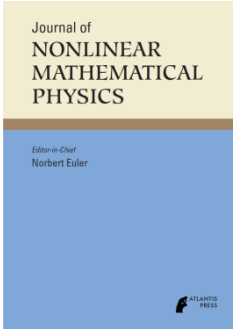

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

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To cite this article: R. Sahadevan, L. Nalinidevi (2010) Integrability of Certain Deformed Nonlinear Partial Differential Equations, Journal of Nonlinear Mathematical Physics 17:3, 379–396, DOI: https://doi.org/10.1142/S1402925110000969

To link to this article: https://doi.org/10.1142/S1402925110000969

Published online: 04 January 2021

ARTICLE

World Scientific
www.worldscientific.com

ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics, Vol. 17, No. 3 (2010) 379–396

DOI: 10.1142/S1402925110000969

INTEGRABILITY OF CERTAIN DEFORMED NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Received 4 December 2009
Accepted 12 March 2010

A systematic investigation of certain higher order or deformed soliton equations with $(1+1)$ dimensions, from the point of complete integrability, is presented. Following the procedure of Ablowitz, Kaup, Newell and Segur (AKNS) we find that the deformed version of Nonlinear Schrodinger equation, Hirota equation and AKNS equation admit Lax pairs. We report that each of the identified deformed equations possesses the Painlevé property for partial differential equations and admits trilinear representation obtained by truncating the associated Painlevé expansions. Hence the above mentioned deformed equations are completely integrable.

Keywords: Integrable equations; nonlinear partial differential equations; soliton equations; deformed equations.

1. Introduction

The investigation of completely integrable higher order nonlinear partial differential equations (PDEs) with $(1+1)$ dimensions admitting solitons has drawn considerable attention in recent years [6–11, 13, 15–18, 21]. For example by extending the Painlevé property for PDE [4, 19, 20], Karasu *et al.* [7] have recently identified a sixth order completely integrable nonlinear PDE

$$u_{6x} + 20u_x u_{4x} + 40u_{xx}u_{xxx} + 120u_x^2 u_{xx} + u_{xxxt} + 4u_t u_{xx} + 8u_x u_{xt} = 0 \tag{1.1}$$

or

$$(\partial_x^3 + 8u_x\partial_x + 4u_{xx})(u_t + u_{xxx} + 6u_x^2) = 0 \tag{1.2}$$

or

$$v_t + v_{xxx} + 12vv_x = g_x(t,x), \tag{1.3a}$$

$$g_{xxx} + 8vg_x + 4v_x g = 0, \tag{1.3b}$$

where subscripts denote partial derivatives, $v = u_x$, and $g(t,x) = u_t + u_{xxx} + 6u_x^2$. Equations (1.3) known as $KdV6$ in the literature and can also be written as

$$u_t - u_{xxx} - 6uu_x = g_x(t,x), \tag{1.4a}$$

$$g_{xxx} + 4ug_x + 2u_x g = 0, \tag{1.4b}$$

which can be viewed as a nonholonomic deformation of the Korteweg–deVries equation [9, 11]. It is appropriate to mention that Eq. (1.2) can also be derived through a different approach [13, 14, 18]. This suggests that one of the possibilities of finding a higher order but scalar completely integrable nonlinear PDEs with $(1+1)$ dimensions possessing solitons is (in a sense) to finding a coupled or deformed equation (with integrability properties) consisting of (i) known nonlinear PDE with $(1+1)$ dimensions possessing solitons with deformed variable and (ii) PDE in the deformed variable. Kundu has shown that Eq. (1.4) admits a Lax pair [9]. Also Kupershmidt [8] has demonstrated that Eq. (1.4) admits a bi-Hamiltonian representation while Ramani *et al.* [15] have shown that it can be written in bilinear form. Recently we have shown that the coupled or deformed equations (1.4) possess other integrability structures such as the existence of infinitely many generalized symmetries, polynomial conserved quantities, nonlocal symmetries, master symmetries and a recursion operator [10, 16] . In this article we report that the deformed NLS, Hirota and AKNS equations with $(1+1)$ dimensions, respectively, given by

$$iu_t - u_{xx} - 2u^2u^* = g(x,t), \tag{1.5a}$$

$$g_x = -2iu\sqrt{c(t)^2 - gg^*}. \tag{1.5b}$$

$$iu_t + i\epsilon(u_{3x} + 6|u|^2u_x) + \tfrac{u_{xx}}{2} + |u|^2u = g(x,t), \tag{1.6a}$$

$$g_x = -2iu\sqrt{c(t)^2 - gg^*}. \tag{1.6b}$$

$$u_t = -u_{xx} + 2u^2v + \tilde{g}(x,t), \tag{1.7a}$$

$$v_t = v_{xx} - 2v^2u + h(x,t), \tag{1.7b}$$

$$\tilde{g}_x = 2u\sqrt{c(t)^2 + \tilde{g}h}, \tag{1.7c}$$

$$h_x = 2v\sqrt{c(t)^2 + \tilde{g}h}, \tag{1.7d}$$

where $*$ denotes complex conjugate, ϵ is a constant and $c(t)$ is an arbitrary function admits Lax pair and possesses the Painlevé property for PDEs. Also we have shown, by truncating the associated Painlevé expansions, that each of them can be written in trilinear form. Thus the deformed NLS, Hirota and AKNS equations are completely integrable.

Note that on eliminating $g(x,t), \tilde{g}(x,t)$ and $h(x,t)$ in the above coupled equations one can obtain higher order nonlinear PDEs. For example, the deformed Hirota equation (1.6) can be written as

$$\left(u\frac{\partial^2}{\partial x^2} - u_x\frac{\partial}{\partial x}\right)\left(iu_t + i\epsilon(u_{3x} + 6|u|^2u_x) + \frac{u_{xx}}{2} + |u|^2u\right) + 2u^2\left[i(uu_t^* + u_tu^*)\right.$$
$$\left. + i\epsilon(uu_{3x}^* + u^*u_{3x}) + 6i\epsilon uu^*(uu_x^* + u^*u_x) - \frac{1}{2}(uu_{xx}^* - u^*u_{xx})\right] = 0$$

which is a fifth order one. The plan of the article is as follows: In Sec. 2, we explain through AKNS procedure that how one can derive Lax matrices for deformed nonlinear PDEs with $(1+1)$ dimensions in general and show that deformed NLS Eq. (1.5), Hirota (1.6) and AKNS equation Eq. (1.7) admit Lax pairs. In Sec. 3, we show explicitly that deformed NLS equation (1.5) possesses the Painlevé property. In Sec. 4, we show by truncating the Painlevé expansions that the deformed NLS admits trilinear representation. In Sec. 5, we give a brief summary of our results. In the Appendix, we show that both the deformed Hirota equation (1.6) and AKNS equation (1.7) possess the Painlevé property and admit trilinear forms.

2. Lax Pair of Nonlinear PDEs: AKNS Procedure

It is well known that the Lax pair of a scalar PDE, for example nonlinear evolution equation of the form

$$u_t = F(u, u_x, u_{xx}, \ldots) \tag{2.1}$$

can be constructed through AKNS procedure [1, 2, 12] in the following manner. Consider a linear system

$$\Psi_x = L\Psi, \quad \Psi_t = M\Psi, \tag{2.2}$$

or equivalently,

$$\begin{bmatrix} \psi_{1x}(\lambda) \\ \psi_{2x}(\lambda) \end{bmatrix} = \begin{bmatrix} L_{11}(\lambda) & L_{12}(\lambda) \\ L_{21}(\lambda) & L_{22}(\lambda) \end{bmatrix} \begin{bmatrix} \psi_1(\lambda) \\ \psi_2(\lambda) \end{bmatrix},$$

$$\begin{bmatrix} \psi_{1t}(\lambda) \\ \psi_{2t}(\lambda) \end{bmatrix} = \begin{bmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix} \begin{bmatrix} \psi_1(\lambda) \\ \psi_2(\lambda) \end{bmatrix},$$

where λ is the spectral parameter and $L_{ij}(\lambda), A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda)$ are functions of dependent variable and their x-derivatives. The compatibility condition of the linear system (2.2) gives

$$L_t - M_x + [L, M] = 0 \tag{2.3}$$

which is usually referred to as Lax equation.

The explicit form of the Lax matrices can be derived in the following way, that is, for a given suitable matrix L the matrix M can be derived by expanding its entries as a polynomial in the spectral parameter λ or $\frac{1}{\lambda}$ satisfying Eq. (2.3). Let us fix the entries of matrix L as

$$L_{11} = i\lambda, \quad L_{12} = iq, \quad L_{21} = ir, \quad L_{22} = -i\lambda$$

or equivalently,

$$\psi_{1x} = i\lambda\psi_1 + iq\psi_2 \tag{2.4}$$

$$\psi_{2x} = ir\psi_1 - i\lambda\psi_2. \tag{2.5}$$

Proceeding further from Eq. (2.3) we find

$$D = -A, \tag{2.6}$$

$$A_x = iqC - irB, \tag{2.7}$$

$$iq_t = B_x - 2i\lambda B + 2iqA, \tag{2.8}$$

$$ir_t = C_x + 2i\lambda C - 2irA. \tag{2.9}$$

Expanding $A(\lambda), B(\lambda)$ and $C(\lambda)$ as polynomials in $(\frac{1}{\lambda})^j, j = -3, -2, -1, 0, 1, 2$ and then equating different powers of λ in the above equations to zero yields the following:

$$A(\lambda) = a_3\lambda^3 + a_2\lambda^2 + \left(-\frac{a_3qr}{2} + a_1\right)\lambda + \left(\frac{ia_3}{4}(rq_x - qr_x) - \frac{a_2qr}{2} + a_0\right) + \frac{A_1}{\lambda} + \frac{A_2}{\lambda^2}, \tag{2.10}$$

$$B(\lambda) = a_3q\lambda^2 + \left(\frac{-ia_3q_x}{2} + a_2q\right)\lambda + \left(\frac{-a_3}{4}(q_{xx} + 2q^2r) - \frac{ia_2q_x}{2} + a_1q\right) + \frac{B_1}{\lambda} + \frac{B_2}{\lambda^2}, \tag{2.11}$$

$$C(\lambda) = a_3r\lambda^2 + \left(\frac{ia_3r_x}{2} + a_2r\right)\lambda + \left(\frac{-a_3}{4}(r_{xx} + 2qr^2) + \frac{ia_2r_x}{2} + a_1r\right) + \frac{C_1}{\lambda} + \frac{C_2}{\lambda^2} \tag{2.12}$$

in addition with

$$iq_t + \frac{a_3}{4}(q_{3x} + 6qrq_x) + \frac{ia_2}{2}q_{xx} + ia_2q^2r - 2ia_0q - a_1q_x = -2iB_1 \tag{2.13}$$

$$ir_t + \frac{a_3}{4}(r_{3x} + 6qrr_x) - \frac{ia_2}{2}r_{xx} - ia_2qr^2 + 2ia_0r - a_1r_x = 2iC_1, \tag{2.14}$$

$$A_{1x} = iqC_1 - irB_1, \quad B_{1x} = 2i(B_2 - qA_1), \quad C_{1x} = -2i(C_2 - rA_1), \tag{2.15}$$

$$A_{2x} = iqC_2 - irB_2, \quad B_{2x} + 2iqA_2 = 0, \quad C_{2x} - 2irA_2 = 0, \tag{2.16}$$

where a_0, a_1, a_2 and a_3 are integration constants. We would like to mention that the Lax matrices of well known soliton equations such as Korteweg–de Vries equation (KdV), modified KdV, etc can be derived by choosing A_i, B_i and $C_i, i = 1, 2$ as zero. This suggests that by choosing nonzero expressions for A_i, B_i and $C_i, i = 1, 2$ satisfying (2.13)–(2.16) one can derive Lax matrices for higher order or coupled or deformed nonlinear PDEs. Some of the identified higher order or deformed PDEs are as follows:

(i) **Deformed NLS Equation**: PDEs (2.13)–(2.14) reduce into

$$iu_t - u_{xx} - 2u^2u^* = g(x, t) \tag{2.17}$$

and its conjugate for the choice

$$a_0 = a_1 = a_3 = 0, \quad a_2 = 2i, \quad q = u, \quad r = u^*, \quad B_1 = \frac{ig}{2}, \quad C_1 = \frac{ig^*}{2}.$$

Proceeding further with the above expressions for B_1 and C_1 we find that (2.15)–(2.16) reduce into

$$b_x = i(ug^* - u^*g), \quad g_x = -2ibu \tag{2.18}$$

where

$$A_1 = \frac{ib}{2}, \quad A_2 = 0, \quad B_2 = 0, \quad C_2 = 0$$

and so the Lax matrices L and M become

$$L = \begin{pmatrix} i\lambda & iu \\ iu^* & -i\lambda \end{pmatrix},$$

$$M = \begin{pmatrix} 2i\lambda^2 - iuu^* & 2i\lambda u + u_x \\ 2i\lambda u^* - u_x^* & -2i\lambda^2 + iuu^* \end{pmatrix} + \frac{i}{2\lambda}\begin{pmatrix} b & g \\ g^* & -b \end{pmatrix}.$$

Note that Eqs. (2.17)–(2.18) become the deformed NLS equation (1.5) when $b(x,t) = \sqrt{c(t)^2 - gg^*}$.

(ii) **Deformed Hirota Equation**: PDEs (2.13)–(2.14) reduce into

$$iu_t + i\epsilon(u_{3x} + 6|u|^2u_x) + \frac{u_{xx}}{2} + |u|^2u = g(x,t) \tag{2.19}$$

and its conjugate for the choice

$$a_0 = a_1 = 0, \quad a_2 = -i, \quad a_3 = 4i\epsilon, \quad q = u, \quad r = u^*, \quad B_1 = \frac{ig}{2}, \quad C_1 = \frac{ig^*}{2}.$$

Proceeding further with the restrictions given above we find that (2.15)–(2.16) reduce into

$$g_x = -2iub \quad b_x = i(ug^* - u^*g) \tag{2.20}$$

where

$$A_1 = \frac{ib}{2}, \quad A_2 = 0, \quad B_2 = 0, \quad C_2 = 0.$$

Note that the deformed Hirota equation (1.6) can be obtained by choosing $b(x,t) = \sqrt{c(t)^2 - gg^*}$ in (2.19)–(2.20) and admits Lax pair with Lax matrices L and M as

$$L = \begin{pmatrix} i\lambda & iu \\ iu^* & -i\lambda \end{pmatrix},$$

$$M = \begin{pmatrix} 4i\epsilon\lambda^3 - i\lambda^2 + \dfrac{iuu^*}{2} - 2i\epsilon\lambda uu^* & 4i\epsilon\lambda^2 u + 2\epsilon\lambda u_x - i\lambda u - i\epsilon u_{xx} \\ +\epsilon(uu_x^* - u^*u_x) & -2i\epsilon u^2u^* - \dfrac{u_x}{2} \\ 4i\epsilon\lambda^2u^* - 2\epsilon\lambda u_x^* - i\lambda u^* & -4i\epsilon\lambda^3 + i\lambda^2 - \dfrac{iuu^*}{2} + 2i\epsilon\lambda uu^* \\ +\dfrac{u_x^*}{2} - i\epsilon u_{xx}^* - 2i\epsilon uu^{*2} & -\epsilon(uu_x^* - u^*u_x) \end{pmatrix} + \frac{i}{2\lambda}\begin{pmatrix} b & g \\ g^* & -b \end{pmatrix}$$

satisfying the Lax equation (2.3).

(iii) **Deformed AKNS Equation**: PDEs (2.13)–(2.14) become

$$u_t = -u_{xx} + 2u^2 v + \tilde{g}, \tag{2.21}$$

$$v_t = v_{xx} - 2v^2 u + h \tag{2.22}$$

for the choice

$$a_0 = a_1 = a_3 = 0, \quad a_2 = 2, \quad q = u, \quad r = v, \quad B_1 = \frac{-\tilde{g}}{2}, \quad C_1 = \frac{-h}{2}.$$

Proceeding further with the restrictions given above we find that (2.15)–(2.16) reduce into

$$\tilde{g}_x = 2ub, \quad h_x = 2vb, \quad b_x = (uh + v\tilde{g}) \tag{2.23}$$

where

$$A_1 = -\frac{ib}{2}, \quad A_2 = 0, \quad B_2 = 0, \quad C_2 = 0.$$

Note that the deformed AKNS equation (1.7) can be obtained by choosing $b(x,t) = \sqrt{c(t)^2 + \tilde{g}h}$ in (2.23) and admits Lax pair with Lax matrices

$$L = \begin{pmatrix} i\lambda & iu \\ -iv & -i\lambda \end{pmatrix},$$

$$M = \begin{pmatrix} 2\lambda^2 + uv & 2\lambda u - iu_x \\ -2\lambda v - iv_x & -2\lambda^2 - uv \end{pmatrix} + \frac{i}{2\lambda} \begin{pmatrix} -b & -\tilde{g} \\ -h & b \end{pmatrix}.$$

satisfying the Lax equation (2.3).

3. Painlevé Analysis of Deformed NLS Equation

We wish to report that each of the above identified deformed equations possesses the Painlevé property for PDEs [4, 19, 20]. In this section, we present the computational details for the deformed NLS equation while the details for the deformed Hirota and AKNS equation are given in the Appendix. We would like to mention that deformed NLS (1.5) can be written as

$$iu_t - u_{xx} - 2u^2 u^* = g(x,t), \tag{3.1a}$$

$$b_x = i(ug^* - u^* g), \quad g_x = -2ibu. \tag{3.1b}$$

In order to extend the Painlevé analysis for PDEs we write the above equation as

$$U_t - V_{xx} - 2U^2 V - 2V^3 = H, \tag{3.2a}$$

$$V_t + U_{xx} + 2U^3 + 2UV^2 = -G, \tag{3.2b}$$

$$G_x = 2VB, \tag{3.2c}$$

$$H_x = -2UB, \tag{3.2d}$$

$$B_x = 2UH - 2VG, \tag{3.2e}$$

where $u = U + iV, g = G + iH$. Obviously $b = B$ is a real valued function.

It is well known that the Painlevé analysis for PDEs consists essentially of three steps [19] namely: (i) determination of the leading-order behavior of the Laurent series solution, (ii) identifying the resonances at which the arbitrary functions enter into the series and (iii) verifying that sufficient number of arbitrary functions exist without the introduction of movable critical singularity manifolds.

Let us assume that the leading order behavior of the solutions of (3.2) be

$$\begin{gathered} U(x,t) \approx U_0\phi^{\alpha_1}, \quad V(x,t) \approx V_0\phi^{\alpha_2}, \quad G(x,t) \approx G_0\phi^{\alpha_3}, \\ H(x,t) \approx H_0\phi^{\alpha_4}, \quad B(x,t) \approx B_0\phi^{\alpha_5}, \end{gathered} \tag{3.3}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 are negative integers to be determined and U_0, V_0, G_0, H_0, and B_0 are functions of (x,t) and $\phi(x,t)$ is the singularity manifold. Substituting (3.3) in (3.2) and equating the most dominant terms we find

$$\alpha_1 = \alpha_2 = -1, \quad \alpha_3 = \alpha_4 = \alpha_5 = -2 \tag{3.4}$$

and

$$\begin{gathered} U_0^2 + V_0^2 = -\phi_x^2 (\text{repeated twice}) \Rightarrow \text{either } U_0 \text{ or } V_0 \text{ is arbitrary} \\ G_0 = \frac{-V_0 B_0}{\phi_x}, \quad H_0 = \frac{U_0 B_0}{\phi_x}, \quad B_0(x,t) - \ \textit{arbitrary}. \end{gathered} \tag{3.5}$$

For finding the powers at which the arbitrary functions enter into the series solution, we substitute

$$\begin{gathered} U(x,t) \approx U_0\phi^{-1} + U_j\phi^{j-1}, \quad V(x,t) \approx V_0\phi^{-1} + V_j\phi^{j-1}, \quad G(x,t) \approx G_0\phi^{-2} + G_j\phi^{j-2}, \\ H(x,t) \approx H_0\phi^{-2} + H_j\phi^{j-2}, \quad B(x,t) \approx B_0\phi^{-2} + B_j\phi^{j-2} \end{gathered} \tag{3.6}$$

into the leading order terms of (3.2), and equating the lowest-order terms to zero we obtain a system of five equations linear in $(U_j, V_j, G_j, H_j, B_j)$. In matrix form it may be conveniently written as

$$\begin{pmatrix} 4U_0V_0 & 4V_0^2 + (j^2-3j)\phi_x^2 & 0 & 0 & 0 \\ 4U_0^2 + (j^2-3j)\phi_x^2 & 4U_0V_0 & 0 & 0 & 0 \\ 0 & 2B_0 & (2-j)\phi_x & 0 & 2V_0 \\ -2B_0 & 0 & 0 & (2-j)\phi_x & -2U_0 \\ 2H_0 & -2G_0 & -2V_0 & 2U_0 & (2-j)\phi_x \end{pmatrix} \begin{pmatrix} U_j \\ V_j \\ G_j \\ H_j \\ B_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.7}$$

Upon evaluation, Eq. (3.7) yields the following resonance values

$$j = -1, 0, 0, 2, 3, 4, 4. \tag{3.8}$$

Obviously, the resonance value at -1 represents the arbitrariness of the singularity manifold $\phi(x,t) = 0$.

To compute the arbitrary functions at the resonance values we now substitute the following Laurent series expansions,

$$U(x,t)=\sum_{j=0}^{4}U_j\phi^{j-1},\quad V(x,t)=\sum_{j=0}^{4}V_j\phi^{j-1},\quad G(x,t)=\sum_{j=0}^{4}G_j\phi^{j-2},$$
$$H(x,t)=\sum_{j=0}^{4}H_j\phi^{j-2},\quad B(x,t)=\sum_{j=0}^{4}B_j\phi^{j-2} \tag{3.9}$$

into (3.2). From the leading-order analysis it is clear that $B_0(x,t)$ and either $U_0(x,t)$ or $V_0(x,t)$ are arbitrary corresponding to the resonance values 0 and 0. Equating the coefficients of $(\phi^{-2},\phi^{-2},\phi^{-2},\phi^{-2},\phi^{-2})$ in (3.2) to zero, we obtain

$$\begin{pmatrix} 4U_0V_0 & 4V_0^2-2\phi_x^2 & 0 & 0 & 0\\ 4U_0^2-2\phi_x^2 & 4U_0V_0 & 0 & 0 & 0\\ 0 & 2B_0 & \phi_x & 0 & 2V_0\\ -2B_0 & 0 & 0 & \phi_x & -2U_0\\ 2H_0 & -2G_0 & -2V_0 & 2U_0 & \phi_x \end{pmatrix}\begin{pmatrix} U_1\\ V_1\\ G_1\\ H_1\\ B_1\end{pmatrix}$$
$$=\begin{pmatrix} -U_0\phi_t+V_0\phi_{xx}+2(V_{0x})\phi_x-H_0\\ V_0\phi_t+U_0\phi_{xx}+2(U_{0x})\phi_x-G_0\\ G_{0x}\\ H_{0x}\\ B_{0x}\end{pmatrix}. \tag{3.10}$$

Solving the above equation (3.10), we obtain

$$U_1(x,t)=\frac{1}{2\phi_x^3}(U_0\phi_{xx}\phi_x-V_0\phi_t\phi_x-2U_{0x}\phi_x^2-V_0B_0),$$
$$V_1(x,t)=\frac{1}{2\phi_x^3}(V_0\phi_{xx}\phi_x+U_0\phi_t\phi_x-2V_{0x}\phi_x^2+U_0B_0),$$
$$G_1(x,t)=\frac{1}{\phi_x^4}(-U_0B_0^2+B_0V_{0x}\phi_x^2+V_0B_{0x}\phi_x^2-2V_0B_0\phi_x\phi_{xx}-U_0B_0\phi_t\phi_x),$$
$$H_1(x,t)=-\frac{1}{\phi_x^4}(V_0B_0^2+B_0U_{0x}\phi_x^2+U_0B_{0x}\phi_x^2-2U_0B_0\phi_x\phi_{xx}+V_0B_0\phi_t\phi_x),$$
$$B_1(x,t)=-\frac{1}{\phi_x^2}(B_{0x}\phi_x-B_0\phi_{xx})$$

so that

$$U_0U_1+V_0V_1=\frac{\phi_{xx}}{2},\quad V_0G_1-U_0H_1=B_1\phi_x.$$

Similarly, by equating the coefficients of $(\phi^{-1}, \phi^{-1}, \phi^{-1}, \phi^{-1}, \phi^{-1})$ to zero, we obtain

$$\begin{aligned} 4U_0V_0U_2 + (4V_0^2 - 2\phi_x^2)V_2 &= U_{0t} - V_{0xx} \\ &\quad - 2V_1\phi_{xx} - 2(U_1^2 + V_1^2)V_0 - H_1, \end{aligned} \tag{3.11a}$$

$$\begin{aligned} (4U_0^2 - 2\phi_x^2)U_2 + 4U_0V_0V_2 &= -(V_{0t} + U_{0xx} + 2U_1\phi_{xx} \\ &\quad + 2(U_1^2 + V_1^2)U_0 + G_1), \end{aligned} \tag{3.11b}$$

$$2V_0B_2 + 2V_2B_0 = -2V_1B_1 + G_{1x}, \tag{3.11c}$$

$$2U_0B_2 + 2U_2B_0 = -2U_1B_1 - H_{1x}, \tag{3.11d}$$

$$2U_2H_0 + 2U_0H_2 - 2V_0G_2 - 2V_2G_0 = B_{1x} + 2V_1G_1 - 2U_1H_1. \tag{3.11e}$$

Solving Eqs. (3.11a) and (3.11b) we obtain

$$\begin{aligned} U_2 &= \frac{1}{6\phi_x^4}[2U_0V_0(-U_{0t} + V_{0xx} + 2V_1\phi_{xx} + H_1) - 2V_0^2(V_{0t} + U_{0xx} + 2U_1\phi_{xx} + G_1) \\ &\quad + \phi_x^2(V_{0t} + U_{0xx} + 2U_1\phi_{xx} + G_1 + 2(U_1^2 + V_1^2)U_0)] \\ V_2 &= \frac{1}{6\phi_x^4}[2U_0V_0(V_{0t} + U_{0xx} + 2U_1\phi_{xx} + G_1) + 2U_0^2(U_{0t} - V_{0xx} - 2V_1\phi_{xx} - H_1) \\ &\quad - \phi_x^2(U_{0t} - V_{0xx} - 2V_1\phi_{xx} - H_1 - 2(U_1^2 + V_1^2)V_0)]. \end{aligned}$$

From Eqs. (3.11c) and (3.11d), we find

$$B_2 = \frac{-1}{2V_0}(2V_2B_0 + 2V_1B_1 - G_{1x}) = \frac{-1}{2U_0}(2U_2B_0 + 2U_1B_1 + H_{1x}).$$

From Eq. (3.11e) we infer that either $G_2(x,t)$ *or* $H_2(x,t)$ is arbitrary corresponding to the resonance at 2.

Now, equating the coefficients of $(\phi^0, \phi^0, \phi^0, \phi^0, \phi^0)$ in (3.2) to zero, we obtain the following equations

$$\begin{aligned} &4U_0U_3 + 4V_0V_3 \\ &\quad = \frac{1}{V_0}[-4(U_1V_0 + U_0V_1)U_2 - 8V_0V_1V_2 - 3\phi_{xx}V_2 - V_{1xx} - H_2 + U_{1t} \\ &\qquad + U_2\phi_t - 2V_{2x}\phi_x - 2(U_1^2 + V_1^2)V_1], \end{aligned} \tag{3.12a}$$

$$\begin{aligned} &4U_0U_3 + 4V_0V_3 \\ &\quad = -\frac{1}{U_0}[4(U_1V_0 + U_0V_1)V_2 + 8U_0U_1U_2 + 3\phi_{xx}U_2 + U_{1xx} + G_2 + V_{1t} + V_2\phi_t \\ &\qquad + 2U_{2x}\phi_x + 2(U_1^2 + V_1^2)U_1], \end{aligned} \tag{3.12b}$$

$$2V_0B_3 + 2B_0V_3 - \phi_xG_3 = -2V_1B_2 - 2V_2B_1 + G_{2x}, \tag{3.12c}$$

$$2U_0B_3 + 2B_0U_3 + \phi_x H_3 = -(2U_1B_2 + 2U_2B_1 + H_{2x}), \tag{3.12d}$$

$$\begin{aligned} &2H_0U_3 + 2U_0H_3 - 2V_0G_3 - 2G_0V_3 \\ &\quad = 2V_1G_2 + 2V_2G_1 - 2U_2H_1 - 2U_1H_2 + B_{2x}. \end{aligned} \tag{3.12e}$$

It is straightforward to check that the right-hand side of Eqs. (3.12a)–(3.12b) are one and the same and hence either $U_3(x,t)$ or $V_3(x,t)$ is arbitrary corresponding to the resonance value at 3. The explicit forms of $B_3(x,t), G_3(x,t)$ and $H_3(x,t)$ can be obtained from the remaining Eqs. (3.12c), (3.12d) and (3.12e).

Equating the coefficient of $(\phi^1, \phi^1, \phi^1, \phi^1, \phi^1)$ in (3.2) to zero leads to a system of five algebraic equations for U_4, V_4, B_4, G_4, H_4 involving lengthy expressions. To solve them without loss of generality we assume that $\phi(x,t) = x + \psi(t)$. Proceeding further, we find that either U_4 or V_4 and B_4 are arbitrary corresponding to the resonance values at $4, 4$. Thus, the deformed NLS equation (3.2) possesses the Painlevé property for PDEs and hence it is expected to be integrable.

4. Trilinear Representation of Deformed NLS Equation

To investigate whether or not the deformed NLS equation admits bilinear or trilinear representation, first we introduce a set of new dependent variables, that is,

$$g \to l_x, \quad b \to k_x, \quad l_x = \frac{\partial l}{\partial x}, \quad k_x = \frac{\partial k}{\partial x}$$

and so the deformed NLS equation (3.1) becomes

$$iu_t - u_{xx} - 2u^2u^* = l_x, \tag{4.1a}$$

$$k_{xx} = i(ul_x^* - u^*l_x), \quad l_{xx} = -2iuk_x. \tag{4.1b}$$

The associated Painlevé expansions of Eq. (4.1) truncated up to constant term read

$$U = \frac{U_0}{\phi} + U_1, \quad V = \frac{V_0}{\phi} + V_1, \quad l_R = \frac{G_0}{\phi} + G_1, \quad l_I = \frac{H_0}{\phi} + H_1, \quad k = \frac{B_0}{\phi} + B_1, \tag{4.2}$$

where $l = l_R + il_I$. Without loss of generality, we consider the vacuum solution

$$U_1 = V_1 = G_1 = H_1 = B_1 = 0.$$

Note that Eq. (4.2) can be rewritten as

$$u = \frac{R}{S}, \quad u^* = \frac{R^*}{S}, \quad l = \frac{Q}{S}, \quad l^* = \frac{Q^*}{S} \quad \text{and} \quad k = \frac{T}{S} \tag{4.3}$$

with $S(x,t)$ and $T(x,t)$ real, and $Q(x,t)$ and $R(x,t)$ are complex valued functions. Substituting (4.3) into (4.1) and rearranging the terms we find that Eq. (4.1) can be written into a trilinear form. They are

$$(iD_t - D_x^2)R \cdot S = D_xQ \cdot S, \tag{4.4a}$$

$$D_x^2 S \cdot S = 2RR^*, \tag{4.4b}$$

$$S(D_x^2 Q \cdot S) - Q(D_x^2 S \cdot S) = -2iR(D_x T \cdot S), \tag{4.4c}$$

$$S(D_x^2 T \cdot S) - T(D_x^2 S \cdot S) = i(R(D_x Q^* \cdot S) - R^*(D_x Q \cdot S)), \tag{4.4d}$$

where D is the Hirota operator defined by [5]

$$D_t^n D_x^m f \cdot g = (\partial/\partial t - \partial/\partial t')^n (\partial/\partial x - \partial/\partial x')^m f(x,t) g(x',t') \quad | \, x = x', t = t'.$$

Expanding the functions $Q(x,t), R(x,t), S(x,t)$ and $T(x,t)$ as power series in $\tilde{\epsilon}$ that is

$$Q = \sum_{n=1}^{\infty} Q_{2n-1} \tilde{\epsilon}^{2n-1}, \quad R = \sum_{n=1}^{\infty} R_{2n-1} \tilde{\epsilon}^{2n-1},$$

$$S = 1 + \sum_{n=1}^{\infty} S_{2n} \tilde{\epsilon}^{2n}, \quad T = T_0 + \sum_{n=1}^{\infty} T^{2n} \tilde{\epsilon}^{2n}$$

and using them in (4.1), one can construct the N-soliton solutions in the usual way. To obtain one-soliton solution, we consider

$$Q = \tilde{\epsilon} Q_1, \quad R = \tilde{\epsilon} R_1, \quad S = 1 + \tilde{\epsilon}^2 S_2, \quad T = T_0 + \tilde{\epsilon}^2 T_2, \tag{4.5}$$

where $\tilde{\epsilon}$ is an arbitrary small parameter. Substituting Eq. (4.5) in Eq. (4.4) and then equating different powers of $\tilde{\epsilon}$ to zero yields an over determined system of linear PDEs. Solving them consistently yields

$$R_1 = e^{\eta_1}, \quad S_2 = e^{\eta_1 + \eta_1^* + A}, \quad Q_1 = \frac{-2i}{p_1^2} c(t) e^{\eta_1},$$

$$T_0 = c(t)x, \quad T_2 = \frac{(p_1 x - 4)}{4p_1^3} c(t) e^{\eta_1 + \eta_1^*}$$

where

$$\eta_1 = p_1 x - i p_1^2 t - \frac{2}{p_1} \int c(t) dt + p_2, \quad e^A = \frac{1}{4p_1^2}$$

p_1 and p_2 are real constants and $c(t)$ is an arbitrary function and so the one-soliton solution of Eq. (4.1) reads

$$u(x,t) = \frac{R}{S} = \frac{e^{\eta_1}}{1 + e^{\eta_1 + \eta_1^* + A}},$$

$$l(x,t) = \frac{Q}{S} = \frac{-2ic(t) e^{\eta_1}}{p_1^2 (1 + e^{\eta_1 + \eta_1^* + A})},$$

$$k(x,t) = \frac{T}{S} = \frac{c(t)(4p_1^3 x + (p_1 x - 4) e^{\eta_1 + \eta_1^*})}{4p_1^3 (1 + e^{\eta_1 + \eta_1^* + A})}.$$

The above form of one-soliton solution implies that the speed of the solution no longer has explicit relation with the amplitude as in the case of NLS equation without deformation.

5. Summary

A systematic investigation of certain higher order or deformed soliton equations with (1+1) dimensions, from the point of complete integrability, is presented. Following the procedure of Ablowitz, Kaup, Newell and Segur (AKNS) we find that the deformed version of Nonlinear Schordinger equation, Hirota equation and AKNS equation admit Lax pairs. It is shown that each of identified deformed equation possesses the Painlevé property for PDEs. Also we have shown by truncating the associated Painlevé expansions that each of the deformed equations admits trilinear representation. Hence they are completely integrable. The analysis shows that the deformation may cause variations in the speed of the soliton solutions.

Appendix: Painlevé Analysis of Deformed Hirota and AKNS Equations

A. Painlevé analysis of deformed Hirota equation

The deformed Hirota equation (1.6) can be written as

$$iu_t + i\epsilon(u_{3x} + 6|u|^2 u_x) + \tfrac{u_{xx}}{2} + |u|^2 u = g(x,t) \tag{A.1a}$$

$$b_x = i(ug^* - u^* g), \quad g_x = -2ibu. \tag{A.1b}$$

In order to extend the Painlevé analysis for PDEs we rewrite the above equations as

$$U_t + \frac{V_{xx}}{2} + (U^2 + V^2)V + 6\epsilon(U^2 + V^2)U_x + \epsilon U_{xxx} = H, \tag{A.2a}$$

$$V_t - \frac{U_{xx}}{2} - (U^2 + V^2)U + 6\epsilon(U^2 + V^2)V_x + \epsilon V_{xxx} = -G, \tag{A.2b}$$

$$G_x = 2VB, \tag{A.2c}$$

$$H_x = -2UB, \tag{A.2d}$$

$$B_x = 2UH - 2VG \tag{A.2e}$$

where $u(x,t) = U(x,t) + iV(x,t)$, $g(x,t) = G(x,t) + iH(x,t)$. Obviously $B(x,t)$ is a real valued function. From the leading order behavior we obtain the following:

$$\begin{gathered} U(x,t) \approx U_0\phi^{-1}, \quad V(x,t) \approx V_0\phi^{-1}, \quad G(x,t) \approx G_0\phi^{-2}, \\ H(x,t) \approx H_0\phi^{-2}, \quad B(x,t) \approx B_0\phi^{-2} \end{gathered} \tag{A.3}$$

and

$$\begin{gathered} U_0^2 + V_0^2 = -\phi_x^2 (\text{repeated twice}) \;\; G_0 = \frac{-V_0 B_0}{\phi_x}, \\ H_0 = \frac{U_0 B_0}{\phi_x}, \quad B_0 - arbitrary. \end{gathered} \tag{A.4}$$

Similarly from the resonance analysis we obtain the following resonance values

$$j = -1, 0, 0, 1, 2, 3, 4, 4, 5.$$

and the resonance value at -1 represents the arbitrariness of the singularity manifold $\phi(x,t) = 0$, while the resonance $0, 0$ are associated with the arbitrariness of U_0 *or* V_0 and B_0.

To compute the arbitrary functions associated with the obtained resonance values we now introduce the following series expansions,

$$\begin{gathered} U(x,t)=\sum_{j=0}^{5}U_j\phi^{j-1},\quad V(x,t)=\sum_{j=0}^{5}V_j\phi^{j-1},\quad G(x,t)=\sum_{j=0}^{5}G_j\phi^{j-2},\\ H(x,t)=\sum_{j=0}^{5}H_j\phi^{j-2},\quad B(x,t)=\sum_{j=0}^{5}B_j\phi^{j-2} \end{gathered} \tag{A.5}$$

into Eq. (A.2). Now collecting the coefficients of $(\phi^{-3},\phi^{-3},\phi^{-2},\phi^{-2},\phi^{-2})$ in (A.2) to zero, we obtain

$$\begin{pmatrix} 2\epsilon U_0\phi_x & 2\epsilon V_0\phi_x & 0 & 0 & 0\\ 2\epsilon U_0\phi_x & 2\epsilon V_0\phi_x & 0 & 0 & 0\\ 0 & 2B_0 & \phi_x & 0 & 2V_0\\ 2B_0 & 0 & 0 & -\phi_x & 2U_0\\ -2H_0 & 2G_0 & 2V_0 & -2U_0 & -\phi_x \end{pmatrix}\begin{pmatrix} U_1\\ V_1\\ G_1\\ H_1\\ B_1 \end{pmatrix}=\begin{pmatrix} \epsilon\phi_x\phi_{xx}\\ \epsilon\phi_x\phi_{xx}\\ G_{0x}\\ -H_{0x}\\ -B_{0x} \end{pmatrix}. \tag{A.6}$$

From Eqs. (A.6) we conclude that either $U_1(x,t)$ or $V_1(x,t)$ is arbitrary corresponding to the resonance value at 1.

Proceeding as before, we find that the functions G_2(or H_2), U_3(or V_3), U_4(or V_4), B_4 and U_5(or V_5) are arbitrary corresponding to the resonance values at $2,3,4,4,5$. Thus the general solution of deformed Hirota (A.2) possesses the required number namely nine arbitrary functions without the introduction of movable critical manifolds. Thus, the deformed Hirota equation (A.2) possesses the Painlevé property for PDEs.

B. Trilinear representation of deformed Hirota equation

To investigate whether or not the deformed Hirota equation admits bilinear or trilinear representation, first we introduce a set of new dependent variables, that is,

$$g\to l_x,\quad b\to k_x$$

and so the deformed Hirota equation (A.1) becomes

$$\begin{gathered} iu_t+i\epsilon(u_{3x}+6|u|^2u_x)+\frac{u_{xx}}{2}+|u|^2u=l_x(x,t)\\ k_{xx}=i(ul_x^*-u^*l_x),\quad l_{xx}=-2iuk_x. \end{gathered} \tag{B.1}$$

The associated Painlevé expansions of Eq. (B.1) truncated up to constant term reads

$$U=\frac{U_0}{\phi}+U_1,\quad V=\frac{V_0}{\phi}+V_1,\quad l_R=\frac{G_0}{\phi}+G_1,\quad l_I=\frac{H_0}{\phi}+H_1,\quad k=\frac{B_0}{\phi}+B_1. \tag{B.2}$$

Without loss of generality, we consider the vacuum solution

$$U_1=V_1=G_1=H_1=B_1=0.$$

Note that Eq. (B.2) can be rewritten as

$$u = \frac{R}{S}, \quad u^* = \frac{R^*}{S}, \quad l = \frac{Q}{S}, \quad l^* = \frac{Q^*}{S} \quad \text{and} \quad k = \frac{T}{S} \tag{B.3}$$

with $S(x,t)$ and $T(x,t)$ real, and $Q(x,t)$ and $R(x,t)$ are complex functions. Substituting (B.3) into (B.1) and rearranging the terms we find that Eq. (B.1) can be written into a tilinear form. They are

$$\begin{aligned} \left(iD_t + \frac{1}{2}D_x^2 + i\epsilon D_x^3\right) R \cdot S &= D_x Q \cdot S, \\ D_x^2 S \cdot S &= 2RR^*, \\ S(D_x^2 Q \cdot S) - Q(D_x^2 S \cdot S) &= -2iR(D_x T \cdot S), \\ S(D_x^2 T \cdot S) - T(D_x^2 S \cdot S) &= i(R(D_x Q^* \cdot S) - R^*(D_x Q \cdot S)). \end{aligned} \tag{B.4}$$

Hence we obtain the one-soliton solution of Eq. (B.1) as

$$\begin{aligned} u(x,t) &= \frac{R}{S} = \frac{e^{\eta_1}}{1 + e^{\eta_1 + \eta_1^* + A}}, \\ l(x,t) &= \frac{Q}{S} = \frac{-2ic(t)e^{\eta_1}}{p_1^2(1 + e^{\eta_1 + \eta_1^* + A})}, \\ k(x,t) &= \frac{T}{S} = \frac{c(t)(4p_1^3 x + (p_1 x - 4)e^{\eta_1 + \eta_1^*})}{4p_1^3(1 + e^{\eta_1 + \eta_1^* + A})} \end{aligned}$$

where

$$\eta_1 = p_1 x + \frac{ip_1^2 t}{2} - \epsilon p_1^3 t - \frac{2}{p_1}\int c(t)dt + p_2, \quad e^A = \frac{1}{4p_1^2},$$

and $c(t)$ is an arbitrary function.

C. Painlevé analysis of deformed AKNS equation (1.7)

The deformed AKNS equation (1.7) can be written as

$$u_t = -u_{xx} + 2u^2 v + \tilde{g}, \tag{C.1a}$$

$$v_t = v_{xx} - 2v^2 u + h, \tag{C.1b}$$

$$\tilde{g}_x = 2ub \tag{C.1c}$$

$$h_x = 2vb \tag{C.1d}$$

$$b_x = uh + v\tilde{g}. \tag{C.1e}$$

From the leading order behavior we obtain the following:

$$\begin{aligned} u(x,t) \approx u_0\phi^{-1}, \quad v(x,t) \approx v_0\phi^{-1}, \quad \tilde{g}(x,t) \approx \tilde{g}_0\phi^{-2}, \\ h(x,t) \approx h_0\phi^{-2}, \quad b(x,t) \approx b_0\phi^{-2} \end{aligned} \tag{C.2}$$

and

$$u_0 v_0 = \phi_x^2 (\text{repeated twice}), \quad \tilde{g}_0 = \frac{h_0 u_0}{v_0}, \quad b_0 = -\frac{\tilde{g}_0 v_0}{\phi_x}. \tag{C.3}$$

Similarly from the resonance analysis, we obtain the following resonance values

$$j = -1, 0, 0, 2, 3, 4, 4.$$

Obviously, the resonance value at -1 represents the arbitrariness of the singularity manifold $\phi(x,t) = 0$, while the resonance $0, 0$ are associated with the arbitrariness of u_0 *or* v_0 and $\tilde{g}_0$ *or* h_0.

To compute the arbitrary functions associated with the obtained resonance values we now introduce the following series expansions,

$$\begin{gathered} u(x,t) = \sum_{j=0}^{4} u_j \phi^{j-1}, \quad v(x,t) = \sum_{j=0}^{4} v_j \phi^{j-1}, \quad \tilde{g}(x,t) = \sum_{j=0}^{4} \tilde{g}_j \phi^{j-2}, \\ h(x,t) = \sum_{j=0}^{4} h_j \phi^{j-2}, \quad b(x,t) = \sum_{j=0}^{4} b_j \phi^{j-2} \end{gathered} \tag{C.4}$$

into Eq. (C.1). Equating the coefficients of $(\phi^{-2}, \phi^{-2}, \phi^{-2}, \phi^{-2}, \phi^{-2})$ in (C.1) to zero, we obtain

$$\begin{pmatrix} 4\phi_x^2 & 2v_0^2 & 0 & 0 & 0 \\ 2v_0^2 & 4\phi_x^2 & 0 & 0 & 0 \\ 2b_0 & 0 & \phi_x & 0 & 2u_0 \\ 0 & 2b_0 & 0 & \phi_x & 2v_0 \\ h_0 & \tilde{g}_0 & v_0 & u_0 & \phi_x \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ \tilde{g}_1 \\ h_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} -u_0\phi_t - u_0\phi_{xx} - 2(u_{0x})\phi_x - \tilde{g}_0 \\ v_0\phi_t - v_0\phi_{xx} - 2(v_{0x})\phi_x + h_0 \\ \tilde{g}_{0x} \\ h_{0x} \\ b_{0x} \end{pmatrix}. \tag{C.5}$$

Solving Eq. (C.5), we obtain the explicit values for $u_1, v_1, \tilde{g}_1, h_1$ and b_1. Proceeding as before, we find that $\tilde{g}_2$(or h_2), u_3(or v_3), u_4(or v_4) and b_4 are arbitrary corresponding to the resonance values at $j = 2, 3, 4, 4$. Thus the general solution of deformed AKNS (C.1) possesses the required number namely seven arbitrary functions with out the introduction of movable critical manifolds. Thus, the deformed AKNS equation (C.1) possesses the Painlevé property for PDEs.

D. Trilinear representation of deformed AKNS equation

To investigate whether or not the deforemed AKNS equation admits bilinear or trilinear representation, first we introduce a set of new dependent variables, that is,

$$u \to iu, \quad v \to iv, \quad \tilde{g} \to il_x, \quad h \to im_x, \quad b \to k_x$$

and so the deformed AKNS equation (D.1) becomes

$$\begin{aligned} u_t + u_{xx} + 2u^2 v &= l_x, \\ v_t - v_{xx} - 2v^2 u &= m_x, \end{aligned}$$

$$\begin{aligned} l_{xx} &= 2uk_x, \\ m_{xx} &= 2vk_x \\ k_{xx} &= -(um_x + vl_x). \end{aligned} \tag{D.1}$$

The associated Painlevé expansions of Eq. (D.1) truncated up to constant term reads

$$u = \frac{u_0}{\phi} + u_1, \quad v = \frac{v_0}{\phi} + v_1, \quad l = \frac{\tilde{g}_0}{\phi} + \tilde{g}_1, \quad m = \frac{h_0}{\phi} + h_1, \quad k = \frac{b_0}{\phi} + b_1. \tag{D.2}$$

Without loss of generality, we consider the vacuum solution

$$u_1 = v_1 = \tilde{g}_1 = h_1 = b_1 = 0.$$

Note that Eq. (D.2) can be rewritten as

$$u = \frac{R}{S}, \quad v = \frac{P}{S}, \quad l = \frac{Q}{S}, \quad m = \frac{M}{S} \quad \text{and} \quad k = \frac{T}{S} \tag{D.3}$$

where $P(x,t), Q(x,t), R(x,t), S(x,t), M(x,t)$ and $T(x,t)$ are real. Substituting and rearranging the terms we find that Eq. (D.1) can be written into a trilinear form. They are

$$\begin{aligned} (D_t + D_x^2)R \cdot S &= D_x Q \cdot S, \\ (D_t - D_x^2)P \cdot S &= D_x M \cdot S, \\ D_x^2 S \cdot S &= 2RP, \\ S(D_x^2 Q \cdot S) - Q(D_x^2 S \cdot S) &= 2R(D_x T \cdot S), \\ S(D_x^2 M \cdot S) - M(D_x^2 S \cdot S) &= 2P(D_x T \cdot S), \\ S(D_x^2 T \cdot S) - T(D_x^2 S \cdot S) &= -(R(D_x M \cdot S) + P(D_x Q \cdot S)). \end{aligned} \tag{D.4}$$

Hence we obtain the one-soliton solution of Eq. (D.1) as

$$\begin{aligned} u(x,t) &= \frac{R}{S} = \frac{e^{\eta_1}}{1 + e^{\eta_1+\eta_2+A}}, \\ v(x,t) &= \frac{P}{S} = \frac{e^{\eta_2}}{1 + e^{\eta_1+\eta_2+A}}, \\ l(x,t) &= \frac{Q}{S} = \frac{2c(t)e^{\eta_1}}{p_1^2(1 + e^{\eta_1+\eta_2+A})}, \\ m(x,t) &= \frac{M}{S} = \frac{2c(t)e^{\eta_2}}{p_1^2(1 + e^{\eta_1+\eta_2+A})}, \\ k(x,t) &= \frac{T}{S} = \frac{c(t)(4p_1^3 x + (p_1 x - 4)e^{\eta_1+\eta_2})}{4p_1^3(1 + e^{\eta_1+\eta_2+A})} \end{aligned}$$

where

$$\eta_1 = p_1 x - p_1^2 t + \frac{2}{p_1} \int c(t) dt + p_2,$$

$$\eta_2 = p_1 x + p_1^2 t + \frac{2}{p_1} \int c(t) dt + p_2,$$

$$e^A = \frac{1}{4p_1^2},$$

and $c(t)$ is an arbitrary function.

Acknowledgments

This work forms part of a research project funded by Council of Scientific and Industrial Research (CSIR), Govt. of India, New Delhi.

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