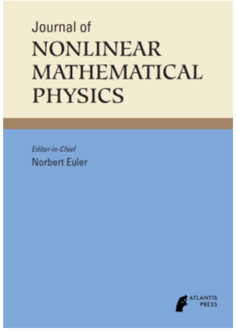

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ISOCHRONOUS OSCILLATORS

F. CALOGERO
Dipartimento di Fisica, Università di Roma "La Sapienza", Italy
Istituto Nazionale di Fisica Nucleare, Sezione di Roma
francesco.calogero@roma1.infn.it
francesco.calogero@uniroma1.it

F. LEYVRAZ
Centro Internacional de Ciencias, Cuernavaca, Mexico
Departamento de Física, Universidad de los Andes, Bogotá, Colombia
fa.leyvraz44@uniandes.edu.co

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We exhibit the solution of the initial-value problem for the system of $2N+2$ oscillators characterized by the Hamiltonian

$$H(\hat{p}_0, \check{p}_0, \underline{\hat{p}}, \underline{\check{p}}; \hat{q}_0, \check{q}_0, \underline{\hat{q}}, \underline{\check{q}}) = \frac{1}{2}[\hat{p}_0^2 - \check{p}_0^2 + \Omega^2(\hat{q}_0^2 - \check{q}_0^2)]$$
$$+ \frac{\hat{p}_0 - \Omega\check{q}_0}{2b}\sum_{n=1}^{N}[\hat{p}_n^2 - \check{p}_n^2 + \omega_n^2(\hat{q}_n^2 - \check{q}_n^2)] + \frac{\check{p}_0 - \Omega\hat{q}_0}{b}\sum_{n=1}^{N}[-\hat{p}_n\check{p}_n + \omega_n^2\hat{q}_n\check{q}_n],$$

where N is an arbitrary positive *integer*, Ω, b and ω_n^2 are $N+2$ arbitrary *real* constants, $\hat{q}_m, \check{q}_m$ with $m = 0, 1, \ldots, N$ are the $2N+2$ canonical coordinates and $\hat{p}_m, \check{p}_m$ the corresponding $2N+2$ canonical momenta. In the classical context the solution is *completely periodic* with period $T = 2\pi/|\Omega|$ (for *arbitrary* initial data). In the quantal context the (infinitely degenerate) spectrum is equispaced, with spacing $\hbar|\Omega|$; all the corresponding eigenfunctions are also exhibited. This finding obtains as special case of a more general (new) class of *isochronous* Hamiltonians.

Keywords: Nonlinear oscillators; isochronous Hamiltonians; quantization; equispaced spectrum; infinite degeneracy.

1. Introduction

The investigation of periodic dynamical systems has been an important thread throughout the evolution of the mathematical formulation of the laws of physics and the development of the theory of differential equations. In the last century a major boost to this type of investigations, but mainly restricted to planar systems with 2 time-dependent variables, was motivated by the famous, still open, 16th Hilbert problem, see for instance the review papers [1–3].

Recently we introduced [4] a new technique, of rather general applicability (also to systems with an *arbitrary* number of degrees of freedom), allowing to modify a Hamiltonian so that, in the *classical* context, the solutions of the initial-value problem of the modified Hamiltonian are *isochronous*: completely periodic with a fixed period independent of the initial data — provided they are in an appropriate — open, fully dimensional — phase space region, which in the case of this new technique [4] actually coincides with the *entire* phase space (for a review of results on *isochronous* systems, including this development, see [5]). A remarkable application of this technique has been discussed, in the context of the general many-body problem, in [6]. Another class of (classical) *isochronous* systems, not covered by the review [5], has been identified quite recently [7]. In the present paper we introduce yet another variant of the approach to generate *isochronous* Hamiltonians, and we report a simple application of this technique, yielding a dynamical system of coupled oscillators whose solutions are *isochronous* for *arbitrary* initial data. We moreover show that, in the *quantal* context, this Hamiltonian features an (infinitely degenerate) *equispaced* spectrum.

The results of this paper are reported in the following Sec. 2; they are then proven in Sec. 3. A very terse Sec. 4, entitled "Outlook", concludes the paper.

2. Results

Proposition 1. *The Hamiltonian*

$$H(p_0, \underline{p}; q_0, \underline{q}) = \frac{1}{2}(p_0^2 + \Omega^2 q_0^2) + \frac{p_0 + i\Omega q_0}{b} h(\underline{p}; \underline{q}) \tag{1}$$

is isochronous: *there is an open, fully dimensional region of its phase space where all its solutions are periodic with period*

$$T = \frac{2\pi}{|\Omega|}, \tag{2a}$$

$$q_m(t+T) = q_m(t), \quad p_m(t+T) = p_m(t), \quad m = 0, 1, \ldots, N. \ \boxdot \tag{2b}$$

Notation and Related Remarks. Here (and in the following Propositions 2 and 3) the context is of course that of classical Hamiltonian dynamics: q_0 and the N components of the N-vector $\underline{q} \equiv (q_1, \ldots, q_N)$ are the $1 + N$ canonical coordinates, likewise p_0 and the N components of the N-vector $\underline{p} \equiv (p_1, \ldots, p_N)$ are the corresponding $1 + N$ canonical momenta; $h(\underline{p}; \underline{q})$ is a, largely *arbitrary*, Hamiltonian (independent of the canonical variables p_0 and q_0; it might of course depend on other, constant, quantities, besides the N canonical coordinates q_n and the corresponding N canonical momenta p_n, with $n = 1, \ldots, N$); the two constants Ω and b are *real* but otherwise *arbitrary* (both, of course, *nonvanishing*: indeed the constant b — with the dimensionality of a momentum — has been introduced merely to keep good track of dimensions and it only plays a quite marginal role, indeed the restriction that it be *real* could be lifted without significant consequences except for some notational complications in some of the following formulae); and i is the imaginary unit, $i^2 = -1$. The independent variable t ("time") characterizing (see below) the evolution, due to this Hamiltonian, of the canonical variables $q_m \equiv q_m(t), p_m \equiv p_m(t)$, is hereafter assumed to be *real* (indeed nonnegative, starting from the initial time $t = 0$); differentiations with respect to it will be denoted below by superimposed dots. On the other hand, due to the *complex* character of this Hamiltonian — caused by the explicit presence of the imaginary unit i in its definition (1) — the canonical variables q_m and p_m are *complex* numbers, and we will use superimposed hats respectively checks to identify their real respectively imaginary parts, according to the following definitions:

$$q_m \equiv \hat{q}_m + i\check{q}_m, \quad p_m \equiv \hat{p}_m + i\check{p}_m, \quad m = 0, 1, \ldots, N. \tag{3}$$

However, as it is well known (see for instance [8]), the same time-evolution produced by the *complex* Hamiltonian (1) for the $2N + 2$ *complex* canonical variables q_m, p_m, and via these formulae for the $4N + 4$ *real* variables $\hat{q}_m, \check{q}_m, \hat{p}_m, \check{p}_m$, is yielded, directly for these $4N + 4$ variables, by the *real* Hamiltonian

$$H_R(\hat{p}_0, \check{p}_0, \hat{\underline{p}}, \check{\underline{p}}; \hat{q}_0, \check{q}_0, \hat{\underline{q}}, \check{\underline{q}}) = \mathrm{Re}[H(\hat{p}_0 - i\check{p}_0, \hat{\underline{p}} - i\check{\underline{p}}; \hat{q}_0 + i\check{q}_0, \hat{\underline{q}} + i\check{\underline{q}})], \tag{4}$$

with the $2N + 2$ *real* quantities $\hat{q}_m, \check{q}_m$ playing now the role of canonical coordinates and the $2N+2$ *real* quantities $\hat{p}_m, \check{p}_m$ being the corresponding canonical momenta (do note the negative signs appearing in the right-hand side of (4)). Here and hereafter whenever we use the indices n respectively m it is understood that they run from 1 to N respectively from 0 to N.

The fact that the validity (proven in the following section) of Proposition 1 requires hardly any restriction on the Hamiltonian $h(\underline{p}; \underline{q})$ might be considered *remarkable* or *trivial*, depending on the level of understanding of the mechanism that underpins this fact (as is indeed the case for any valid mathematical finding). This mechanism is analogous to — yet different from — that reported in previous papers [4–6]: indeed the *isochronous* Hamiltonian, (1), introduced here is to some extent simpler than that introduced previously, in particular it involves $h(\underline{p}; \underline{q})$ *linearly*, see (1), rather than *quadratically*, see [4] (another Hamiltonian involving $h(\underline{p}; \underline{q})$ linearly was introduced in [6], see Eq. (42) there, but the property of periodicity of its generic solution is incorrectly attributed there to this Hamiltonian). On the other hand the Hamiltonian (1) has the disadvantage of being *complex*, yielding a *complex* dynamics; as indicated above this can be remedied, allowing to return to a context of *real* Hamiltonian dynamics, but at the cost of doubling the number of canonical variables and possibly of causing the resulting Hamiltonian (involving the real and imaginary parts of $h(\hat{\underline{p}} - i\check{\underline{p}}; \hat{\underline{q}} + i\check{\underline{q}})$: see (4), (3) and (1)) to be much uglier — although the example on which we focus in this paper shows that this need not happen, see below.

Proposition 2. *The solution of the initial-value problem for the complex Hamiltonian*

$$H(p_0, \underline{p}; q_0, \underline{q}) = \frac{1}{2}(p_0^2 + \Omega^2 q_0^2) + \frac{p_0 + i\Omega q_0}{b} \sum_{n=1}^{N} h_n(p_n, q_n), \tag{5a}$$

$$h_n(p_n, q_n) = \frac{1}{2}(p_n^2 + \omega_n^2 q_n^2), \tag{5b}$$

is provided by the following formulae:

$$q_0(t) = \left[q_0(0) + i\frac{E}{b\Omega}\right]\cos(\Omega t) + \left[p_0(0) + \frac{E}{b}\right]\frac{\sin(\Omega t)}{\Omega} - i\frac{E}{b\Omega}, \tag{6a}$$

$$p_0(t) = \left[p_0(0) + \frac{E}{b}\right]\cos(\Omega t) - \left[\Omega q_0(0) + i\frac{E}{b}\right]\sin(\Omega t) - \frac{E}{b}, \tag{6b}$$

$$E = \frac{1}{2}\sum_{n=1}^{N}\{[p_n(0)]^2 + \omega_n^2[q_n(0)]^2\}; \tag{6c}$$

$$q_n(t) = q_n(0)\cos[\omega_n\tau(t)] + p_n(0)\frac{\sin[\omega_n\tau(t)]}{\omega_n}, \quad n = 1,\ldots,N \tag{7a}$$

$$p_n(t) = p_n(0)\cos[\omega_n\tau(t)] - q_n(0)\omega_n\sin[\omega_n\tau(t)], \quad n = 1,\ldots,N \tag{7b}$$

$$\tau(t) = \frac{p_0(0) + i\Omega q_0(0)}{b}\frac{\exp(i\Omega t) - 1}{i\Omega}. \ \boxdot \tag{7c}$$

Notation. The N constants ω_n^2 are arbitrary, and the other notation has been defined above — but let us note that, for the validity of this Proposition 2, no condition is required on the reality of any of the quantities appearing in these formulae.

Proposition 3. *The solution of the initial-value problem for the real Hamiltonian*

$$H(\hat{p}_0, \check{p}_0, \underline{\hat{p}}, \underline{\check{p}}; \hat{q}_0, \check{q}_0, \underline{\hat{q}}, \underline{\check{q}}) = \frac{1}{2}[\hat{p}_0^2 - \check{p}_0^2 + \Omega^2(\hat{q}_0^2 - \check{q}_0^2)]$$
$$+ \frac{\hat{p}_0 - \Omega\check{q}_0}{b}\sum_{n=1}^{N}\hat{h}_n(\hat{p}_n, \check{p}_n; \hat{q}_n, \check{q}_n) + \frac{\check{p}_0 - \Omega\hat{q}_0}{b}\sum_{n=1}^{N}\check{h}_n(\hat{p}_n, \check{p}_n; \hat{q}_n, \check{q}_n), \tag{8a}$$

$$\hat{h}_n(\hat{p}, \check{p}; \hat{q}, \check{q}) = \frac{1}{2}[\hat{p}^2 - \check{p}^2 + \omega_n^2(\hat{q}^2 - \check{q}^2)], \tag{8b}$$

$$\check{h}_n(\hat{p}, \check{p}; \hat{q}, \check{q}) = -\hat{p}\check{p} + \omega_n^2\hat{q}\check{q}, \tag{8c}$$

is provided by the following formulae:

$$\hat{q}_m(t) = \mathrm{Re}[q_m(t)], \quad \check{q}_m(t) = \mathrm{Im}[q_m(t)], \tag{9a}$$

$$\hat{p}_m(t) = \mathrm{Re}[p_m(t)], \quad \check{p}_m(t) = \mathrm{Im}[p_m(t)], \tag{9b}$$

where $q_m(t), p_m(t)$ *are given by the explicit expressions* (6) (*for* $m = 0$) *and* (7) (*for* $m = n$), *of course with* $q_m(0) = \hat{q}_m(0) + i\check{q}_m(0), p_m(0) = \hat{p}_m(0) + i\check{p}_m(0),$ *see* (3). ⊡

Note that the assumed property of the constant Ω to be *real* and *nonvanishing* guarantees that, for *arbitrary* initial data, the quantities $q_m(t), p_m(t)$ given by these formulae are *all* periodic with period T, see (2a); but this property does not require that the $N+1$ constants ω_n^2 and b be *real* — although this requirement is indeed essential to guarantee that the Hamiltonian (8) be *real*. Of course the specific evolution of these solutions within the period T depends quite significantly on the sign of the real quantities ω_n^2 and as well on the magnitude of these quantities and of Ω.

In the following, to treat the *quantal* case, we assume for simplicity Ω, b and all the circular frequencies ω_n to be *real*.

Proposition 4. *The stationary Schrödinger equation obtained, in the quantal context, by applying the standard quantization prescription*

$$\hat{q}_m \Rightarrow x_m, \quad \check{q}_m \Rightarrow y_m, \quad \hat{p}_m \Rightarrow -i\hbar\frac{\partial}{\partial x_m}, \quad \check{p}_m \Rightarrow -i\hbar\frac{\partial}{\partial y_m}; \quad m = 0,\ldots,N, \tag{10}$$

to the real Hamiltonian (8), *features an* (*infinitely degenerate*) *equispaced spectrum with spacing* $\hbar\Omega$*:*

$$\left\{\frac{1}{2}\left[\hbar^2\left(-\frac{\partial^2}{\partial x_0^2}+\frac{\partial^2}{\partial y_0^2}\right)+\Omega^2(x_0^2-y_0^2)\right]\right.$$
$$-b^{-1}\left(i\hbar\frac{\partial}{\partial x_0}+\Omega y_0\right)\frac{1}{2}\sum_{n=1}^{N}\left[\hbar^2\left(-\frac{\partial^2}{\partial x_n^2}+\frac{\partial^2}{\partial y_n^2}\right)+\omega_n^2(x_n^2-y_n^2)\right]$$
$$\left.-b^{-1}\left(i\hbar\frac{\partial}{\partial y_0}+\Omega x_0\right)\sum_{n=1}^{N}\left[\hbar^2\frac{\partial^2}{\partial x_n\partial y_n}+\omega_n^2 x_n y_n\right]\right\}\Psi$$
$$=k\,\hbar\Omega\,\Psi,\quad k=0,\pm1,\pm2,\ldots.\ \boxdot \tag{11}$$

Explicit expressions of the eigenfunctions $\Psi \equiv \Psi(x_0, y_0; \underline{x}, \underline{y}; k)$ are easily obtained from the proof of this proposition, as provided in the following section; they allow a precise accounting of the infinite degeneracy of the energy level corresponding to each (arbitrary integer) value of the quantum number k.

3. Proofs

The point of departure to prove the two Propositions 1 and 2 are the equations of motion implied by the Hamiltonian (1), reading

$$\dot{q}_0 = p_0 + \frac{E}{b}, \quad \dot{p}_0 = -\Omega\left(\Omega q_0 + i\frac{E}{b}\right), \tag{12a}$$

$$\dot{q}_n = \frac{p_0 + i\Omega q_0}{b}\frac{\partial h(\underline{p};\underline{q})}{\partial p_n}, \quad \dot{p}_n = -\frac{p_0 + i\Omega q_0}{b}\frac{\partial h(\underline{p};\underline{q})}{\partial q_n}. \tag{12b}$$

To write these formulae, and below, we use the fact — obviously implied by the definition of the Hamiltonian (1) — that $h(\underline{p}, \underline{q})$ is a constant of motion, here denoted by E, which can of course be identified with the *initial* value of $h(\underline{p}, \underline{q})$,

$$E = h(\underline{p}(0), \underline{q}(0)). \tag{13}$$

It is then clear (from (12a)) that the time evolution of the two canonical variables $q_0(t)$ and $p_0(t)$ is given by (6), implying

$$\frac{p_0(t) + i\Omega q_0(t)}{b} = \frac{p_0(0) + i\Omega q_0(0)}{b}\exp(i\Omega t) = \dot{\tau}(t). \tag{14}$$

Here the second equality is clearly implied by the definition (7c) of τ; and it implies that the equations of motion (12b) can now be reformulated as follows:

$$\xi_n' = \frac{\partial h(\underline{\eta};\underline{\xi})}{\partial \eta_n}, \quad \eta_n' = -\frac{\partial h(\underline{\eta};\underline{\xi})}{\partial \xi_n}, \tag{15}$$

via the formal change of dependent variables

$$\underline{q}(t) = \underline{\xi}(\tau), \quad \underline{p}(t) = \underline{\eta}(\tau), \tag{16a}$$

with (since $\tau(0) = 0$, see (7c))

$$\underline{q}(0) = \underline{\xi}(0), \quad \underline{p}(0) = \underline{\eta}(0). \tag{16b}$$

Note that these equations of motion, (15), are just the standard equations of motion yielded by the Hamiltonian $h(\underline{\eta};\underline{\xi})$: of course the appended primes in the left-hand sides of (15) denote differentiations with respect to the (*complex*) independent variable τ, and we are assuming that this makes good sense, namely that $h(\underline{\eta};\underline{\xi})$ is an *analytic* (but not necessarily *holomorphic*) function of its $2N$ arguments, namely of the $2N$ components of the two N-vectors $\underline{\eta}$ and $\underline{\xi}$.

We therefore conclude that the solution of the initial-value problem for the equations of motion (12b) is given, via the change of independent variable (16) with (7c), by the solution of the initial-value problem — with the same initial data — for the equations of motion (15). Hence — see (7c) — the solution $\underline{q}(t), \underline{p}(t)$ of (12b) is *completely periodic* in the real independent variable t — see (2b) — if the corresponding solution $\underline{\xi}(\tau), \underline{\eta}(\tau)$ of (15) is *holomorphic*, as a function of the *complex* variable τ, in the disk D enclosed, in the *complex* τ-plane, by the circle C traveled periodically — with period T, see (2a) — by the variable $\tau = \tau(t)$ when the real variable t progresses continuously from its initial value $t = 0$. This circle C — see (7c) — has, in the *complex* τ-plane, a diameter one end of which is at the origin, $\tau = 0$, and the other end is at the point

$$d = 2[ip_0(0) - \Omega q_0(0)]/(b\Omega); \tag{17}$$

hence it is enclosed inside the circular disk $\tilde{D}$ centered at the origin of the complex τ-plane and having radius $|d|$. Since every solution $\underline{\xi}(\tau), \underline{\eta}(\tau)$ of the equations of motion (15) is *holomorphic* — as a function of the complex variable τ — in the neighborhood of the origin $\tau = 0$ where the initial data $\underline{\xi}(0) = \underline{q}(0), \underline{\eta}(0) = \underline{p}(0)$ (see (16b)) are assigned (provided of course these initial data are not assigned where the right-hand sides of the equations of motion (15) are singular, see for instance [9, Sec. 12.21]), and, since the extent of the region around $\tau = 0$ where the solution $\underline{\xi}(\tau), \underline{\eta}(\tau)$ is *holomorphic* depends on the initial data $\underline{\xi}(0) = \underline{q}(0), \underline{\eta}(0) = \underline{p}(0)$ of the equations of motion (12) (see for instance [9, Sec. 12.21]) while the radius $|d|$ of the disk $\tilde{D}$ can be made arbitrarily small by an appropriate choice of the initial data $p_0(0)$ and $q_0(0)$, see (17), it is clear that there is an open, fully dimensional set of these initial data $\underline{\xi}(0) = \underline{q}(0), \underline{\eta}(0) = \underline{p}(0)$ yielding *isochronous* solutions, see (2b), for any arbitrary (nonvanishing) assignment of the real constant Ω. Proposition 1 is thus proven.

To prove Proposition 2 we note that the Hamiltonian (5) corresponds to the Hamiltonian (1) with

$$h(\underline{p}, \underline{q}) = \sum_{n=1}^{N} h_n(p_n, q_n), \tag{18}$$

see (5b). Note that, via (13), this entails (6c), while the equations of motion (15) now read, via (18) and (16), as follows:

$$\xi_n' = \eta_n, \quad \eta_n' = -\omega_n^2 \xi_n, \tag{19}$$

entailing

$$\xi_n(\tau) = \xi_n(0)\cos(\omega_n\tau) + \eta_n(0)\frac{\sin(\omega_n\tau)}{\omega_n}, \tag{20a}$$

$$\eta_n(\tau) = \eta_n(0)\cos(\omega_n\tau) - \xi_n(0)\omega_n\sin(\omega_n\tau). \tag{20b}$$

These formulae yield, via (16) and (7c), the three formulae (7), while the three formulae (6) are implied by our previous treatment, see (12a) with (13) and (18). This completes the proof of Proposition 2.

The proof of Proposition 3 is an immediate consequence of Proposition 2, see the remarks made in the last part of the paragraph entitled *Notation and related remarks* reported in Sec. 2 after Proposition 1: note that indeed $h_n(\hat{p} - i\check{p}, \hat{q} + i\check{q}) = \hat{h}_n(\hat{p}, \check{p}; \hat{q}, \check{q}) + i\check{h}_n(\hat{p}, \check{p}; \hat{q}, \check{q})$, see (5b), (8b) and (8c).

The first step to prove Proposition 4 is to observe that the $1{+}2N$ operators obtained from the $1{+}2N$ Hamiltonians $H(\hat{p}_0, \check{p}_0, \underline{\hat{p}}, \underline{\check{p}}; \hat{q}_0, \check{q}_0, \underline{\hat{q}}, \underline{\check{q}})$, $\hat{h}_n(\hat{p}_n, \check{p}_n; \hat{q}_n, \check{q}_n)$ and $\check{h}_n(\hat{p}_n, \check{p}_n; \hat{q}_n, \check{q}_n)$ via the standard quantization prescription (10) commute among themselves; note that neither here nor below there is any ordering problem in the transition from the classical to the quantal contexts. Next we note that the following formula holds:

$$H(\hat{p}_0, \check{p}_0, \underline{\hat{p}}, \underline{\check{p}}; \hat{q}_0, \check{q}_0, \underline{\hat{q}}, \underline{\check{q}}) = \frac{1}{2}(\hat{P}^2 + \Omega^2\hat{Q}^2) - \frac{1}{2}(\check{P}^2 + \Omega^2\check{Q}^2), \tag{21a}$$

$$\hat{P} = \hat{p}_0 + \frac{1}{b}\sum_{n=1}^{N}\hat{h}_n(\hat{p}_n, \check{p}_n; \hat{q}_n, \check{q}_n), \quad \hat{Q} = \hat{q}_0 - \frac{1}{\Omega b}\sum_{n=1}^{N}\check{h}_n(\hat{p}_n, \check{p}_n; \hat{q}_n, \check{q}_n), \tag{21b}$$

$$\check{P} = \check{p}_0 - \frac{1}{b}\sum_{n=1}^{N}\check{h}_n(\hat{p}_n, \check{p}_n; \hat{q}_n, \check{q}_n), \quad \check{Q} = \check{q}_0 + \frac{1}{\Omega b}\sum_{n=1}^{N}\check{h}_n(\hat{p}_n, \check{p}_n; \hat{q}_n, \check{q}_n), \tag{21c}$$

and that these definitions entail that the quantities $\hat{Q}$ and $\check{Q}$ can be considered — in the classical context — as two independent canonical variables, with the two quantities $\hat{P}$ and $\check{P}$ the corresponding two canonical momenta. Hence in the quantal context these quantities satisfy the standard canonical commutation rules. It is then clear that the Hamiltonian $H(\hat{p}_0, \check{p}_0, \underline{\hat{p}}, \underline{\check{p}}; \hat{q}_0, \check{q}_0, \underline{\hat{q}}, \underline{\check{q}})$ has a discrete spectrum with eigenvalues $(\hat{k} - \check{k})\hbar\Omega = k\hbar\Omega$, with the two quantum numbers $\hat{k}$ and $\check{k}$ required to be *nonnegative integers*, hence the quantum number $k = \hat{k} - \check{k}$ required to be an *arbitrary integer*. This essentially completes the proof of Proposition 4. The explicit construction of the eigenfunctions Ψ — hence the corresponding accounting of the infinite degeneracy of the energy level $k\hbar\Omega$, see (11) — can be left as an exercise for the diligent reader, who shall take advantage of the property of the quantities $\hat{h}_n(\hat{p}_n, \check{p}_n; \hat{q}_n, \check{q}_n)$ and $\check{h}_{n'}(\hat{p}_{n'}, \check{p}_{n'}; \hat{q}_{n'}, \check{q}_{n'})$ to commute among themselves and with $\hat{Q}$, $\check{Q}$, $\hat{P}$, $\check{P}$ and H $(\hat{p}_0, \check{p}_0, \underline{\hat{p}}, \underline{\check{p}}; \hat{q}_0, \check{q}_0, \underline{\hat{q}}, \underline{\check{q}})$, as well as the fact that the N quantities $\hat{h}_n(\hat{p}, \check{p}; \hat{q}, \check{q})$ — being just the difference among two standard harmonic oscillator Hamiltonians — have a discrete spectrum, while the N quantities $\check{h}_n(\hat{p}, \check{p}; \hat{q}, \check{q})$ have a continuous spectrum. This latter fact is conveniently seen — and the corresponding eigenfunctions obtained — by performing the following simple (*linear*) canonical transformation,

$$p_\pm = \frac{\hat{p} \pm \check{p}}{\sqrt{2}}, \quad q_\pm = \frac{\hat{q} \pm \check{q}}{\sqrt{2}}, \tag{22a}$$

yielding

$$\check{h}_n(\hat{p}, \check{p}; \hat{q}, \check{q}) = \frac{1}{2}(p_-^2 - \omega_n^2 q_-^2) - \frac{1}{2}(p_+^2 - \omega_n^2 q_+^2). \tag{22b}$$

It is thus seen that the hamiltonian $\check{h}_n(\hat{p}, \check{p}; \hat{q}, \check{q})$ is again the difference among two simple Hamiltonians, each however representing now an antiharmonic oscillator (note the minus signs in front of ω_n^2), hence characterized by a continuous spectrum. But let us reemphasize that these feature, while relevant to identify the degeneracy of the spectrum of the Hamiltonian $H(\hat{p}_0, \check{p}_0, \underline{\hat{p}}, \underline{\check{p}}; \hat{q}_0, \check{q}_0, \underline{\hat{q}}, \underline{\check{q}})$ and correspondingly its eigenfunctions (which can be identified with a certain freedom, due to this degeneracy), do not affect the conclusion of Proposition 4 concerning its spectrum.

Finally let us note that the approach, via appropriate canonical transformations, employed here to prove Proposition 4, could obviously have been as well used to prove the three previous propositions.

4. Outlook

The explicit solvability of the dynamical system characterized by the Hamiltonian (8) in both the *classical* and the *first quantized* contexts raises the challenge of its possible solvability in a *second quantized* context.

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