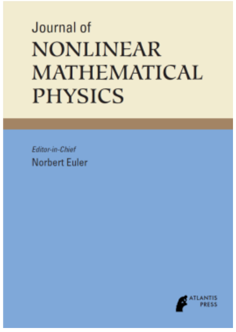

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P. G. Estévez, J. D. Lejarreta, C. Sardón

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NON-ISOSPECTRAL 1 + 1 HIERARCHIES ARISING FROM A CAMASSA HOLM HIERARCHY IN 2 + 1 DIMENSIONS

P. G. ESTÉVEZ*, J. D. LEJARRETA† and C. SARDÓN
Departamento de Fisica Fundamental
Universidad de Salamanca
Salamanca, 37008, Spain
**pilar@usal.es*
†*leja@usal.es*

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The non-isospectral problem (Lax pair) associated with a hierarchy in $2+1$ dimensions that generalizes the well known Camassa–Holm hierarchy is presented. Here, we have investigated the non-classical Lie symmetries of this Lax pair when the spectral parameter is considered as a field. These symmetries can be written in terms of five arbitrary constants and three arbitrary functions. Different similarity reductions associated with these symmetries have been derived. Of particular interest are the reduced hierarchies whose $1+1$ Lax pair is also non-isospectral.

Keywords: Lie symmetries; reductions; Camassa–Holm hierarchy.

2000 Mathematics Subject Classification: 35C06, 35P30, 35051

1. Introduction

The identification of the Lie symmetries of a given partial differential equation (PDE) is an instrument of primary importance in order to solve such an equation [21]. A standard method for finding solutions of PDEs is that of reduction using Lie symmetries: each Lie symmetry allows a reduction of the PDE to a new equation with the number of independent variables reduced by one [6, 18]. To a certain extent this procedure gives rise to the ARS conjecture [2], which establishes that a PDE is integrable in the sense of Painlevé [19] if all its reductions pass the Painlevé test [22]. This means that the solutions of a PDE can be achieved by solving its reductions to ordinary differential equations (ODE). Classical [21] and non-classical [6, 18] Lie symmetries are the usual way for identifying the reductions.

2. The $2+1$ Camassa–Holm Hierarchy

Lax pair

A generalization to $2+1$ dimensions of the celebrated Camassa–Holm hierarchy (henceforth CHn$2+1$) was presented in [10]. By using reciprocal transformations, this hierarchy was proved to be equivalent to n copies of the AKNS equation in $2+1$ variables [9, 16]. It is well known that the $2+1$ AKNS equation has the Painlevé property, and its non-isospectral Lax pair can be obtained by means of the singular manifold method [9]. Therefore we can use the inverse reciprocal transformation to obtain the Lax pair of CHn$2+1$ [10]. This Lax pair is also a non-isospectral one that can be written in terms of $n+1$ fields as follows:

$$\begin{aligned} &\psi_{xx} - \left(\frac{1}{4} - \frac{\lambda}{2}M\right)\psi = 0 \\ &\psi_y - \lambda^n\psi_t + \hat{A}\psi_x - \frac{\hat{A}_x}{2}\psi = 0, \end{aligned} \tag{2.1}$$

where

$$\hat{A} = \sum_{j=1}^{n} \lambda^{(n-j+1)} U^{[j]} \tag{2.2}$$

and

$$M = M(x,y,t), \quad U^{[j]} = U^{[j]}(x,y,t), \;\; j = 1,\dots,n.$$

Non-isospectrality and equations

The compatibility condition between Eqs. (2.1) yields the non-isospectral condition

$$\lambda_y - \lambda^n\lambda_t = 0, \quad \lambda_x = 0, \tag{2.3}$$

as well as the equations

$$\begin{aligned} &M_y = U^{[n]}_x - U^{[n]}_{xxx} \\ &M_t = U^{[1]}M_x + 2MU^{[1]}_x \\ &U^{[j]}M_x + 2MU^{[j]}_x = U^{[j-1]}_x - U^{[j-1]}_{xxx}, \quad j = 2,\dots,n. \end{aligned} \tag{2.4}$$

Recursion operator and hierarchy

The above equations can be written in more compact form by defining the operators:

$$J = \frac{\partial}{\partial x} - \frac{\partial}{\partial x^3}, \quad K = M\frac{\partial}{\partial x} + \frac{\partial}{\partial x}M. \tag{2.5}$$

Equations (2.4) are therefore:

$$\begin{aligned} M_y &= JU^{[n]} \\ M_t &= KU^{[1]} \\ KU^{[j]} &= JU^{[j-1]}, \quad j = 2,\dots,n, \end{aligned} \tag{2.6}$$

which yields the hierarchy:

$$M_y = R^n M_t \tag{2.7}$$

where the recursion operator is:

$$R = JK^{-1}. \tag{2.8}$$

Solutions of these equations were studied in [11]. The positive and negative, [1, 5], $1+1$ Camassa–Holm hierarchies can be obtained by setting $\frac{\partial}{\partial y} = \frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y} = \frac{\partial}{\partial t}$, respectively [10].

The $n = -1$ case of (2.5) has been considered in [5] and [14]. There are also different generalizations of the Camassa–Holm hierarchy to $2+1$ dimensions arising from different (although isospectral) spectral problems [15, 20].

3. Non-Classical Symmetries of the CHn$2+1$ Spectral Problem

Lie point symmetries

Here, we are interested in the Lie symmetries of the Lax pair (2.1). Naturally, the symmetries of equations (2.4) are interesting in themselves, but we also wish to know how **the eigenfunction and the spectral parameter transform under the action of a Lie symmetry**. More precisely, we wish to know what these fields look like under the reduction associated with each symmetry. This is why we shall proceed to write the infinitesimal Lie point transformation of the variables and fields that appear in the spectral problem (2.1). We have proved the benefits of such a procedure [17] in a previous paper [12].

In the present case, it is important to note that the spectral parameter $\lambda(y,t)$ is not a constant, and therefore that it should be considered as an additional field satisfying (2.3). This means that we are actually looking for the Lie point symmetries of equations (2.1) together with (2.3).

The infinitesimal form of the Lie point symmetry that we are considering is:

$$\begin{aligned}
x' &= x + \varepsilon\, \xi_1(x,y,t,\lambda,\psi,M,U^{[j]}) + O(\varepsilon^2) \\
y' &= y + \varepsilon\, \xi_2(x,y,t,\lambda,\psi,M,U^{[j]}) + O(\varepsilon^2) \\
t' &= t + \varepsilon\, \xi_3(x,y,t,\lambda,\psi,M,U^{[j]}) + O(\varepsilon^2) \\
\psi' &= \psi + \varepsilon\, \phi_1(x,y,t,\lambda,\psi,M,U^{[j]}) + O(\varepsilon^2) \\
\lambda' &= \lambda + \varepsilon\, \phi_2(x,y,t,\lambda,\psi,M,U^{[j]}) + O(\varepsilon^2) \\
M' &= M + \varepsilon\, \Theta_0(x,y,t,\lambda,\psi,M,U^{[j]}) + O(\varepsilon^2) \\
(U^{[i]})' &= U^{[i]} + \varepsilon\, \Theta_i(x,y,t,\lambda,\psi,M,U^{[j]}) + O(\varepsilon^2), \quad i,j = 1,\dots,n,
\end{aligned} \tag{3.1}$$

where ϵ is the group parameter. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form:

$$X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \xi_3 \frac{\partial}{\partial t} + \phi_1 \frac{\partial}{\partial \psi} + \phi_2 \frac{\partial}{\partial \lambda} + \Theta_0 \frac{\partial}{\partial M} + \sum_{j=1}^{n} \Theta_j \frac{\partial}{\partial U^{[j]}}. \tag{3.2}$$

We also need to know how the derivatives of the fields transform under the Lie symmetry. This means that we have to introduce the "prolongations" of the action of the group to the different derivatives that appear in (2.1) and (2.3). Exactly how to calculate the prolongations is a very well known procedure whose technical details can be found in [21].

It is therefore necessary that the Lie transformation should leave (2.1) and (2.3) invariant. This yields an overdetermined system of equations for the infinitesimals $\xi_1(x,y,t,\lambda,\psi,M,U^{[j]})$, $\xi_2(x,y,t,\lambda,\psi,M,U^{[j]})$, $\xi_3(x,y,t,\lambda,\psi,M,U^{[j]})$, $\phi_1(x,y,t,\lambda,\psi,M,U^{[j]})$, $\phi_2(x,y,t,\lambda,\psi,M,U^{[j]})$, and $\Theta_i(x,y,t,\lambda,\psi,M,U^{[j]})$.

This is the **classical method** [21] of finding Lie symmetries, and it can be summarized as follows:

(1) Calculation of the prolongations of the derivatives of the fields that appear in (2.1) and (2.3).
(2) Substitution of the transformed fields (3.1) and their derivatives in (2.1) and (2.3).
(3) Set all the coefficients in ϵ at 0.
(4) Substitution of the prolongations.
(5) ψ_{xx}, ψ_y and λ_y can be substituted by using (2.1) and (2.3).
(6) The system of equations for the infinitesimals can be obtained by setting each coefficient in the different remaining derivatives of the fields at zero.

Non-classical symmetries

There is a generalization of the classical method that determines the **non-classical or conditional symmetries** [6, 18]. In this case we are looking for symmetries that leave invariant not only the equations but also the so called "invariant surfaces", which in our case are:

$$\begin{aligned}
\phi_1 &= \xi_1\psi_x + \xi_2\psi_y + \xi_3\psi_t\\
\phi_2 &= \xi_2\lambda_y + \xi_3\lambda_t\\
\Theta_0 &= \xi_1 M_x + \xi_2 M_y + \xi_3 M_t\\
\Theta_j &= \xi_1 U_x^{[j]} + \xi_2 U_y^{[j]} + \xi_3 U_t^{[j]}, \quad j = 1, \ldots, n.
\end{aligned} \tag{3.3}$$

These non-classical symmetries are the symmetries that we address below. The method for calculating these symmetries is the same as the one we have described for the classical ones complemented with Eqs. (3.3), that must also be combined with step 4 to eliminate as many derivatives of the fields as possible, depending on whether all of the ξ_i are different from zero or not. This is why we have to distinguish three different types of non-classical symmetries.

- $\xi_3 = 1$.
- $\xi_3 = 0,\ \xi_2 = 1$.
- $\xi_3 = 0,\ \xi_2 = 0,\ \xi_1 = 1$.

Note that owing to (3.3), there is no restriction in selecting $\xi_j = 1$ when $\xi_j \neq 0$ [18]. In the following sections we shall determine these three types of symmetries of the Lax pair and its reduction to $1+1$ dimensions by solving the characteristic equation

$$\frac{dx}{\xi_1} = \frac{dy}{\xi_2} = \frac{dt}{\xi_3} = \frac{d\psi}{\phi_1} = \frac{d\lambda}{\phi_2} = \frac{dM}{\Theta_0} = \frac{dU^{[j]}}{\Theta_j}. \tag{3.4}$$

The advantage of our approach of working with the Lax pair instead of the equations of the hierarchy lies in the fact that we can obtain the reduced eigenfuntion and the reduced spectral parameter at the same time, which as we shall see, in many cases is not a trivial matter. The equations of the reduced hierarchies can be explicitly obtained from the reduced spectral problem and we shall write them in all the cases.

Of course the calculation of the symmetries is tedious, and we have used the MAPLE symbolic package to handle these calculations. For the benefit of the reader, we shall omit the technical details.

4. Non-Classical Symmetries for $\xi_3 = 1$

Calculation of symmetries

In this case (3.3) allows us to eliminate the derivatives with respect to t

$$\begin{aligned}\psi_t &= \phi_1 - \xi_1\psi_x - \xi_2\psi_y \\ \lambda_t &= \phi_2 - \xi_2\lambda_y \\ M_t &= \Theta_0 - \xi_1 M_x - \xi_2 M_y \\ U_t^{[j]} &= \Theta_j - \xi_1 U_x^{[j]} - \xi_2 U_y^{[j]}, \quad j = 1, \dots, n.\end{aligned} \tag{4.1}$$

If we add (4.1) to the five steps listed above for the calculation of non-classical symmetries, we obtain (after long but straightforward calculations) the following symmetries:

$$\begin{aligned}\xi_1 &= \frac{S_1}{S_3} \\ \xi_2 &= \frac{S_2}{S_3} \\ \xi_3 &= 1 \\ \phi_1 &= \frac{1}{S_3}\left(\frac{1}{2}\frac{\partial S_1}{\partial x} + a_0\right)\psi \\ \phi_2 &= \frac{1}{S_3}\left(\frac{a_3 - a_2}{n}\right)\lambda \\ \Theta_0 &= \frac{1}{S_3}\left(-2\frac{\partial S_1}{\partial x} + \frac{a_2 - a_3}{n}\right)M \\ \Theta_1 &= \frac{1}{S_3}\left(U^{[1]}\left(\frac{\partial S_1}{\partial x} - a_3\right) - \frac{\partial S_1}{\partial t}\right) \\ \Theta_j &= \frac{1}{S_3}\left(\frac{\partial S_1}{\partial x} - a_2\frac{j-1}{n} - a_3\frac{n-j+1}{n}\right)U^{[j]}, \quad j = 2, \dots, n,\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}S_1 &= S_1(x,t) = A_1(t) + B_1(t)\,e^{x} + C_1(t)\,e^{-x}, \\ S_2 &= S_2(y) = a_2 y + b_2, \\ S_3 &= S_3(t) = a_3 t + b_3.\end{aligned} \tag{4.3}$$

$A_1(t), B_1(t), C_1(t)$ are arbitrary functions of t. Furthermore, a_0, a_2, b_2, a_3, b_3 are arbitrary constants, such that a_3 and b_3 cannot at the same time be 0.

Classification of the reductions

We have, therefore, several different reductions depending on which arbitrary functions and/or constants are or are not zero. We shall use the following classification:

- Type I: Corresponding to selecting $A_1(t) \neq 0$, $B_1(t) = C_1(t) = 0$.
- Type II: Corresponding to selecting $B_1(t) \neq 0$, $A_1(t) = C_1(t) = 0$. As we shall show in Appendix I, this case yields the same reduced spectral problems as those obtained for Type I, although the reductions are different.

- Type III: Corresponding to selecting $C_1(t) \neq 0$, $A_1(t) = B_1(t) = 0$. It is easy to see that this case is equivalent to II owing to the invariance of the Lax pair under the transformation $x \to -x$, $y \to -y$, $t \to -t$. Below we only consider Cases I and II.

In each of the cases listed before we have different subcases, depending on the values of the constants a_j and b_j. We have the following 5 independent possibilities:

- Case 1: $a_2 = 0, \; a_3 = 0; \, b_2 = 0$.
- Case 2: $a_2 = 0, \; a_3 = 0; \, b_2 \neq 0$.
- Case 3: $a_2 = 0, \; a_3 \neq 0; \, b_2 = 0$.
- Case 4: $a_2 = 0, \; a_3 \neq 0; \, b_2 \neq 0$.
- Case 5: $a_2 \neq 0$;

We can obtain 5 different non-trivial reductions: (I.1) $i = 1, \ldots, 5$. We shall see each reduction separately by obtaining the reduced variables, the reduced fields, the transformation of the spectral parameter and the eigenfuction and, finally, the reduced spectral problem and the corresponding reduced hierarchy. Furthermore, there are several interesting reductions, especially those that also have a non-isospectral parameter in $1+1$ dimensions. Let us summarize the results:

(I.1) $B_1(t) = C_1(t) = 0,\ A_1(t) \neq 0,\ a_2 = 0,\ a_3 = 0,\ b_2 = 0$

By solving the characteristic equation (3.4), we have the following results

- Reduced variables: $z_1 = x - \frac{1}{b_3} \int A_1(t)\, dt$, $z_2 = y$
- Spectral parameter: $\lambda(y,t) = \lambda_0$
- Reduced fields:

$$\begin{aligned}
\psi(x,y,t) &= e^{\left(\frac{a_0 t}{b_3}\right)} e^{\left(\frac{\lambda_0^n a_0 z_2}{b_3}\right)} \Phi(z_1,z_2) \\
M(x,y,t) &= H(z_1,z_2) \\
U^{[1]}(x,y,t) &= V^{[1]}(z_1,z_2) - \frac{A_1}{b_3} \\
U^{[j]}(x,y,t) &= V^{[j]}(z_1,z_2)
\end{aligned}$$

- Reduced spectral problem:

$$\begin{aligned}
&\Phi_{z_1 z_1} - \left(\frac{1}{4} - \frac{\lambda_0}{2} H\right)\Phi = 0 \\
&\Phi_{z_2} + \hat{\mathcal{B}}\Phi_{z_1} - \frac{\hat{\mathcal{B}}_{z_1}}{2}\Phi = 0
\end{aligned} \tag{4.4}$$

  where

$$\hat{\mathcal{B}} = \sum_{j=1}^{n} \lambda_0^{(n-j+1)} V^{[j]}(z_1,z_2). \tag{4.5}$$

- Reduced hierarchy: The compatibility condition of (4.4) yields:

$$
\begin{aligned}
&\frac{\partial^3 V^{[n]}}{\partial z_1^3} - \frac{\partial V^{[n]}}{\partial z_1} + \frac{\partial H}{\partial z_2} = 0 \\
&2H\frac{\partial V^{[1]}}{\partial z_1} + V^{[1]}\frac{\partial H}{\partial z_1} = 0 \\
&2H\frac{\partial V^{[j+1]}}{\partial z_1} + V^{[j+1]}\frac{\partial H}{\partial z_1} + \frac{\partial^3 V^{[j]}}{\partial z_1^3} - \frac{\partial V^{[j]}}{\partial z_1} = 0
\end{aligned}
\tag{4.6}
$$

which is **the positive Camassa–Holm hierarchy**, whose first component ($n = 1$) is a modified Dym equation [1, 10].

(I.2) $B_1(t) = C_1(t) = 0,\ A_1(t) \neq 0,\ a_2 = 0,\ a_3 = 0,\ b_2 \neq 0$

By solving the characteristic equation (3.4), we have the following results

- Reduced variables: $z_1 = x - \frac{1}{b_3}\int A_1(t)\,dt,\ z_2 = \frac{y}{b_2} - \frac{t}{b_3}$
- Spectral parameter: $\lambda(y,t) = (\frac{b_3}{b_2})^{(\frac{1}{n})}\lambda_0$
- Reduced fields:

$$
\begin{aligned}
\psi(x,y,t) &= e^{(\frac{a_0 t}{b_3})}\, e^{(\frac{\lambda_0^n a_0 z_2}{1+\lambda_0^n})}\,\Phi(z_1,z_2) \\
M(x,y,t) &= \left(\frac{b_2}{b_3}\right)^{(\frac{1}{n})} H(z_1,z_2) \\
U^{[1]}(x,y,t) &= \left(\frac{1}{b_3}\right) V^{[1]}(z_1,z_2) - \frac{A_1}{b_3} \\
U^{[j]}(x,y,t) &= \left(\frac{1}{b_3}\right)\left(\frac{b_2}{b_3}\right)^{(\frac{1-j}{n})} V^{[j]}(z_1,z_2)
\end{aligned}
$$

- Reduced spectral problem:

$$
\begin{aligned}
&\Phi_{z_1 z_1} - \left(\frac{1}{4} - \frac{\lambda_0}{2}H\right)\Phi = 0 \\
&\Phi_{z_2}(1+\lambda_0^n) + \hat{\mathcal{B}}\Phi_{z_1} - \frac{\hat{\mathcal{B}}_{z_1}}{2}\Phi = 0
\end{aligned}
\tag{4.7}
$$

where

$$
\hat{\mathcal{B}} = \sum_{j=1}^{n} \lambda_0^{(n-j+1)} V^{[j]}(z_1,z_2) \tag{4.8}
$$

- Reduced hierarchy: The compatibility condition of (4.7) yields:

$$
\begin{aligned}
&\frac{\partial^3 V^{[n]}}{\partial z_1^3} - \frac{\partial V^{[n]}}{\partial z_1} + \frac{\partial H}{\partial z_2} = 0 \\
&2H\frac{\partial V^{[1]}}{\partial z_1} + V^{[1]}\frac{\partial H}{\partial z_1} + \frac{\partial H}{\partial z_2} = 0 \\
&2H\frac{\partial V^{[j+1]}}{\partial z_1} + V^{[j+1]}\frac{\partial H}{\partial z_1} + \frac{\partial^3 V^{[j]}}{\partial z_1^3} - \frac{\partial V^{[j]}}{\partial z_1} = 0.
\end{aligned}
\tag{4.9}
$$

(I.3) $B_1(t) = C_1(t) = 0,\ A_1(t) \neq 0,\ a_2 = 0,\ a_3 \neq 0,\ b_2 = 0$

By solving the characteristic equation (3.4), we have the following results

- Reduced variables: $z_1 = x - \int \frac{A_1(t)}{S_3(t)}\,dt$, $z_2 = a_3 y$
- Spectral parameter: In this case the reduction of the spectral parameter is a non-trivial one that yields

$$\lambda(y,t) = S_3^{(\frac{1}{n})}\Lambda(z_2)$$

where $\Lambda(z_2)$ is the reduced spectral parameter.
- Reduced fields:

$$
\begin{aligned}
\psi(x,y,t) &= \Lambda(z_2)^{(\frac{a_0 n}{a_3})} S_3^{(\frac{a_0}{a_3})}\Phi(z_1,z_2) \\
M(x,y,t) &= S_3^{(-\frac{1}{n})} H(z_1,z_2) \\
U^{[1]}(x,y,t) &= \frac{a_3}{S_3}V^{[1]}(z_1,z_2) - \frac{A_1}{S_3} \\
U^{[j]}(x,y,t) &= a_3\, S_3^{(\frac{j-1}{n}-1)} V^{[j]}(z_1,z_2).
\end{aligned}
$$

- Reduced spectral problem:

$$
\begin{aligned}
&\Phi_{z_1 z_1} - \left(\frac{1}{4} - \frac{\Lambda(z_2)}{2}H\right)\Phi = 0 \\
&\Phi_{z_2} + \hat{\mathcal{B}}\Phi_{z_1} - \frac{\hat{\mathcal{B}}_{z_1}}{2}\Phi = 0
\end{aligned}
\tag{4.10}
$$

where

$$\hat{\mathcal{B}} = \sum_{j=1}^{n} \Lambda(z_2)^{(n-j+1)} V^{[j]}(z_1,z_2) \tag{4.11}$$