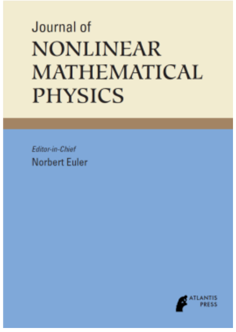

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NOVIKOV SUPER-ALGEBRAS WITH ASSOCIATIVE NON-DEGENERATE SUPER-SYMMETRIC BILINEAR FORMS

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Novikov super-algebras are related to quadratic conformal super-algebras which correspond to Hamiltonian pairs and play fundamental role in completely integrable systems. In this paper, we focus on quadratic Novikov super-algebras, which are Novikov super-algebras with associative non-degenerate super-symmetric bilinear forms. We show that quadratic Novikov super-algebras are associative and the associated Lie-super algebras of quadratic Novikov super-algebras are 2-step nilpotent. Moreover, we give some properties on quadratic Novikov super-algebras and classify the associated Lie-super algebras of quadratic Novikov super-algebras up to dimension 7.

Keywords: Novikov super-algebra; Novikov algebra; quadratic Novikov super-algebra; quadratic Novikov algebra; Lie-super algebra.

1. Introduction

Novikov super-algebras are super variant of Novikov algebras. They are closely connected to popular algebraic objects such as conformal super-algebras [5], vertex operator super-algebras [8] and super Gelfand–Dorfman bialgebras [7], which play important role in quantum field theory and the theory of completely integrable systems.

A Novikov super-algebra A is a $\mathbb{Z}_2$-graded vector space $A = A_0 + A_1$ with a bilinear product $(x, y) \mapsto xy$ for any $x \in A_i, y \in A_j, z \in A$ satisfying

$$(x, y, z) = (-1)^{ij}(y, x, z), \tag{1.1}$$

$$(zx)y = (-1)^{ij}(zy)x, \tag{1.2}$$

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where $(x,y,z) = (xy)z - x(yz)$. The even part of a given Novikov super-algebra is what is said to be a Novikov algebra introduced in connection with the Poisson brackets of hydrodynamic type [1] and Hamiltonian operators in the formal variational calculus [2–4, 9, 10].

The super-commutator

$$[x,y] = xy - (-1)^{ij}yx, \quad \text{for any } x \in A_i,\ \ y \in A_j$$

makes any Novikov super-algebra A a Lie-super algebra denoted $\mathfrak{g}(A)$ in what follows. We call $\mathfrak{g}(A)$ the associated Lie-super algebra of A. As usual, a form $f : A \times A \to \mathbb{F}$ is said to be super-symmetric if

$$f(x,y) = (-1)^{ij}f(y,x), \quad \text{for any } x \in A_i,\ \ y \in A_j, \tag{1.3}$$

and non-degenerate if

$$f(x,y) = 0 \quad \text{for any } y \in A \Rightarrow x = 0 \quad \text{and} \quad f(y,x) = 0 \quad \text{for any } y \in A \Rightarrow x = 0, \tag{1.4}$$

and even if

$$f(x,y) = 0, \quad \text{for any } x \in A_0,\ \ y \in A_1. \tag{1.5}$$

In this paper, we introduce the term quadratic Novikov super-algebra for denoting the pair (A, f) where A is a Novikov super-algebra and the bilinear form f on A is non-degenerate, super-symmetric and associative, i.e.,

$$f(xy,z) = f(x,yz), \quad \text{for any } x,y,z \in A. \tag{1.6}$$

The motivation for studying quadratic Novikov super-algebras comes from the fact that Lie or associative algebras with forms have important applications in several areas of mathematics and physics, such as the structure theory of finite-dimensional semi-simple Lie algebras, the theory of complete integrable Hamiltonian systems and the classification of statistical models over two-dimensional graphs.

The main goal of this paper is to study quadratic Novikov super-algebras and their associated Lie-super algebras. The paper is organized as follows. In Sec. 2, we show that A is associative and the associated Lie-super algebra $\mathfrak{g}(A)$ is 2-step nilpotent if (A, f) is a quadratic Novikov super-algebra. Also we show that $(\mathfrak{g}(A), f)$ is a quadratic Lie-super algebra. In Sec. 3, assume that f is even. We obtain some properties on quadratic Novikov super-algebras and their associated Lie-super algebras. In Sec. 4, we give the classification of the associated Lie-super algebras of quadratic Novikov super-algebras with even forms up to dimension 7.

Throughout this paper we assume that the algebras are of finite dimension over $\mathbb{C}$.

2. Quadratic Novikov Super-Algebras

Firstly, we give some definitions. Let A be a Novikov super-algebra. Define $Z(A) = \{x \in A | xy = yx = 0, \text{ for any } y \in A\}$. As usual, the pair $(\mathfrak{g}, f)$ is called a quadratic Lie-super algebra if $\mathfrak{g}$ is a Lie-super algebra and the bilinear form f on $\mathfrak{g}$ is non-degenerate, super-symmetric and $\mathfrak{g}$-invariant, i.e.,

$$f(x,[y,z]) = f([x,y],z), \quad \text{for any } x,y,z \in \mathfrak{g}.$$

Let $(\mathfrak{g}, f)$ be a quadratic Lie-super algebra and H be an ideal of $\mathfrak{g}$. As usual, H is said to be isotropic if $f|_{H\times H} = 0$ and non-degenerate if $f|_{H\times H}$ is non-degenerate.

Theorem 2.1. *Let (A, f) be a quadratic Novikov super-algebra. Then A is associative.*

Proof. For any $x, y, z, d \in A$,

$$\begin{aligned} f((x,y,z),d) &= f((xy)z - x(yz), d) \\ &= f((xy)z, d) - f(x(yz), d) \\ &= f(xy, zd) - f(x, (yz)d) \\ &= f(x, y(zd) - (yz)d) \\ &= -f(x, (y,z,d)). \end{aligned}$$

Thus for any $x \in A_i, y \in A_j, z \in A_k, d \in A_m$, we have that

$$\begin{aligned} f((x,y,z),d) = -f(x,(y,z,d)) &= (-1)^{1+i(j+k+m)+jk} f((z,y,d),x) \\ &= (-1)^{i(j+k+m)+jk} f(z,(y,d,x)), \\ &= (-1)^{i(j+k+m)+jk+k(i+j+m)} f((y,d,x),z), \\ &= (-1)^{i(j+k+m)+jk+k(i+j+m)+jm} f((d,y,x),z), \\ &= (-1)^{1+i(j+k+m)+jk+k(i+j+m)+jm} f(d,(y,x,z)), \\ &= (-1)^{1+i(j+k+m)+jk+k(i+j+m)+jm+m(i+j+k)} f((y,x,z),d), \\ &= (-1)^{1+i(j+k+m)+jk+k(i+j+m)+jm+m(i+j+k)+ij} f((x,y,z),d), \\ &= (-1)^{1+2(ij+ik+im+jk+mk+jm)} f((x,y,z),d), \\ &= -f((x,y,z),d). \end{aligned}$$

It follows that $(x,y,z) = 0$ by the non-degeneracy of f. □

Theorem 2.2. *Let (A, f) be a quadratic Novikov super-algebra. Then $[x,y] \in Z(A)$ for any $x, y \in A$. In particular, $[x,[y,z]] = 0$ for any $x, y, z \in A$.*

Proof. For any $x \in A_j, y \in A_k, z \in A$, by Theorem 2.1, we have that

$$\begin{aligned} z[x,y] &= z(xy) - (-1)^{jk} z(yx) = (zx)y - (-1)^{jk}(zy)x \\ &= (zx)y - (-1)^{jk+jk}(zx)y = 0. \end{aligned}$$

Then for any $x \in A_j, y \in A_k, z \in A_i, d \in A_m$, we have that

$$f([x,y]z, d) = f([x,y], zd) = (-1)^{(i+m)(j+k)} f(z, d[x,y]) = 0.$$

It follows that $[x,y]z = 0$ by the non-degeneracy of f. Then $[x,y] \in Z(A)$ for any $x, y \in A$. □

Proposition 2.3. *Let (A, f) be a quadratic Novikov super-algebra and $\mathfrak{g}(A)$ be the associated Lie-super algebra of A. Then $(\mathfrak{g}(A), f)$ is a quadratic Lie-super algebra.*

Proof. Since (A, f) is a quadratic Novikov super-algebra, we have that

$$\begin{aligned} f([x,y],z) &= f(xy-(-1)^{ij}yx,z) = f(x,yz)-(-1)^{ij+k(i+j)}f(z,yx) \\ &= f(x,yz)-(-1)^{ij+k(i+j)}f(zy,x) = f(x,yz)-(-1)^{kj}f(x,zy) \\ &= f(x,[y,z]) \end{aligned}$$

for any $x \in A_i, y \in A_j, z \in A_k$. Namely $(\mathfrak{g}(A), f)$ is a quadratic Lie-super algebra. □

Let $C(\mathfrak{g})$ denote the center of a Lie-super algebra $\mathfrak{g}$, i.e.,

$$C(\mathfrak{g}) = \{x \in \mathfrak{g} | [x,y] = 0, \text{ for any } y \in \mathfrak{g}\}.$$

Proposition 2.4. *Let $(\mathfrak{g}, f)$ be a quadratic Lie-super algebra. Then we have*:

(1) $C(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]^{\perp}$.
(2) *Let H be an ideal of $\mathfrak{g}$. Then $H^{\perp}$ is an ideal of $\mathfrak{g}$. Furthermore, assume that H is non-degenerate. Then $H^{\perp}$ is also non-degenerate and $\mathfrak{g} = H \oplus H^{\perp}$.*

3. Properties on Quadratic Novikov Super-Algebras with Even Forms

Let (A, f) be a quadratic Novikov super-algebra with f even and $\mathfrak{g}(A)$ be the associated Lie-super algebra of A. Let H be a subspace of A, define

$$H^{\perp} = \{x \in A | f(x,y) = 0, \text{ for any } y \in H\}.$$

In the following, we assume that f is even without special statements although most of the results exist when f is not even.

Lemma 3.1. *Let (A, f) be a quadratic Novikov super-algebra. Then $Z(A) = (AA)^{\perp}$. Moreover*, $\dim Z(A) + \dim AA = \dim A$.

Proof. For any $x \in Z(A)$ and $y, z \in A$, we have that $f(x, yz) = f(xy, z) = 0$. Namely, $Z(A) \subseteq (AA)^{\perp}$. For any $x \in (AA)^{\perp}$ and $y, z \in A$, $f(x, yz) = 0$. So $f(xy, z) = 0$. It follows that $xy = 0$ by the non-degeneracy of f. By $f(yx, z) = f(y, xz) = 0$, $yx = 0$. Thus, $x \in Z(A)$. Then $Z(A) = (AA)^{\perp}$. Clearly $\dim Z(A) + \dim AA = \dim A$ since f is non-degenerate. □

Proposition 3.2. *Let (A, f) be a quadratic Novikov super-algebra. Then* $\dim \mathfrak{g}(A) \geq 2\dim[\mathfrak{g}(A), \mathfrak{g}(A)]$.

Proof. By Proposition 2.4, we have $\dim C(\mathfrak{g}(A)) + \dim[\mathfrak{g}(A), \mathfrak{g}(A)] = \dim \mathfrak{g}$. By Theorem 2.2, $[\mathfrak{g}(A), \mathfrak{g}(A)] \subseteq C(\mathfrak{g}(A))$. Then the theorem follows. □

Proposition 3.3. *Let (A, f) be a quadratic Novikov super-algebra. Then $[\mathfrak{g}(A), \mathfrak{g}(A)] \subseteq AA \subseteq C(\mathfrak{g}(A))$ and $[\mathfrak{g}(A), \mathfrak{g}(A)] \subseteq Z(A) \subseteq C(\mathfrak{g}(A))$.*

Proof. We only need to prove that $AA \subseteq C(\mathfrak{g}(A))$. For any $x, y, z, d \in A$,

$$f(x, [y, zd]) = f([x, y], zd]) = f([x, y]z, d]) = 0.$$

It follows that $[y, zd] = 0$ by the non-degeneracy of f. So $AA \subseteq C(\mathfrak{g}(A))$. □

Proposition 3.4. *Let (A, f) be a quadratic Novikiv super-algebra. If $C(\mathfrak{g}(A))$ is isotropic, then $[\mathfrak{g}(A), \mathfrak{g}(A)] = C(\mathfrak{g}(A))$ and $Z(A) = AA$. Furthermore $\dim \mathfrak{g}(A)$ is even.*

Proof. If $C(\mathfrak{g}(A))$ is isotropic, then $C(\mathfrak{g}(A)) \subseteq C(\mathfrak{g}(A))^{\perp} = [\mathfrak{g}(A), \mathfrak{g}(A)]$. By Proposition 3.3, we have $[\mathfrak{g}(A), \mathfrak{g}(A)] = C(\mathfrak{g}(A))$ and $Z(A) = AA$. Then $\dim \mathfrak{g}(A)$ is even by Proposition 2.4. □

4. The Associated Lie-super Algebras of Quadratic Novikov Super-algebras

By Theorem 2.2, we know that the associated Lie-super algebra of a quadratic Novikov super-algebra is 2-step nilpotent. On the other hand:

Theorem 4.1. *Let $(\mathfrak{g}, f)$ be a 2-step nilpotent quadratic Lie-super algebra. Then $\mathfrak{g}$ admits a quadratic Novikov super-algebra structure.*

Proof. Define a bilinear product on $\mathfrak{g}$ by $xy = \frac{1}{2}[x, y]$. Under the product, $\mathfrak{g}$ is a Novikov super-algebra since $\mathfrak{g}$ is 2-step nilpotent as a Lie-super algebra. Moreover for any $x, y, z \in \mathfrak{g}$, $f(xy, z) = f(\frac{1}{2}[x, y], z) = f(x, \frac{1}{2}[y, z]) = f(x, yz)$. Namely, $\mathfrak{g}$ admits a quadratic Novikov super-algebra structure. □

Thus to get the classification of associated Lie-super algebras of quadratic Novikov super-algebras, it is enough to get the classification of 2-step nilpotent quadratic Lie-super algebras. By Proposition 2.4, we have:

Statement 1. Let (A, f) be a quadratic Novikov super-algebra. Then its associated Lie-super algebra $\mathfrak{g}(A)$ is a direct sum $\mathfrak{g}(A) = \mathfrak{g}_a \oplus \mathfrak{g}_i$, where $\mathfrak{g}_a$ is an Abelian ideal with non-degenerate restriction f on it and $\mathfrak{g}_i$ is an ideal with an isotropic center.

The statement 1 formulated above shows that the classification under consideration reduces to classification of quadratic Lie-super algebra with isotropic center. In the following, we will discuss the classification up to dimension 7 with even forms. Firstly, we have a well-known fact:

Statement 2. Let $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ be a Lie-super algebra and $x = x_0 + x_1$ be an element of $C(\mathfrak{g})$ with $x_0 \in \mathfrak{g}_0, x_1 \in \mathfrak{g}_1$. Then $x_0, x_1 \in C(\mathfrak{g})$.

Theorem 4.2. *Let (A, f) be a quadratic Novikov super-algebra with f even and $\dim A \leq 7$ such that the center of the associated Lie-super algebra $\mathfrak{g}$ is isotropic. Then $\mathfrak{g}$ is one of the following cases:*

(1) $\dim \mathfrak{g} = 4$ *and there exists a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$ such that $[e_4, e_4] = e_1, [e_2, e_4] = -[e_4, e_2] = e_3$, where $e_1, e_2 \in \mathfrak{g}_0$ and $e_3, e_4 \in \mathfrak{g}_1$;*
(2) $\dim \mathfrak{g} = 6$ *and there exists a basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ of $\mathfrak{g}$ such that $[e_2, e_5] = e_3, [e_2, e_6] = e_4, [e_5, e_5] = e_1, [e_6, e_6] = ke_1$, where $e_1, e_2 \in \mathfrak{g}_0$ and $e_3, e_4, e_5, e_6 \in \mathfrak{g}_1$;*

(3) $\dim \mathfrak{g} = 6$ *and there exists a basis* $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ *of* $\mathfrak{g} = \mathfrak{g}_0$ *such that* $[e_1, e_2] = e_4,\ [e_2, e_3] = e_5,\ [e_3, e_1] = e_6$.

Proof. Since $C(\mathfrak{g})$ is isotropic, we have that $\dim \mathfrak{g}$ is even by Proposition 3.4. That is, $\dim \mathfrak{g} = 2, 4$ or 6.

Case 1. It is easy to check that $\dim \mathfrak{g} \neq 2$.

Case 2. $\dim \mathfrak{g} = 4$. Since $C(\mathfrak{g})$ is isotropic, we have that $\dim C(\mathfrak{g}) = 2$.

If $\dim C(\mathfrak{g}) \cap \mathfrak{g}_0 = 0$, then $C(\mathfrak{g}) \subseteq \mathfrak{g}_1$. Since f is non-degenerate on $\mathfrak{g}_1$, we have that $\dim \mathfrak{g}_1 = 4$. Then $[\mathfrak{g}, \mathfrak{g}] = 0$ which contradicts to $\dim C(\mathfrak{g}) = \dim[\mathfrak{g}, \mathfrak{g}] = 2$.

Assume that $\dim C(\mathfrak{g}) \cap \mathfrak{g}_0 = 2$. Then $C(\mathfrak{g}) \subseteq \mathfrak{g}_0$. Similarly, we have that $\dim \mathfrak{g}_0 = 4$ since f is non-degenerate on $\mathfrak{g}_0$. Then $\mathfrak{g}_0$ is a 2-step nilpotent Lie algebra, that is, $\mathfrak{g}$ is Abelian. It contradicts to $\dim C(\mathfrak{g}) = 2$.

So we must have that $\dim C(\mathfrak{g}) \cap \mathfrak{g}_0 = 1$. Thus there exists a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$ such that $f(e_1, e_2) = f(e_2, e_1) = 1$ and $f(e_3, e_4) = -f(e_4, e_3) = 1$, where $e_1, e_3 \in C(\mathfrak{g})$, $e_1, e_2 \in \mathfrak{g}_0$ and $e_3, e_4 \in \mathfrak{g}_1$. We have that $[e_4, e_4] = ae_1$ since $[e_4, e_4] \in \mathfrak{g}_0$ and $\mathfrak{g}$ is 2-step nilpotent. Similarly, $[e_2, e_4] = ce_3$. Thus

$$a = f(e_2, [e_4, e_4]) = f([e_2, e_4], e_4) = f(ce_3, e_4) = c.$$

Moreover $a = c \neq 0$ since $\dim[\mathfrak{g}, \mathfrak{g}] = 2$. We can assume that $a = c = 1$ by replacing e_1 by ae_1 and e_3 by ae_3.

Case 3. $\dim \mathfrak{g} = 6$. Since $C(\mathfrak{g})$ is isotropic, we have that $\dim C(\mathfrak{g}) = 3$.

If $\dim C(\mathfrak{g}) \cap \mathfrak{g}_0 = 0$, then $C(\mathfrak{g}) \subseteq \mathfrak{g}_1$. Thus $\dim \mathfrak{g}_1 = 6$ since f is even and non-degenerate on $\mathfrak{g}_1$. It is a contradiction.

Support that $\dim C(\mathfrak{g}) \cap \mathfrak{g}_0 = 1$. Since f is even and non-degenerate on $\mathfrak{g}$, we have that there exists a basis $\{e_1, e_2, \ldots, e_6\}$ such that

$$f(e_1, e_2) = f(e_2, e_1) = 1, \quad f(e_3, e_5) = -f(e_5, e_3) = f(e_4, e_6) = -f(e_6, e_4) = 1,$$

where $e_1, e_3, e_4 \in C(\mathfrak{g})$, $e_1, e_2 \in \mathfrak{g}_0$ and $e_3, \ldots, e_6 \in \mathfrak{g}_1$. Let V be a vector subspace of $\mathfrak{g}_1$ extended by e_5 and e_6. Assume that $[x, x] = 0$ for any $x \in V$. Then for any $y, z \in V$,

$$[y, z] = [z, y] = \frac{1}{2}([y + z, y + z] - [y, y] - [z, z]) = 0.$$

It contradicts $C(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$. Thus there exists a basis $\{\eta_5, \eta_6\}$ of V such that

$$[\eta_5, \eta_5] = e_1, \quad [\eta_5, \eta_6] = [\eta_6, \eta_5] = 0, \quad [\eta_6, \eta_6] = ke_1.$$

At this time, $[e_2, \eta_5] = a_1 e_3 + a_2 e_4$, $[e_2, \eta_6] = b_1 e_3 + b_2 e_4$. Since $\dim[\mathfrak{g}, \mathfrak{g}] = 3$, we have that $\eta_3 = [e_2, e_6], \eta_4 = [e_2, e_5]$ are linear independent. That is, $\{e_1, e_2, \eta_3, \eta_4, \eta_5, \eta_6\}$ is a basis of $\mathfrak{g}$.

If $\dim C(\mathfrak{g}) \cap \mathfrak{g}_0 = 2$, then $\dim \mathfrak{g}_0 = 4$ since f is non-degenerate and even. Thus $\mathfrak{g}_0$ is a 2-step nilpotent quadratic Lie algebra of dimension 4. Then $\mathfrak{g}_0$ is Abelian. It follows that $\dim[\mathfrak{g}, \mathfrak{g}] \leq 2$. It contradicts to $\dim[\mathfrak{g}, \mathfrak{g}] = 3$.

If $\dim C(\mathfrak{g}) \cap \mathfrak{g}_0 = 3$, then $\dim \mathfrak{g}_0 = 6$. Namely, $\mathfrak{g} = \mathfrak{g}_0$ is a 2-step nilpotent quadratic Lie algebra. Moreover $\mathfrak{g}$ is not Abelian since $\dim[\mathfrak{g}, \mathfrak{g}] = 3$. Clearly, there exists a basis $\{e_1, \ldots, e_6\}$ such that $[e_1, e_2] = e_4,\ [e_2, e_3] = e_5,\ [e_3, e_1] = e_6$. □

By Theorem 4.2 and Statement 1, it is easy to classify the associated Lie-super algebras of quadratic Novikov super-algebras with even forms up to dimension 7.

5. Further Discussion

Propositions 3.2–3.4 mean that if $C(\mathfrak{g})$ is isotropic, then it is a maximal isotropic subspace in $\mathfrak{g}$ and $C(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$. Together with even form f, we can choose a basis $v_1, \dots, v_n, w_1, \dots, w_n$ in $\mathfrak{g}$ such that $w_i \in C(\mathfrak{g})$ for any i, the subspace spanned by $v_1, \dots, v_n$ is isotropic, and

$$f(v_i, w_j) = \delta_{ij} = (-1)^{p(v_i)} f(w_j, v_i),$$

where $p(v)$ denotes the parity of v. Then $[v_i, v_j] = \sum_s c_{ij}^s w_s$, where $c_{ij}^k = -(-1)^{p(v_i)p(v_j)} c_{ji}^k$ and $c_{ij}^k \neq 0$ only if $p(v_i) + p(v_j) = p(v_k)$. Since f is $\mathfrak{g}$-invariant, we have the following equalities:

$$f([v_i, v_j], v_k) = (-1)^{p(v_k)} c_{ij}^k = f(v_i, [v_j, v_k]) = c_{jk}^i = -(-1)^{p(v_j)p(v_k)} c_{kj}^i$$

or $c_{ij}^k = -(-1)^{(p(v_k)+1)p(v_j)} c_{kj}^i$.

Then the classification of quadratic Lie-super algebras with isotropic center reduces to a problem of Linear Super-algebra on a canonical form of a non-degenerate tensor c_{ij}^k on a linear super-space V with certain symmetry of each pair of indices. In particular, in the presence of a non-degenerate form on V, the symmetric in the purely even cases reduces to classification of 3-vectors, i.e., antisymmetric indices tensors c_{ijk}. This problem is completely solved in [6]. But it remains unknown in the absence of a non-degenerate form on V.

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