.3}$$

such that $\det x(0) = 1$*. Furthermore, the above Lie system is related to a non-solvable Vessiot–Guldberg Lie algebra isomorphic to* $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$*.*

Corollary 5.1. *Given two Riccati equations associated with curves* $a'(t)$ *and* $a(t)$ *in* $\mathfrak{sl}(2,\mathbb{R})$ *there always exists a curve* $\bar{A}(t)$ *in* $SL(2,\mathbb{R})$ *transforming the Riccati equation related to* $a(t)$ *into the one associated with* $a'(t)$*. If furthermore* $\bar{A}(0) = I$*, this curve is uniquely defined.*

Proof. Given a matrix $A(0) \in SL(2,\mathbb{R})$ and an element $x(0)$ related to it, according to the theorem of existence and uniqueness of solutions for differential equations, the system (5.3), with the chosen $a'(t)$ and $a(t)$, admits a solution $x(t)$ with initial condition $x(0)$. As $\det(x(t)) = \det(x(0)) = 1$, such a solution, considered as a matrix $\bar{A}(t)$, belongs to $SL(2,\mathbb{R})$ and therefore there exists a solution $x(t)$ for the system (5.3) with initial condition $x(0)$ related to $\bar{A}(0)$. This proves the first statement of our corollary.

If the curve $\bar{A}(t)$ connecting two curves in $\mathfrak{sl}(2,\mathbb{R})$ satisfies $\bar{A}(0) = I$, it is the curve $x(t)$ in $\mathbb{R}^4$ being the solution of system (5.3) with initial condition $x(0) = (1,0,0,1)$, which is uniquely determined because of the theorem of existence and uniqueness of solutions of systems of first-order differential equations. □

Even if we know that given two equations on the Lie group $SL(2,\mathbb{R})$ there always exists a transformation relating both, in order to obtain such a curve we need to solve the Lie system (5.3). Unfortunately, such a Lie system is associated with a non-solvable Lie algebra and it is not easy in general to find its solutions, i.e. it is not integrable by quadratures and therefore such a curve cannot be easily found in the general case.

Nevertheless, we will explain many known properties and obtain new integrability conditions for Riccati equations by means of Theorem 5.1. Furthermore, the procedure to obtain the Lie system (5.3) can be generalized to deal with any Lie system related to a Lie group G with Lie algebra $\mathfrak{g}$. In this general case, relation (4.3) implies that

$$\dot{\bar{A}}(t) = R_{\bar{A}(t)*I}a'(t) - L_{\bar{A}(t)*I}a(t).$$

As $X^{\mathrm{R}}(t,\bar{A}) = R_{\bar{A}*I}a'(t)$ is a t-dependent right-invariant vector field on G and $X^{\mathrm{L}}(t,\bar{A}) = -L_{\bar{A}*I}a(t)$ a left-invariant one, the above system is the equation determining the integral curves of a time-dependent vector field with values in the linear space spanned by right- and left-invariant vector fields on G. Note that the family of left-invariant (right-invariant) vector fields on G spans a Lie algebra isomorphic to $\mathfrak{g}$ and, as right- and left-invariant vector fields commute among them, the set of vector fields spanned by both families is a Lie algebra of vector fields isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$. In this way, we get that the above system, relating two Lie systems associated with curves $a(t)$ and $a'(t)$ in $\mathfrak{g}$, is a Lie system related to a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$.

6. Lie Systems and Integrability Conditions

In this section some integrability conditions are analyzed from the perspective of the theory of Lie systems with $SL(2,\mathbb{R})$ as associated Lie group, with the aim of giving a unified approach to the reduction and transformations procedures described in [5, 8]. More explicitly, these methods are related to conditions for the existence of a curve in a previously chosen family of curves in $SL(2,\mathbb{R})$ connecting a curve $a(t) \subset \mathfrak{sl}(2,\mathbb{R})$ with a curve $a'(t)$ in a solvable Lie subalgebra of $\mathfrak{sl}(2,\mathbb{R})$. It is also shown that this viewpoint enables us to explain many of the previous results scattered in the literature about this topic and to prove other new properties.

As it was shown in Sec. 4, if the curve $\bar{A}(t) \subset SL(2,\mathbb{R})$ transforms the equation on this Lie group defined by the curve $a(t)$ into another one characterized by $a'(t)$ and $A'(t)$ is a solution for the equation similar to (3.6) for the primed system, i.e. characterized by $a'(t)$, then $A(t) = \bar{A}^{-1}(t)A'(t)$ is a solution for the equation in $SL(2,\mathbb{R})$ characterized by $a(t)$. Moreover, if $a'(t)$ lies in a solvable Lie subalgebra of $\mathfrak{sl}(2,\mathbb{R})$, we can obtain $A'(t)$ in many ways, e.g. by quadratures or by other methods as those used in [8]. Then, once $A'(t)$ is obtained, the knowledge of the curve $\bar{A}(t)$ transforming the curve $a(t)$ into $a'(t)$ provides the curve $A(t)$.

Therefore if we begin with a curve $a'(t)$ in a solvable Lie subalgebra of $\mathfrak{sl}(2,\mathbb{R})$ and consider the solutions for the system (5.3) in a subset of $SL(2,\mathbb{R})$, we can relate the curve $a'(t)$, and therefore its Riccati equation, to other possible curves $a(t)$, finding in this way a family of Riccati equations that can be exactly solved. Note that, if we do not consider solutions of the system (5.3) in a subset of $SL(2,\mathbb{R})$, it is generally difficult to check whether a particular Riccati equation belongs to the family of integrable Riccati equations so obtained.

Suppose we impose some restrictions on the family of curves solutions of the system (5.3), for instance $\beta = \gamma = 0$. Consequently, the system may not have solutions compatible with such restrictions, i.e. it may be impossible to connect the curves $a(t)$ and $a'(t)$ by a curve in $SL(2,\mathbb{R})$ satisfying the assumed restrictions. This gives rise to some compatibility conditions for the existence of these special solutions, some of them algebraic and other differential ones, between the t-dependent coefficients of $a'(t)$ and a(t). It will be shown later on that such restrictions correspond to integrability conditions previously proposed in the literature.

Therefore, there are two ingredients to take into account:

(1) *The equations on the Lie group characterized by curves $a'(t)$ for which we can obtain an explicit solution.* We always suppose that $a'(t)$ is related to a solvable Lie subalgebra of $\mathfrak{sl}(2,\mathbb{R})$ and we leave open other possible restrictions for further study.

(2) *The conditions imposed on the solutions of system* (5.2). We follow two principal approaches in next sections where the solutions of this system are related to curves in certain one-parameter or two-parameter subsets of $SL(2,\mathbb{R})$.

Consider the next example of our theory: suppose we try to connect any $a(t)$ with a final curve of the form $a'(t) = -D(t)(c_0a_0 + c_1a_1 + c_2a_2)$, where c_0, c_1 and c_2 are real numbers. In this way, the system (5.2) describing the curve $\bar{A}(t) \subset SL(2,\mathbb{R})$ connecting these curves is

$$\frac{dx}{dt} = \sum_{j=0}^{2}(b_j(t)N_j(x) + D(t)c_jN'_j(x)) = N(t,x). \tag{6.1}$$

Now, as the vector field

$$N' = \sum_{j=0}^{2} c_j N'_j,$$

is such that

$$[N_j, N'] = 0, \quad j = 0, 1, 2,$$

the Lie system (6.1) is related to a non-solvable Lie algebra of vector fields isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$. Hence, it is not integrable by quadratures and the solution cannot be easily found in the general case. Nevertheless, note that system (6.1) always has a solution.

In this way, we can consider some particular cases of Lie system (6.1) for which the resulting system of differential equations can be easily integrated. As a first instance, take x related to a one-parameter family of elements of $SL(2, \mathbb{R})$. Such a restriction implies that system (6.1) has not always a solution because sometimes it is not possible to connect a(t) and $a'(t)$ by means of the chosen family of curves. This fact induces differential and/or algebraic restrictions on the initial t-dependent functions b_j, with $j = 0, 1, 2$, that describe some known integrability conditions and may be some new ones developing the ideas of [5]. From this viewpoint we can obtain new integrability conditions that can be used, for instance, to obtain exact solutions.

Otherwise, if we choose a two-parameter set for the restriction, we find in some cases that we need a particular solution of the initial Riccati equation to obtain the reduction of the given Riccati equation into an integrable one. This is the point of view shown in [8] where integrability conditions were related to reduction methods.

7. Description of Known Integrability Conditions

Let us first remark that Lie systems on G of the form (3.6) and determined by a constant curve, $a = -\sum_{j=0}^{2} c_j M_j$, are integrable and consequently the same happens for curves of the form $a(t) = -D(t)(\sum_{j=0}^{2} c_j M_j)$, where D is any non-vanishing function, because a time-reparametrization reduces the problem to the previous one.

Our aim in this section is to determine the curves $\bar{A}(t)$ in $SL(2, \mathbb{R})$ relating two equations on $SL(2, \mathbb{R})$ characterized by the curves $a(t)$ and $a'(t) = -D(t)(c_0 M_0 + c_1 M_1 + c_2 M_2)$ with $D(t)$ a non-vanishing function and c_0, c_1 and c_2 real constants such that $c_0 c_2 \neq 0$. As the final equation is integrable, the transformation establishing the relation to such a final integrable equation allows us to find by quadratures the solution of the initial equation and, therefore, the solution for its associated Riccati equation. In order to get such a transformation, we look for curves $\bar{A}(t)$ in $SL(2, \mathbb{R})$ satisfying certain conditions in order to get an integrable equation (6.1). Nevertheless, under the assumed restrictions, we may obtain a system of differential equations admitting no solution. As an application, we show that many known results can be recovered and explained in this way.

We have already showed that the Riccati equations (1.1) with either $b_0 \equiv 0$ or $b_2 \equiv 0$ are reducible to linear differential equations and therefore they are always integrable. Hence, they are not interesting in our study and we focus our attention on reducing a Riccati equation (1.1), with $b_0 b_2 \neq 0$ in an open interval in t, into an integrable one by means of the action of a curve in $SL(2,\mathbb{R})$. With this aim, we consider the family of curves in $SL(2,\mathbb{R})$ with $\beta = 0$ and $\gamma = 0$, i.e. we take curves of the form

$$\bar{A}(t) = \begin{pmatrix} \alpha(t) & 0 \\ 0 & \delta(t) \end{pmatrix} \in SL(2,\mathbb{R}), \quad \alpha(t)\delta(t) = 1.$$

We already pointed out that a curve $\bar{A}(t)$ in $SL(2,\mathbb{R})$ induces a t-dependent change of variables in $\bar{\mathbb{R}}$ given by $y'(t) = \Phi(\bar{A}(t), y(t))$. In view of (3.5) and as $\alpha\delta = 1$, we get that, in our case, such a change of variables is given by

$$y' = \alpha^2(t)y = G(t)y, \quad G(t) \equiv \frac{\alpha(t)}{\delta(t)} > 0. \tag{7.1}$$

In view of the relations (4.1), the initial Riccati equation is transformed by means of the curve $\bar{A}(t)$ into the new Riccati equation with t-dependent coefficients

$$b'_2 = \delta^2\, b_2, \quad b'_1 = \alpha\,\delta\, b_1 + \dot{\alpha}\,\delta - \alpha\,\dot{\delta}, \quad b'_0 = \alpha^2\, b_0.$$

Furthermore, the functions α and δ are solutions of system (5.2), which in this case reads

$$\begin{pmatrix} \dot{\alpha} \\ 0 \\ 0 \\ \dot{\delta} \end{pmatrix} = \begin{pmatrix} \dfrac{b'_1 - b_1}{2} & b_2 & b'_0 & 0 \\ -b_0 & \dfrac{b'_1 + b_1}{2} & 0 & b'_0 \\ -b'_2 & 0 & -\dfrac{b'_1 + b_1}{2} & b_2 \\ 0 & -b'_2 & -b_0 & -\dfrac{b'_1 - b_1}{2} \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \delta \end{pmatrix}. \tag{7.2}$$

The existence of particular solutions for the above system related to elements of $SL(2,\mathbb{R})$ and satisfying the required conditions determines integrability conditions for Riccati equations by the described method. Thus, let us analyze the existence of such solutions to get these integrability conditions.

From some of the relations of the system (7.2), we get that

$$-b_0\,\alpha + b'_0\,\delta = 0, \quad -b'_2\,\alpha + b_2\,\delta = 0.$$

As $\alpha(t)\delta(t) = 1$, the above relations imply that $b_0 b_2 = b'_0 b'_2$ and

$$\alpha^2 = \frac{b'_0}{b_0} = \frac{b_2}{b'_2} \equiv G > 0.$$

Hence, the transformation formulas (4.1) reduce to

$$b'_2 = \alpha^{-2}\, b_2, \quad b'_1 = b_1 + 2\frac{\dot{\alpha}}{\alpha}, \quad b'_0 = \alpha^2 b_0. \tag{7.3}$$

Then, in order to exist a t-dependent function D and two real constants c_0 and c_2, with $c_0 c_2 \neq 0$, such that $b'_2 = Dc_2$ and $b'_0 = Dc_0$, the function D must be given by

$$D^2 c_0 c_2 = b_0 b_2 \Rightarrow D = \pm\sqrt{\frac{b_0 b_2}{c_0 c_2}},$$

where we have used that $b'_0 b'_2 = b_0 b_2$. On the other hand, as $b'_0/b_0 = \alpha^2 > 0$, we have to fix the sign κ of the function D in order to satisfy this relation, i.e. $\mathrm{sg}(c_0 D) = \mathrm{sg}(b_0)$. Therefore,

$$\kappa = \mathrm{sg}(D) = \mathrm{sg}(b_0/c_0).$$

Also, as $b_0 b_2 = b'_0 b'_2$, we get that $\mathrm{sg}(b_0 b_2) = \mathrm{sg}(c_0 c_2 D^2) = \mathrm{sg}(c_0 c_2)$. Furthermore, in view of the relations (7.3), α is determined, up to a sign, by

$$\alpha = \sqrt{\frac{Dc_0}{b_0}} = \left(\frac{c_0}{c_2}\,\frac{b_2}{b_0}\right)^{1/4}, \tag{7.4}$$

and therefore the change of variables (7.1) reads:

$$y' = \frac{D(t)c_0}{b_0(t)}y. \tag{7.5}$$

Finally, as a consequence of (7.3), in order for b'_1 to be the product $b'_1 = c_1\,D$, we see that

$$b_1 + 2\,\frac{\dot\alpha}{\alpha} = \kappa c_1\sqrt{\frac{b_0 b_2}{c_0 c_2}}. \tag{7.6}$$

Using (7.4) we get

$$4\,\frac{\dot\alpha}{\alpha} = \frac{1}{\alpha^4}\,\frac{d\alpha^4}{dt} = \frac{b_0}{b_2}\,\frac{d}{dt}\left(\frac{b_2}{b_0}\right) = \frac{b_0}{b_2}\,\frac{\dot b_2 b_0 - \dot b_0 b_2}{b_0^2} = \frac{\dot b_2}{b_2} - \frac{\dot b_0}{b_0},$$

and replacing $2\dot\alpha/\alpha$ in (7.6) for the value obtained above, we see that the required integral condition is

$$\sqrt{\frac{c_0 c_2}{b_0 b_2}}\left[b_1 + \frac{1}{2}\left(\frac{\dot b_2}{b_2} - \frac{\dot b_0}{b_0}\right)\right] = \kappa c_1.$$

Conversely, it can be verified that if the above integrability condition holds and $D^2 c_0 c_2 = b_0 b_2$, then the change of variables (7.5) transforms the Riccati equation (1.1) into $dy'/dt = D(t)(c_0 + c_1 y' + c_2 y'^2)$, with $c_0 c_2 \neq 0$.

In summary:

Theorem 7.1. *The necessary and sufficient condition for the existence of a transformation*

$$y' = G(t)y, \quad G(t) > 0,$$

relating the Riccati equation

$$\frac{dy}{dt} = b_0(t) + b_1(t)y + b_2(t)y^2, \quad b_0 b_2 \neq 0,$$

to an integrable one given by

$$\frac{dy'}{dt} = D(t)(c_0 + c_1 y' + c_2 y'^2), \quad c_0 c_2 \neq 0 \tag{7.7}$$

where c_0, c_1, c_2 *are real numbers and* $D(t)$ *is a non-vanishing function, are*

$$D^2 c_0 c_2 = b_0 b_2, \qquad \left(b_1 + \frac{1}{2}\left(\frac{\dot b_2}{b_2} - \frac{\dot b_0}{b_0}\right)\right)\sqrt{\frac{c_0 c_2}{b_0 b_2}} = \kappa c_1, \tag{7.8}$$

where $\kappa = \mathrm{sg}(D) = \mathrm{sg}(b_0/c_0)$*. The transformation is then uniquely defined by*

$$y' = \sqrt{\frac{b_2(t) c_0}{b_0(t) c_2}}\, y.$$

As a consequence of Theorem 7.1, given a Riccati equation

$$\frac{dy}{dt} = b_0(t) + b_1(t) y + b_2(t) y^2, \quad b_0(t) b_2(t) \neq 0,$$

if there are real constants c_0, c_1 and c_2, with $c_0 c_2 \neq 0$, such that

$$\sqrt{\frac{c_0 c_2}{b_0 b_2}}\left(b_1 + \frac{1}{2}\left(\frac{\dot b_2}{b_2} - \frac{\dot b_0}{b_0}\right)\right) = \kappa c_1,$$

there exists a t-dependent linear change of variables transforming the given equation into an integrable Riccati equation of the form

$$\frac{dy'}{dt} = D(t)(c_0 + c_1 y' + c_2 y'^2), \quad c_0 c_2 \neq 0, \tag{7.9}$$

and the function D is given by (7.8) with the sign determined by κ.

From the previous results, it can be derived the following corollary.

Corollary 7.1. *A Riccati equation* (1.1) *with* $b_0 b_2 \neq 0$ *can be transformed into a Riccati equation of the form* (7.9) *by a* t*-dependent change of variables* $y' = G(t) y$*, with* $G(t) > 0$*, if and only if*

$$\frac{1}{\sqrt{|b_0 b_2|}}\left(b_1 + \frac{1}{2}\left(\frac{\dot b_2}{b_2} - \frac{\dot b_0}{b_0}\right)\right) = K, \tag{7.10}$$

for a certain real constant K*. In such a case, the Riccati equation* (1.1) *is integrable by quadratures.*

According to Theorem 7.1, if we start with the integrable Riccati equation (7.9), we can obtain the set of all Riccati equations that can be reached from it by means of a transformation of the form (7.1).

Corollary 7.2. *Given an integrable Riccati equation*

$$\frac{dy}{dt} = D(t)(c_0 + c_1 y + c_2 y^2), \quad c_0 c_2 \neq 0,$$

with $D(t)$ a non-vanishing function, the set of Riccati equations which can be obtained with a transformation $y' = G(t)y$, with $G(t) > 0$, are those of the form:

$$\frac{dy'}{dt} = b_0(t) + \left(\frac{\dot b_0(t)}{b_0(t)} - \frac{\dot D(t)}{D(t)} + c_1 D(t)\right) y' + \frac{D^2(t)c_0c_2}{b_0(t)} y'^2,$$

with

$$G = \frac{Dc_0}{\sqrt{b_0}}.$$

Therefore starting with an integrable equation we can generate a family of solvable Riccati equations whose coefficients are parametrized by a non-vanishing function b_0. Moreover, the integrability condition to check if a Riccati equation belongs to this family can be easily verified.

These results can now be used for a better understanding of some integrability conditions found in the literature.

- *The case of Allen and Stein*:

The results of the paper by Allan and Stein [1] can be recovered through our general approach. In that work, a Riccati equation (1.1) with $b_0b_2 > 0$ and b_0, b_2 differentiable functions satisfying the condition

$$\frac{b_1 + \frac{1}{2}\left(\frac{\dot b_2}{b_2} - \frac{\dot b_0}{b_0}\right)}{\sqrt{b_0b_2}} = C, \tag{7.11}$$

where C is a real constant, was transformed into the integrable one

$$\frac{dy'}{dt} = \sqrt{b_0(t)b_2(t)}(1 + Cy' + y'^2), \tag{7.12}$$

through the t-dependent linear transformation

$$y' = \sqrt{\frac{b_2(t)}{b_0(t)}}y.$$

If integrability condition (7.11) is satisfied by a Riccati equation, such an equation also holds the assumptions of the Corollary 7.1 and, therefore, the integrability condition given in Theorem 7.1 with

$$c_0 = 1 = c_2, \quad c_1 = C, \quad D = \sqrt{b_0b_2}.$$

Consequently, the corresponding transformation given by Theorem 7.1 reads

$$y' = \sqrt{\frac{b_2(t)}{b_0(t)}}y,$$

showing that the transformation in [1] is a particular case of our results. This is not an unexpected result because Theorem 7.1 shows that if such a time-dependent change of variables is used to transform a Riccati equation (1.1) into one of the form (7.7), this change of variables must be of the form (7.5) and the initial Riccati equation must hold the integrability conditions (7.8).

- *The case of Rao and Ukidave*:

Rao and Ukidave stated in their work [37] that the Riccati equation (1.1), with $b_0b_2 > 0$, can be transformed into an integrable Riccati equation of the form

$$\frac{dy'}{dt} = \sqrt{cb_0b_2}\left(1 - ky' + \frac{1}{c}y'^2\right),$$

through a t-dependent linear transformation

$$y' = \frac{1}{v(t)}y,$$

if there exist real constants c and k such that following integrability condition holds

$$b_2 = \frac{b_0}{cv^2}, \tag{7.13}$$

with v being a solution of the differential equation

$$\frac{dv}{dt} = b_1(t)v + kb_0(t). \tag{7.14}$$

Note that, in view of (7.13), necessarily $c > 0$ and if the integrability conditions (7.13) and (7.14) hold with constants c and k and a negative solution $v(t)$, the same conditions hold for the constants c and $-k$ and a positive solution $-v(t)$. Consequently, we can restrict ourselves to studying the integrability conditions (7.13) and (7.14) for positive solutions $v(t) > 0$. In such a case, the previous method uses a t-dependent linear change of coordinates of the form (7.1) and the final Riccati equation are of the type described in our work (7.7), therefore the integrability conditions derived by Rao and Ukidave must be a particular instance of the integrable cases provided by Theorem 7.1.

Using the value of $v(t)$ in terms of the constant c and the functions b_0 and b_2 obtained from formula (7.13) and Eq. (7.14), we get that

$$\frac{1}{\sqrt{|b_0b_2|}}\left(b_1 + \frac{1}{2}\left(\frac{\dot b_2}{b_2} - \frac{\dot b_0}{b_0}\right)\right) = -k\,\mathrm{sg}(b_0)\sqrt{c}.$$

Hence, the Riccati equations holding conditions (7.13) and (7.14) satisfy the integrability conditions of Corollary 7.1. Moreover, if we choose

$$D^2 = cb_0b_2, \quad c_0 = 1, \quad c_1 = -k, \quad c_2 = c^{-1},$$

then $D = \sqrt{cb_0b_2}$ and the only possible transformation (7.1) given by Theorem 7.1 reads

$$y' = \alpha^2(t)y = \sqrt{\frac{cb_2(t)}{b_0(t)}}y,$$

and then

$$\frac{1}{v} = \sqrt{\frac{cb_2}{b_0}}.$$

In this way, we recover one of the results derived by Rao and Ukidave in [37].

- *The case of Kovalevskaya*:

Kovalevskaya showed in the paper [25] that the Riccati equation

$$\frac{dy}{dt} = F(t) + \left(L + \frac{\dot{F}(t)}{F(t)}\right) y - \frac{K}{F(t)} y^2,$$

where K and L are real constant, can be integrated through quadratures. It can be verified that the above family of Riccati equations holds the assumption of Corollary 7.1. Indeed, taking $c_0 = 1$, $c_2 = -K$, $c_1 = L$ we get that $\kappa = 1$,

$$\sqrt{\frac{c_0 c_2}{b_0 b_2}} \left(b_1 + \frac{1}{2}\left(\frac{\dot{b}_2}{b_2} - \frac{\dot{b}_0}{b_0}\right)\right) = L = c_1,$$

and $D = \sqrt{b_2 b_2 / c_2 c_0} = 1$. Therefore, Theorem 7.1 shows that the above family of Riccati equations can be integrated. Moreover, taking the above values of the constants c_0, c_1, c_2 and the function $b_0(t) = F(t)$, Corollary 7.2 reproduces the family of Riccati equations analyzed by Kovalevskaya.

- *The Case of Hong-Xiang*:

As a final example we can consider the Riccati equation

$$\frac{dy}{dt} = -y^2 - \left(2bG(t) - \frac{\dot{G}(t)}{G(t)}\right) y - cG^2(t),$$

used to analyze a certain integrable linear differential equation in [39] which was also analyzed by Hong-Xiang [17]. The above Riccati equation satisfies the integrability condition (7.10) and hence it can be integrated. Indeed, we have

$$\begin{cases} b_0(t) = -cG^2(t), \\ b_1(t) = -\left(2bG(t) - \dfrac{\dot{G}(t)}{G(t)}\right), \\ b_2(t) = -1, \end{cases}$$

and therefore we get that

$$\frac{b_1 + \frac{1}{2}\left(\frac{\dot{b}_2}{b_2} - \frac{\dot{b}_0}{b_0}\right)}{\sqrt{|b_0 b_2|}} = \text{const.}$$

In summary, many integrability conditions shown in the literature are equivalent to or particular instances of those given in our more general statements.

8. Integrability and Reduction

In this section we develop a procedure that is similar to the one derived throughout the previous sections but we here consider solutions of system (5.2) in two-parameter subsets of $SL(2,\mathbb{R})$. In this case, we recover some known integrability conditions, e.g. a certain kind of integrability used in [8]. More specifically, we try to relate a Riccati equation (1.1) to an integrable one associated, as a Lie system, with a curve of the form $a'(t) = -D(t)(c_0a_0 + c_1a_1 + c_2a_2)$, with $c_2 \neq 0$ and a non-vanishing function $D = D(t)$. Furthermore, we consider solutions of system (5.3) with $\gamma = 0$ and $\alpha > 0$ related to elements of $SL(2,\mathbb{R})$, i.e. we analyze transformations

$$y' = \frac{\alpha(t)}{\delta(t)}y + \frac{\beta(t)}{\delta(t)} = \alpha^2(t)\,y + \alpha(t)\beta(t).$$

In this case, using the expression in coordinates (5.2) of system (5.3), we get that

$$\begin{pmatrix} \dot\alpha \\ \dot\beta \\ 0 \\ \dot\delta \end{pmatrix} = \begin{pmatrix} \frac{b_1'-b_1}{2} & b_2 & b_0' & 0 \\ -b_0 & \frac{b_1'+b_1}{2} & 0 & b_0' \\ -b_2' & 0 & -\frac{b_1'+b_1}{2} & b_2 \\ 0 & -b_2' & -b_0 & -\frac{b_1'-b_1}{2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ 0 \\ \delta \end{pmatrix}, \tag{8.1}$$

where $b_j' = D\,c_j$ and $c_j \in \mathbb{R}$ for $j = 0, 1, 2$. As we suppose $b_2' \neq 0$, the third equation of the above system implies

$$\frac{\alpha}{\delta} = \frac{b_2}{b_2'}.$$

As $\alpha\delta = 1$ in order to obtain a solution of (5.3) related to an element of $SL(2,\mathbb{R})$ and $b_2' = Dc_2$, we get

$$\alpha^2 = \frac{b_2}{Dc_2}. \tag{8.2}$$

Hence, α is determined, up to a sign, by the values of $b_2(t), D$ and c_2. In this way, if we take α to be positive, the first differential equation of system (8.1) gives us the value of β in terms of the related initial and final Riccati equation, i.e.

$$\beta = \frac{1}{b_2}\left(\dot\alpha - \frac{b_1'-b_1}{2}\alpha\right).$$

Taking into account the relation (8.2), the above expression is equivalent to the differential equation

$$\frac{dD}{dt} = \left(b_1(t) + \frac{\dot b_2(t)}{b_2(t)}\right)D - c_1D^2 - 2b_2(t)D\beta\left(\frac{c_2D}{b_2(t)}\right)^{1/2},$$

and, as $\alpha\delta = 1$, we can define $M = \beta/\alpha$ and rewrite the above expression as follows

$$\frac{dD}{dt} = \left(b_1(t) + \frac{\dot{b}_2(t)}{b_2(t)}\right) D - c_1 D^2 - 2b_2(t)MD.$$

Considering the differential equation in $\dot{\beta}$ in terms of M, we get the equation

$$\frac{dM}{dt} = -b_0(t) + \frac{c_0 c_2}{b_2(t)} D^2 + b_1(t)M - b_2(t)M^2.$$

Finally, as $\delta\alpha = 1$ is a first-integral of system (5.3), if the system for the variables M and D and all the obtained conditions are satisfied, the value $\delta = \alpha^{-1}$ satisfies its corresponding differential equation of the system (8.1). To sum up, we have obtained the following result.

Theorem 8.1. *Given a Riccati equation* (1.1), *there exists a transformation*

$$y' = G(t)y + H(t), \quad G(t) > 0,$$

relating it to the integrable equation

$$\frac{dy'}{dt} = D(t)(c_0 + c_1 y' + c_2 y'^2), \tag{8.3}$$

with $c_2 \neq 0$ *and* D *a non-vanishing function, if and only if there exist functions* D *and* M *satisfying the following system*

$$\begin{cases} \dfrac{dD}{dt} = \left(b_1(t) + \dfrac{\dot{b}_2(t)}{b_2(t)}\right) D - c_1 D^2 - 2b_2(t)MD, \\ \dfrac{dM}{dt} = -b_0(t) + \dfrac{c_0 c_2}{b_2(t)} D^2 + b_1(t)M - b_2(t)M^2. \end{cases}$$

The transformation is then given by

$$y' = \frac{b_2(t)}{D(t)c_2}(y + M(t)). \tag{8.4}$$

Consider $c_0 = 0$ in Eq. (8.3). Thus, the system determining the curve in $SL(2, \mathbb{R})$ performing the transformation of Theorem 8.1 is

$$\begin{cases} \dfrac{dD}{dt} = \left(b_1(t) + \dfrac{\dot{b}_2(t)}{b_2(t)}\right) D - c_1 D^2(t) - 2b_2(t)MD, \\ \dfrac{dM}{dt} = -b_0(t) + b_1(t)M - b_2(t)M^2. \end{cases} \tag{8.5}$$

On one hand, this system does not involve any integrability condition because, as a consequence of the Theorem of existence and uniqueness of solutions, there always exists a solution for every initial condition. On the other hand, such solutions can be as difficult to be found as the general solution of the initial Riccati equation. Hence, in order to find a particular solution, we need to look for some simplifications. For instance, we can consider the case in which $M = b_1/b_2$. In this case, the first differential equation of the above system does not depend on M and reads

$$\frac{dD}{dt} = \left(-b_1(t) + \frac{\dot{b}_2(t)}{b_2(t)}\right) D - c_1 D^2$$

and it is integrable by quadratures. Its solution reads

$$D(t) = \frac{\exp\left(\int_0^t A(t')dt'\right)}{C + c_1 \int_0^t \exp\left(\int_0^{t''} A(t')dt'\right) dt''}, \quad A(t) = \left(-b_1(t) + \frac{\dot{b}_2(t)}{b_2(t)}\right).$$

Meanwhile, the condition for $M = b_1/b_2$ to be a solution of the second equation in (8.5) is

$$\frac{d}{dt}\left(\frac{b_1}{b_2}\right) = -b_0,$$

giving rise to an integrability condition. This summarizes one of the integrability conditions considered in [36].

Next, we recover from this new viewpoint the well-known result that the knowledge of a particular solution of the Riccati equation allows us to solve the system (8.5). In fact, under the change of variables $M \mapsto -y$, system (8.5) becomes

$$\begin{cases} \dfrac{dD}{dt} = \left(b_1(t) + \dfrac{\dot{b}_2(t)}{b_2(t)}\right) D - c_1 D^2 + 2b_2(t)yD, \\ \dfrac{dy}{dt} = b_0(t) + b_1(t)y + b_2(t)y^2. \end{cases} \tag{8.6}$$

Note that each particular solution of the above system is the form $(D_p(t), y_p(t))$, with $y_p(t)$ a particular solution of the Riccati equation (1.1). Therefore, given such a particular solution $y_p(t)$, the function $D_p = D_p(t)$, corresponding to the particular solution $(D_p(t), y_p(t))$ of system (8.6), holds the integrable equation

$$\frac{dD_p}{dt} = \left(b_1(t) + \frac{\dot{b}_2(t)}{b_2(t)} + 2b_2(t)y_p(t)\right) D_p - c_1 D_p^2. \tag{8.7}$$

Hence, the knowledge of a particular solution $y_p(t)$ of the Riccati equation (1.1) enables us to get a particular solution $(D_p(t), y_p(t))$ of system (8.6) and, taking into account the change of variables $y \mapsto -M$, a particular solution $(D_p(t), M_p(t)) = (D_p(t), -y_p(t))$ of system (8.5). Finally, the functions $M_p(t)$ and $D(t)$ determines a change of variables (8.4) given by Theorem 8.1 transforming the initial Riccati equation (1.1) into another one related, as a Lie system, to a solvable Lie algebra of vector fields. In this way, we describe a reduction process similar to that one pointed out in [8]. Nevertheless, we here directly obtain a reduction to a Riccati equation related, as a Lie system, to a one-dimensional Lie subalgebra of $\mathfrak{sl}(2, \mathbb{R})$ through one of its particular solutions.

There exists many ways to impose conditions on the coefficients of the second equation of (8.6) for being able to obtain one of its particular solutions easily. Now, we give some particular examples of this.

If there exists a real constant c such that for the time-dependent functions b_0, b_1 and b_2 we have that $b_0 + b_1 c + b_2 c^2 = 0$, then c is a particular solution. This resumes some cases found in [8, 42]. For instance:

(1) $b_0 + b_1 + b_2 = 0$ means that $c = 1$ is a particular solution.
(2) $c_1^2 b_0 + c_1 c_2 b_1 + c_2^2 b_2 = 0$ means that $c = c_2/c_1$ is a particular solution.

In these particular instances, we can find D through the first differential equation of (8.6).

As a first application of this last case we can integrate the Riccati equation

$$\frac{dy}{dt} = -\frac{n}{t} + \left(1 + \frac{n}{t}\right) y - y^2 \tag{8.8}$$

related to Hovy's equation [39]. This Riccati equation admits the particular constant solution $y_p(t) = 1$. Using such a particular solution in Eq. (8.7) and fixing, for instance, $c_1 = 0$, we can obtain a particular solution for Eq. (8.7), e.g. $D_p(t) = t^n e^{-t}$. Therefore, we get that $(t^n e^{-t}, 1)$ is a solution of the system (8.6) related to Eq. (8.8) and $(t^n e^{-t}, -1)$ is a solution of the system (8.5). In this way, Theorem 8.1 states that the transformation (8.4), determined by the $D_p(t) = t^n e^{-t}$ and $M_p(t) = -1$, of the form

$$y' = -t^{-n} e^t c_2^{-1}(y - 1), \tag{8.9}$$

relates the solutions of Eq. (8.8) to the integrable one

$$\frac{dy'}{dt} = e^{-t} t^n (c_0 + c_2 y'^2).$$

If we fix $c_0 = 1$ and $c_2 = 1$ the solution for the above equation is

$$y'(t) = -\frac{1}{-K + \Gamma(1 + n, t)},$$

where K is an integration constant and $\Gamma(a, b)$ is the incomplete Euler's Gamma function

$$\Gamma(a, t) = \int_t^\infty t'^{a-1} e^{-t'} dt'.$$

In view of the change of variables (8.9), the solutions $y(t)$ of the Riccati equation (8.8) and $y'(t)$ are related by means of the expression $y'(t) = -t^{-n} e^t c_2^{-1}(y(t) - 1)$. Therefore, if we substitute the general solution $y'(t)$ in this expression we can derive the general solution for the Riccati equation (8.8), that is,

$$y(t) = 1 - \frac{e^{-t} t^n}{\Gamma(n + 1, t) + K}.$$

Another approach that can be summarized by Theorem 8.1 is the factorization method developed in [39] to explain an integrability process for second-order differential equations. In that work, it was analyzed the differential equation:

$$\frac{d^2y}{dt^2} + 2P(t)\frac{dy}{dt} + \left(\frac{dP}{dt} + P^2(t) - \frac{d\phi}{dt} - \phi^2(t)\right) y = 0. \tag{8.10}$$

We know that invariance under dilations leads to consider an adapted variable z, such that $y = e^z$. Under this change of variables the equation obtained for $\psi = \dot{z}$ is the Riccati equation

$$\frac{d\psi}{dt} = -\psi^2 - 2P(t)\psi - \left(\frac{dP}{dt}(t) + P^2(t) - \frac{d\phi}{dt}(t) - \phi^2(t)\right). \tag{8.11}$$

This equation was integrated through a factorization method in [39]. Nevertheless, we can also integrate this equation if we take into account that $\psi_p(t) = \phi(t) - P(t)$ is a particular solution of the above differential equation and then applying the same procedure as for Eq. (8.8). Indeed, as $\psi_p(t)$ is a particular solution for the Riccati equation (8.11), we can obtain a particular solution $D_p = D_p(t)$ for Eq. (8.7) and by means of the functions $M_p(t) = -\psi_p(t)$ and $D_p(t)$ we can obtain the solution of the Riccati equation (8.11). Finally, inverting the change of variables used to relate Eqs. (8.10) to (8.11) we obtain the solution for Eq. (8.10).

9. Linearization of Riccati Equations

One can also study the problem of the linearization of Riccati equations through the linear fractional transformations (7.1). This set of time-dependent transformations is general enough to include many of the time-dependent or time-independent changes of variables already used to study Riccati equations, e.g. it allows us to recover the results of [38]. As a main result, we state in this section some integrability conditions to be able to transform a t-dependent Riccati equation into a linear one by means of a diffeomorphism on $\overline{\mathbb{R}}$ associated with certain linear fractional transformations.

As a first insight in the linearization process, note that Corollary 5.1 shows that there always exists a curve in $SL(2,\mathbb{R})$, and then a t-dependent linear fractional transformation on $\overline{\mathbb{R}}$, transforming a given Riccati equation into any other one. In particular, if we fix $b_2' = 0$ in the final Riccati equation, we obtain that there is a t-dependent linear fractional change of variables transforming any Riccati equation (1.1) into a linear one. Nevertheless, as the Lie system (5.3) describing such a transformation is not related to a solvable Lie algebra of vector fields, it is not easy to find such a transformation in the general case.

Let us try to relate a Riccati equation (1.1) to a linear differential equation by means of a linear fractional transformation (3.5) determined by a vector $(\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4$ with $\alpha\delta - \beta\gamma = 1$. In this case, the existence of solutions of the system (5.2) performing such a transformation is an easy task and we can look for integrability conditions to get the corresponding change of variables. Note that as $(\alpha, \beta, \gamma, \delta)$ is a constant, we have $\dot{\alpha} = \dot{\beta} = \dot{\gamma} = \dot{\delta} = 0$ and, in view of (5.2), the diffeomorphism on $\overline{\mathbb{R}}$ performing the transformation is related to a vector in the kernel of the matrix

$$B = \begin{pmatrix} \frac{b_1' - b_1}{2} & b_2 & b_0' & 0 \\ -b_0 & \frac{b_1' + b_1}{2} & 0 & b_0' \\ 0 & 0 & -\frac{b_1' + b_1}{2} & b_2 \\ 0 & 0 & -b_0 & -\frac{b_1' - b_1}{2} \end{pmatrix}, \tag{9.1}$$

where we assume $b_0 b_2 \neq 0$ in an open interval in the variable t. We leave out the study of the case $b_0 b_2 = 0$ in an open interval because, as it was shown in Sec. 2, this case is known to be integrable.

The necessary and sufficient condition for a non-trivial $\ker B$ is $\det B = 0$ and, therefore, a short calculation shows that $\dim \ker B > 0$ if and only if $(-b_1^2 + b_1'^2(t) + 4b_0b_2)^2 = 0$. Thus, $b_1' = \pm\sqrt{b_1^2 - 4b_0b_2}$ and b_1' is fixed, but a sign, by the values of b_0, b_1 and b_2. Let us study the kernel of the matrix B in the positive and negative cases for b_1'.

- Positive case: The kernel of the matrix (9.1) is given by the vectors

$$\left(\delta\frac{b_0'}{b_0} + \beta\frac{b_1 + \sqrt{b_1^2 - 4b_0b_2}}{2b_0}, \beta, -\delta\frac{-b_1 + \sqrt{b_1^2 - 4b_0b_2}}{2b_0}, \delta\right), \quad \delta, \beta \in \mathbb{R}.$$

Recall that we only consider the constant elements of $\ker B$, therefore there should be two real constants K_1 and K_2 such that

$$K_1 = \delta\frac{b_0'}{b_0} + \beta\frac{b_1 + \sqrt{b_1^2 - 4b_0b_2}}{2b_0}, \quad K_2 = \frac{-b_1 + \sqrt{b_1^2 - 4b_0b_2}}{2b_0}.$$

Moreover, in order to relate these vectors to elements in $SL(2, \mathbb{R})$ we have to impose that $\det(K_1, \beta, -\delta K_2, \delta) = \delta(K_1 + \beta K_2) = 1$.

The second condition imposes a restriction on the coefficients of the initial Riccati equation to be linearizable by a constant linear fractional transformation (3.5). Then, if this condition is satisfied we can fix β, γ, K_1 and b_0' to satisfy the other conditions. Thus, the only linearization condition is the condition on K_2.

- Negative case: In this case, $\ker B$ reads

$$\left(\delta\frac{b_0'}{b_0} + \beta\frac{b_1 - \sqrt{b_1^2 - 4b_0b_2}}{2b_0}, \beta, -\delta\frac{-b_1 - \sqrt{b_1^2 - 4b_0b_2}}{2b_0}, \delta\right), \quad \delta, \beta \in \mathbb{R},$$

and now the new conditions reduce to the existence of two real constants K_1 and K_2 such that

$$K_1 = \delta\frac{b_0'}{b_0} + \beta\frac{b_1 - \sqrt{b_1^2 - 4b_0b_2}}{2b_0}, \quad K_2 = \frac{-b_1 - \sqrt{b_1^2 - 4b_0b_2}}{2b_0},$$

with $\delta(K_1 + \beta K_2) = 1$. If the condition in K_2 is satisfied we can proceed as in the positive case to obtain the transformation performing the linearization of the initial Riccati equation.

In summary:

Theorem 9.1. *The necessary and sufficient condition for the existence of a diffeomorphism on* $\bar{\mathbb{R}}$ *of linear fractional type associated with a transformation of* $SL(2,\mathbb{R})$ *transforming the Riccati equation* (1.1) *into a linear differential equation is the existence of a real constant* K *such that*

$$K = \frac{-b_1 \pm \sqrt{b_1^2 - 4b_0b_2}}{2b_0}. \tag{9.2}$$

As a Riccati equation (1.1) holds condition (9.2) if and only if K is a constant particular solution, we get the following corollary:

Corollary 9.1. *A Riccati equation can be linearized by means of a diffeomorphism on* $\overline{\mathbb{R}}$ *of the form* (3.5) *if and only if it admits a constant particular solution.*

Ibragimov showed that a Riccati equation (1.1) is linearizable by means of a change of variables $z = z(y)$ if and only if the Riccati equation admits a constant solution [21]. Additionally, we have proved that in such a case, the change of variables can be described by means of a transformation of the type (3.5).

Now, it can be checked that the example given in [21, 38] satisfies the above integrability condition. In this work, the differential equation

$$\frac{dy}{dt} = P(t) + Q(t)y + k(Q(t) - kP(t))y^2,$$

was studied. The only interesting case is that with $k \neq 0$ because the other ones are linear. In this latter case, $b_0(t) = P(t)$, $b_1(t) = Q(t)$ and $b_2(t) = k(Q(t) - kP(t))$. Hence,

$$\frac{-b_1 - \sqrt{b_1^2 - 4b_0b_2}}{2b_0} = -k,$$

and the integrability condition (9.2) holds. Now we may fix $K_1 = 0$ and we look for a solution for the condition $\det(K_1, \beta, -\delta K_2, \delta) = 1$ reading $k\delta\beta = -1$. As $k \neq 0$, we can take $\beta = -1/k$ to get from the above condition that $\delta = 1$. Thus the transformation is that one associated with the vector $(0, -1/k, k, 1)$, i.e. the linear fractional transformation

$$y' = \frac{-1/k}{ky + 1}$$

that is the same found in [38]. In this way we only have to obtain b_0' from the condition

$$K_1 = 0 = \delta\frac{b_0'}{b_0} + \beta\frac{b_1 + \sqrt{b_1^2 - 4b_0b_2}}{2b_0},$$

to get the final linear differential equation, that is,

$$\frac{dy'}{dt} = \frac{Q(t)}{P(t)} + (Q(t) - 2P(t)k)y',$$

as it appears in [5].

10. Conclusions and Outlook

It has been shown that previous works about the integrability of the Riccati equation can be explained from the unifying viewpoint of Lie systems. The transformations used in the study of the integrability condition for these equations have been understood as induced by curves in $SL(2,\mathbb{R})$.

We have investigated a Lie system characterizing the t-dependent fractional transformations relating different Riccati equations associated, as Lie systems, with curves in $\mathfrak{sl}(2,\mathbb{R})$. We have used this differential equation and considered some simple instances. These simplifications have been used to analyze known integrability conditions and provide new ones.

We have also shown that the system (5.2) is a good way to describe linear fractional time-dependent transformations and found necessary and sufficient conditions for the linearizability or simplification of a Riccati equation through time-independent and time-dependent transformations obtained from curves in $SL(2,\mathbb{R})$.

There are many ways of simplifying (5.3) and some of them have been developed here. Other ways can be used to obtain new integrability conditions.

Finally, the theory used here can be extended to any other Lie system to provide new or recover known integrability conditions. This fact is to be developed in forthcoming works.

Acknowledgments

Partial financial support by research projects MTM2009-11154, MTM2009-08166-E and E24/1 (DGA) are acknowledged. JdL also acknowledges a F.P.U. grant from Ministerio de Educación y Ciencia.

References

[1] J. L. Allen and F. M. Stein, On solutions of certain Riccati differential equations, *Amer. Math. Monthly* **71** (1964) 1113–1115.
[2] J. F. Cariñena, Sections along maps in Geometry and Physics, *Rend. Sem. Mat. Univ. Pol. Torino* **54** (1996) 245–256.
[3] J. F. Cariñena and J. de Lucas, Quantum Lie systems and integrability conditions, *Int. J. Geom. Methods Mod. Phys.* **6** (2009) 1235–1252.
[4] J. F. Cariñena and J. de Lucas, Applications of Lie systems in dissipative Milne–Pinney equations, *Int. J. Geom. Meth. Modern Phys.* **7** (2009) 683–699.
[5] J. F. Cariñena, J. de Lucas and A. Ramos, A geometric approach to integrability conditions for Riccati Equations, *Electron. J. Diff. Eqns.* **122** (2007) 1–14.
[6] J. F. Cariñena, J. de Lucas and M. F. Rañada, Recent applications of the theory of Lie systems in Ermakov systems, *SIGMA* **4** (2008) 031.
[7] J. F. Cariñena, J. de Lucas and M. F. Rañada, Lie systems and integrability conditions for t-dependent frequency harmonic oscillators, *Int. J. Geom. Methods Mod. Phys.* **7**(2) (2010) 289–310.
[8] J. F. Cariñena, J. Grabowski and A. Ramos, Reduction of t-dependent systems admitting a superposition principle, *Acta Appl. Math.* **66** (2001) 67–87.
[9] J. F. Cariñena, G. Marmo and J. Nasarre, The nonlinear superposition principle and the Wei-Norman method, *Int. J. Mod. Phys. A* **13** (1998) 3601–3627.
[10] J. F. Cariñena and A. Ramos, Integrability of the Riccati equation from a group theoretical viewpoint, *Int. J. Mod. Phys. A* **14** (1999) 1935–1951.
[11] J. F. Cariñena and A. Ramos, A new geometric approach to Lie systems and physical applications, *Acta Appl. Math.* **70** (2002) 43–69.

[12] J. F. Cariñena and A. Ramos, Lie systems and connections in fibre bundles: Applications in quantum mechanics, *9th Int. Conf. Diff. Geom and Appl.*, (2004) 437–452, eds. J. Bures *et al.* (Matfyzpress, Prague, 2005).

[13] J. F. Cariñena and A. Ramos, Riccati equation, factorization method and shape invariance, *Rev. Math. Phys.* **12** (2000) 1279–1304.

[14] N. L. Chuprikov, Hybrid Taylor-WKB series (2009) arxiv:0911.3217.

[15] I. A. García, J. Giné and J. Llibre, Liénard and Riccati differential equations related via Lie algebras, *Discrete and Continuous Dynamical Systems — Series B* **10** (2008) 485–494.

[16] A. Guldberg, Sur les équations différentielles ordinaires qui possèdent un système fondamental d'intégrales, *C. R. Math. Acad. Sci. Paris* **116** (1893) 964–965.

[17] L. Hong-Xiang, Elementary quadratures of ordinary differential equations, *Amer. Math. Monthly* **89** (1982) 198–208.

[18] N. H. Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 3: New Trends in Theoretical Developments and Computational Methods* (CRC Press Inc., Boca Raton, 1996).

[19] N. H. Ibragimov, *Elementary Lie Group Analysis and Ordinary Differential Equations* (John Wiley & Sons, Chichester, 1999).

[20] N. H. Ibragimov, *Premier on Group Analysis* (Znanie, No. 8, Moscow, 1989) (Russian). Revised edition in English: *Introduction to Modern Group Analysis* (Tau, Ufa, 2000).

[21] N. H. Ibragimov, Memoir on integration of ordinary differential equations by quadrature, *Archives ALGA* **5** (2008) 27–62.

[22] N. H. Ibragimov, Integration of systems of first-order differential equations admitting nonlinear superposition, *J. Non. Math. Phys.* **16** (2009) 137–147.

[23] N. H. Ibragimov, Utilization of canonical variables for integration of systems of first-order differential equations, *Archives ALGA* **6** (2009) 1–18.

[24] E. Kamke, *Differentialgleichungen: Lösungsmethoden und Lösungen* (Akademische Verlagsgesellschaft, Leipzig, 1959).

[25] N. M. Kovalevskaya, On some cases of integrability of a general Riccati equation (2006) arXiv:math.CA/0604243 v1.

[26] P. Libermann and Ch.-M. Marle, *Symplectic Geometry and Analytical Mechanics* (Reidel, Dordrecht, 1987).

[27] S. Lie and G. Scheffers, *Vorlesungen Über Continuierliche Gruppen Mit Geometrischen Und Anderen Anwendungen*, ed. and revised by G. Scheffers (Teubner, Leipzig, 1893).

[28] Z. Linlong, A new integrability condition for Riccati differential equation, *Chinese Sci. Bull.* **43** (1998) 439–440.

[29] Z. Linlong, The integrable conditions of Riccati differential equation, *Chinese Quart. Math.* **14** (1999) 67–70.

[30] M. K. Mak and T. Harko, Relativistic compact objects in isotropic coordinates, *Pramana* **65** (2005) 185–192.

[31] G. M. Murphy, *Ordinary Differential Equations and their Solutions* (Van Nostrand, New York, 1960).

[32] J. Napora, On integrable by quadratures generalized Riccati–Abel equations: Differential-geometric and Lie algebraic analysis, *Rep. Math. Phys.* **44** (1999) 149–158.

[33] J. Napora, The Moser type reduction of integrable Riccati differential equations and its Lie-algebraic structure, *Rep. Math. Phys.* **46** (2000) 211–216.

[34] A. K. Rajagopal, On the generalized Riccati equation, *Amer. Math. Monthly* **68** (1961) 777–779.

[35] S. S. Rajah and S. D. Maharaj, A Riccati equation in radiative stellar collapse, *J. Math. Phys.* **49** (2008) 012501.

[36] P. R. P. Rao, The Riccati differential equation, *Amer. Math. Monthly* **69** (1962) 995–996.

[37] P. R. P. Rao and V. H. Ukidave, Some separable forms of the Riccati equation, *Amer. Math. Monthly* **75** (1968) 38–39.

[38] S. Rezzag, R. Dridi and A. Makhlouf, Sur le principe de superposition et l'èquation de Riccati, *C. R. Math. Acad. Sci. Paris.* **340** (2005) 799–802.

[39] W. Robin, Operator factorization and the solution of second-order linear evolution differential equations, *Int. J. Math. Educ. Sci. Technol.* **38** (2007) 189–211.

[40] N. Saad, R. L. Hall and H. Ciftci, Solutions for certain classes of Riccati differential equations, *J. Phys. A: Math. Theor.* **40** (2007) 10903–10914.

[41] D. Schuch, Riccati and Ermakov equations in time dependent and time independent quantum systems, *SIGMA* **4** (2008) 043.

[42] V. M. Strelchenya, A new case of integrability of the general Riccati equation and its application to relaxation problems, *J. Phys. A: Math. Gen.* **24** (1991) 4965–4967.

[43] M. E. Vessiot, Sur une classe d'équations différentielles, *Ann. Sci. École Norm. Sup.* **10** (1893) 53–64.

[44] P. Winternitz, Lie groups and solutions of nonlinear differential equations, in *Nonlinear Phenomena*, ed. K. B. Wolf, *Lecture Notes in Physics* **189** (Springer-Verlag, New York, 1983).