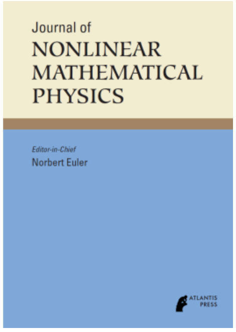

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THE PERIODIC μ-b-EQUATION AND EULER EQUATIONS ON THE CIRCLE

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In this paper, we study the μ-variant of the periodic b-equation and show that this equation can be realized as a metric Euler equation on the Lie group $\mathrm{Diff}^\infty(\mathbb{S})$ if and only if $b = 2$ (for which it becomes the μ-Camassa–Holm equation). In this case, the inertia operator generating the metric on $\mathrm{Diff}^\infty(\mathbb{S})$ is given by $L = \mu - \partial_x^2$. In contrast, the μ-Degasperis–Procesi equation (obtained for $b = 3$) is not a metric Euler equation on $\mathrm{Diff}^\infty(\mathbb{S})$ for any regular inertia operator $A \in \mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$. The paper generalizes some recent results of [13, 16, 24].

Keywords: μ-b-equation; diffeomorphism group of the circle; metric and non-metric Euler equations.

2000 Mathematics Subject Classification: 35Q35, 58D05

For the mathematical modelling of fluids, the so-called family of b-equations

$$m_t = -(m_x u + b m u_x), \quad m = u - u_{xx}, \tag{1}$$

attracted a considerable amount of attention in recent years. Here, b stands for a real parameter, [17]. Each of these equations models the unidirectional irrotational free surface flow of a shallow layer of an inviscid fluid moving under the influence of gravity over a flat bed. In this model $u(t, x)$ represents the wave's height at time $t \geq 0$ and position x above the flat bottom. If the wave profile is assumed to be periodic, $x \in \mathbb{S} \simeq \mathbb{R}/\mathbb{Z}$; otherwise $x \in \mathbb{R}$. For further details concerning the hydrodynamical relevance we refer to [10, 21, 22]. As shown in [11, 18, 20, 28], the b-equation is asymptotically integrable which is a necessary condition for complete integrability, but only for $b = 2$ and $b = 3$ for which it becomes the Camassa–Holm (CH) equation

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0$$

and the Degasperis–Procesi (DP) equation

$$u_t - u_{txx} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0$$

respectively. The Cauchy problems for CH and DP have been studied in detail: for the CH, there are global strong as well as global weak solutions. In addition, CH allows for finite time blow-up solutions which can be interpreted as breaking waves and there are no shock waves (see, e.g., [4–6]). Some recent global well-posedness results for strong and weak solutions, precise blow-up scenarios and wave breaking for the DP are discussed in [14, 15, 30–32].

Besides the various common properties of the CH and the DP there are also significant differences to report on, e.g., when studying geometric aspects of the family (1). The periodic equation (1) reexpresses a geodesic flow on the group $\mathrm{Diff}^\infty(\mathbb{S})$ of smooth and orientation preserving diffeomorphisms of the circle, cf. [13]. If $b = 2$, the geodesic flow corresponds to the right-invariant metric induced by the inertia operator $1 - \partial_x^2$ whereas for $b \neq 2$, Eq. (1) can only be realized as a non-metric Euler equation, i.e., as geodesic flow with respect to a linear connection which is not Riemannian in the sense that it is compatible with a right-invariant metric, cf. [8, 9, 16, 24].

The idea of studying Euler's equations of motion for perfect (i.e., incompressible, homogeneous and inviscid) fluids as a geodesic flow on a certain diffeomorphism group goes back to [1, 12] and in a recent work [13], Escher and Kolev show that the theory is also valid for the general b-equation.

In this paper, we are interested in the following variant of the periodic family (1). Let $\mu(u) = \int_{\mathbb{S}} u(t,x)\,\mathrm{d}x$ and $m = \mu(u) - u_{xx}$ in (1) to obtain the family of μ-b-equations, cf. [27]. The study of the μ-variant of (1) is motivated by the following key observation: Letting $m = -\partial_x^2 u$, Eq. (1) for $b = 2$ becomes the Hunter–Saxton (HS) equation, cf. [19], which possesses various interesting geometric properties, cf. [25, 26], whereas the choice $m = (1 - \partial_x^2)u$ leads to the CH as explained above. In the search for integrable equations that are obtained by a perturbation of $-\partial_x^2$, the μ-b-equation has been introduced and it could be shown that it behaves quite similarly to the b-equation; cf. [27] where the authors discuss local and global well-posedness as well as finite time blow-up and peakons. Peakons are peculiar wave forms: they are travelling wave solutions which are smooth except at their crests; the lateral tangents exist, are symmetric but different. Such wave forms are known to characterize the steady water waves of greatest height, [3, 7, 29], and were first shown to arise for the CH in [2].

The goal of this paper is to extend the work done in [16] to the family of μ-b-equations. Our main result is that the periodic μ-b-equation can be realized as a metric Euler equation on $\mathrm{Diff}^\infty(\mathbb{S})$ if and only if $b = 2$, for which it becomes the μCH equation. The corresponding regular inertia operator is $\mu - \partial_x^2$. Before we give a proof, we begin with some introductory remarks about Euler equations on $\mathrm{Diff}^\infty(\mathbb{S})$. In a first step, we comment on the operator $\mu - \partial_x^2$.

Lemma 1. *The bilinear map*

$$\langle \cdot, \cdot \rangle_\mu : C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}) \to \mathbb{R}, \quad \langle u, v \rangle_\mu = \mu(u)\mu(v) + \int_{\mathbb{S}} u_x(x) v_x(x)\, dx$$

defines an inner product on $C^\infty(\mathbb{S})$.

Proof. Clearly, $\langle \cdot, \cdot \rangle_\mu$ is a symmetric bilinear form and $\langle u, u \rangle_\mu \geq 0$. If $u \in C^\infty(\mathbb{S})$ satisfies $\langle u, u \rangle_\mu = 0$, then $u_x = 0$ on $\mathbb{S}$ and hence u is constant. The fact that $\mu(u) = 0$ implies $u = 0$. ∎