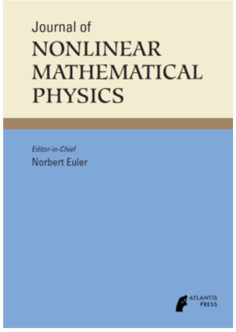

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SOLVABLE SYSTEMS OF ISOCHRONOUS, MULTI-PERIODIC OR ASYMPTOTICALLY ISOCHRONOUS NONLINEAR OSCILLATORS

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A simple technique is identified to manufacture *solvable* nonlinear dynamical systems, and in particular three classes whose *generic* solutions are, respectively, *isochronous*, *multi-periodic*, or *asymptotically isochronous*.

Keywords: Nonlinear oscillators; isochronous; multi-periodic; asymptotically isochronous; matrix ODEs.

1. Introduction

Purpose and scope of this paper is to describe a simple technique to manufacture three classes of *solvable* dynamical systems, characterized by the fact that the solution of the initial-value problem with *generic* initial data is, for the first class, *isochronous*, for the second *multi-periodic*, and for the third *asymptotically isochronous*. These three classes appear as special cases of a class of *solvable* dynamical systems.

Terminology. In this paper a dynamical system is identified as *solvable* if the solution of its initial-value problem can be achieved by solving systems of *linear* algebraic equations: so that the solution can in fact be provided in explicit form, in term of determinants. It is called (i) *isochronous*, (ii) *multi-periodic*, respectively (iii) *asymptotically isochronous*, if the solution of its initial-value problem (with *generic* initial data, as explained below) is: (i) *completely periodic* (periodic in *all* degrees of freedom) with a *fixed* period independent of the initial data, (ii) expressed in terms of a finite set of functions of time each of which is periodic with a fixed period independent of the initial data (iff *all* these periods are *congruent* the system is *isochronous*), (iii) neither *isochronous* nor *multi-periodic* but becomes *isochronous* in the remote future, up to corrections vanishing *exponentially* as the time diverges to *positive* infinity.

Our main motivation to exhibit these three classes of dynamical systems is in order to offer to practitioners — in physics, chemistry, population dynamics, whatever — a potentially useful tool to model observed phenomena or to perform appropriately designed experiments.

Isochronous systems have been investigated since the beginnings of the modern theory of dynamics by Newton, Huygens and by too many others to allow proper accounting here; recently there has been a revival of interest in such systems, for a review see [1]. The results reported below are — to the best of our knowledge — new, although an analogous — albeit less general — finding is provided by the *solvable* matrix equation displayed as Eq. (5.4.3-24) in [2], which also underpins the, quite different, class of *solvable* dynamical systems introduced and discussed in [3].

2. A Trivially Solvable Linear Matrix ODE

In this section we review, mainly to establish our notation, well-known results concerning the *linear*, *constant-coefficient* — hence *solvable* — matrix ODE of order $N+1$

$$V^{(N+1)} + \sum_{n=0}^{N} a_n V^{(n)} = 0. \tag{1a}$$

Notation. Here and hereafter N is an arbitrary positive integer (except when we restrict it to some specific, small, value: see below), the dependent variable $V \equiv V(t)$ is a matrix of arbitrary order (the order might be just unity, in which case V is just a scalar), t is the independent variable ("time"), and

$$V^{(m)}(t) \equiv \left(\frac{d}{dt}\right)^m V(t). \tag{1b}$$

The time variable t is of course *real*, and we generally assume it goes onwards from the *initial* time $t = 0$ at which the *initial* data $V^{(m)}(0)$, with $m = 0, \ldots, N$, are assigned; the matrix $V(t)$ shall be hereafter assumed to be *complex*, except in special cases as discussed below. We also assume the $N+1$ scalar coefficients a_n to be *complex* numbers — although we eventually focus below on systems of *real* (first-order, nonlinear) evolution ODEs, which are presumably more relevant for applications.

The general solution of (1a) reads of course

$$V(t) = \sum_{n=1}^{N+1} V^{[n]} \exp(ik_n \omega t), \tag{2a}$$

where the $N+1$ numbers k_n are the $N+1$ roots of the algebraic equation, of order $N+1$ in k,

$$k^{N+1} + \sum_{n=0}^{N} [a_n (i\omega)^{n-N-1} k^n] = 0, \tag{2b}$$

so that

$$k^{N+1} + \sum_{n=0}^{N} [a_n (i\omega)^{n-N-1} k^n] = \prod_{n=1}^{N+1} (k - k_n); \tag{2c}$$

and the $N + 1$ constant matrices $V^{[n]}$ are identified, in the context of the initial-value problem, as the solutions of the system of N *linear* algebraic (matrix) equations

$$\sum_{n=1}^{N+1} [V^{[n]} (ik_n\omega)^m] = V^{(m)}(0), \quad m = 0, 1, \ldots, N. \tag{2d}$$

Note that, here and hereafter, we restrict attention to the (generic) case when the $N + 1$ numbers k_n are *all different* among themselves.

Here and hereafter ω is a *real nonvanishing* scaling constant having the dimensions of inverse time, which is hereafter assumed to be adjusted so as to guarantee that one of the numbers k_n be unity, say $k_1 = 1$. Note that this entails, see (2b), for a *real* ω, the following restriction — hereafter assumed to hold — on the N coefficients a_n:

$$(i\omega)^{N+1} + \sum_{n=0}^{N} [a_n (i\omega)^n] = 0. \tag{3}$$

The Eqs. (2b) or (2c) institute a one-to-one correspondence between the ordered set of the $N + 1$ coefficients a_n and the unordered set of the $N + 1$ (different!) roots k_n (with $k_1 = 1$ via an appropriate assignment of ω); hereafter whenever using these quantities it will be understood that such a correspondence holds. And let us emphasize that, while in order to compute the $N + 1$ roots k_n from the $N + 1$ coefficients a_n the algebraic equation of order $N + 1$ (2b) must be solved, well known *explicit* formulas — implied by (2c) — are instead available to compute the set of coefficients a_n corresponding to an *assigned* set of numbers k_n. This will be our main point of view in the following: for instance

$$a_0 = (-i\omega)^{N+1} \prod_{n=2}^{N+1} k_n, \quad a_N = -i\omega \sum_{n=1}^{N+1} k_n = -i\omega \left[1 + \sum_{n=2}^{N+1} k_n\right]. \tag{4}$$

Finally let us note that the *general solution* (2a) of the ODE (1a) is clearly *isochronous* with period T,

$$T = \frac{2\pi}{|\omega|}, \tag{5a}$$

$$V(t + T) = V(t), \tag{5b}$$

if the $N + 1$ (different!) numbers k_n are all *real integers* (positive, negative or zero), it is *isochronous* with a larger period than T if all the numbers k_n are *rational*, it is *multi-periodic* if the $N+1$ (different!) numbers k_n are all *real* and at least one of them is *irrational* (recall that $k_1 = 1$), and it is *asymptotically isochronous* if none of the N quantities $k_j\omega$ (with $j = 2, \ldots, N + 1$) has a *negative* imaginary part, at least one of them has a *positive* imaginary part, and the *real* numbers k_j are *integers* (or *rationals*): say,

$$\text{Im}[\omega k_j] > 0 \quad \text{for } 2 \le j \le L + 1; \quad k_\ell = \text{real integer or rational}, \quad L + 2 \le \ell \le N + 1. \tag{6}$$

3. A Class of Solvable Nonlinear Dynamical Systems

Now introduce the set of matrices $U_m \equiv U_m(t)$ via the definition

$$U_m(t) = \left[\left(\frac{\partial}{\partial t} \right)^{m+1} V(t) \right] [V(t)]^{-1}, \quad m = 0, 1, 2 \ldots, \tag{7a}$$

so that, in particular,

$$U_0(t) = \left[\left(\frac{\partial}{\partial t} \right) V(t) \right] [V(t)]^{-1}. \tag{7b}$$

Then to the solution $V(t)$ (2a) of the linear matrix ODE (1a) there correspond the following expression of $U_m(t)$:

$$U_m(t) = \left\{ \sum_{n=1}^{N+1} [V^{[n]} (ik_n \omega)^{m+1} \exp(ik_n \omega t)] \right\} \left[\sum_{n=1}^{N+1} V^{[n]} \exp(ik_n \omega t) \right]^{-1}. \tag{8}$$

We now note that the definition (7) entails the relations

$$\dot{U}_m(t) = U_{m+1}(t) - U_m(t) U_0(t), \quad m = 0, 1, 2, \ldots. \tag{9}$$

Here and hereafter superimposed dots denote differentiations with respect to the time t.

It is moreover plain that the differential equation (1) entails, via (7), the relation

$$U_N(t) = -\sum_{n=1}^{N} a_n U_{n-1}(t) - a_0. \tag{10}$$

The insertion of this relation in the ODE (9) with $m = N - 1$ yields the ODE

$$\dot{U}_{N-1} = -a_0 - \sum_{m=0}^{N-1} a_{m+1} U_m - U_{N-1} U_0, \tag{11a}$$

and this ODE, together with the $N - 1$ ODEs (9) with $m = 0, \ldots, N - 2$,

$$\dot{U}_m = U_{m+1} - U_m U_0, \quad m = 0, \ldots, N - 2 \tag{11b}$$

(which are of course only present if $N \geq 2$), constitute a system of N first-order nonlinear ODEs for the N matrix variables $U_m \equiv U_m(t)$, $m = 0, \ldots, N - 1$. The initial data for this dynamical system are the N matrices $U_m(0)$ with $m = 0, \ldots, N - 1$; and clearly the solution of its initial-value problem is provided, via (7), by the solution (2a) of the initial-value problem for the ODE (1), with the constant matrices $V^{[n]}$ related via (2d) to the initial data

$$V^{(0)}(0) \equiv V(0) = \mathbf{1}; \quad V^{(n)}(0) = U_{n-1}(0), \quad n = 1, \ldots, N, \tag{12}$$

where $\mathbf{1}$ is of course the unit matrix having the same order as $V(t)$.

It is therefore plain (see (7)) that the N matrices $U_m(t)$, with $m = 0, \ldots, N - 1$, satisfying the system of N *nonlinear* matrix ODEs (11) inherit from the matrix $V(t)$, solution of the *linear* matrix ODE (1), of order $N + 1$, analogous (albeit not quite identical, see below) properties of *isochrony*, *multi-periodicity* or *asymptotic isochrony* — determined, as explained above, by the values of the $N+1$ numbers k_n, themselves related to the $N+1$ scalar coefficients a_n. With one proviso: while *all* solutions of the *linear* matrix ODE of order $N+1$ (1) are *nonsingular* — see (2a) — the solutions of the system of N *nonlinear* ODEs (11) might instead become *singular.* This obviously happens, see (7), whenever the determinant of the matrix $V(t)$ vanishes, entailing that this matrix is not invertible. As long as one works with *complex* matrices $V(t)$ whose determinants are *complex* numbers, this may happen, throughout the time evolution of $V(t)$, only for some *exceptional* set of initial data; it does *not* happen for *generic* initial data. Hence *in that context* it can be concluded that, for *generic* initial data, the solution of the initial-value problem for the system of N *nonlinear* first-order ODEs (11) inherits from the *linear* matrix ODE of order $N+1$ (1) analogous (albeit not quite identical, see below) properties — *isochrony*, *multi-periodicity* or *asymptotic isochrony* — as determined by the $N+1$ numbers k_n, or equivalently by the corresponding $N+1$ coefficients a_n.

Another case we will not dwell on — but we mention it here as also potentially of interest — is that in which one of the numbers $k_n\omega$, having a *negative* imaginary part, dominates the asymptotic behavior of $V(t)$ in the remote future: say, $\mathrm{Im}[k_{N+1}\omega] < \mathrm{Im}[k_n\omega]$, $n = 1,\ldots,N$ (implying $\mathrm{Im}[k_{N+1}\omega] < 0$ since $\mathrm{Im}[k_1\omega] = 0$). It is then clear from (2a) and (7) that the *generic* solution of the system of nonlinear matrix ODEs (11) (i.e., any solution such that in the corrresponding $V(t)$ that dominant component is present) has the following asymptotic property:

$$\lim_{t\to\infty}[U_m(t)] = (ik_{N+1}\omega)^{m+1}, \quad m = 0,\ldots,N-1. \tag{13}$$

These asymptotic values correspond of course to the equilibrium configuration of the dynamical system (11).

In the following sections we investigate the *solvable* systems of type (11) featuring the three kinds of time evolution mentioned above — *isochrony*, *multi-periodicity*, *asymptotic isochrony* — but focusing on systems characterized by evolution equations which are *real.* Such *real* dynamical systems are provided directly by the system of (first-order, nonlinear) evolution equations (11) when all the coefficients a_m are *real*, but in such a case one cannot be certain that a *real* matrix $V(t)$, solution of the corresponding linear matrix ODE (1a) with *generic* initial data, is invertible for all time, hence that the corresponding *generic* solution of the dynamical system (11) does not run into singularities; although there are cases — as shown below — in which no singularities arise for an open subset of initial data in phase space.

Hence our main device to manufacture *real* dynamical systems whose *generic* solutions are *nonsingular* for all (*real*) time, is to start from the system of *complex* ODEs (11) — where we now generally assume also the coefficients a_n to be *complex*,

$$a_n = \alpha_n + i\beta_n, \quad n = 0,\ldots,N, \tag{14}$$

— and to consider as dependent variables the real and imaginary parts of the matrices $U_m(t)$,

$$U_m(t) = X_m(t) + iY_m(t), \quad m = 0,\ldots,N-1. \tag{15}$$

We thereby obtain from (11) the following system of $2N$ first-order *real* matrix ODEs for the $2N$ dependent variables $X_m(t)$ and $Y_m(t)$:

$$\dot{X}_{N-1} = -\alpha_0 - \sum_{m=0}^{N-1} [\alpha_{m+1} X_m - \beta_{m+1} Y_m] - X_{N-1} X_0 + Y_{N-1} Y_0, \tag{16a}$$

$$\dot{Y}_{N-1} = -\beta_0 - \sum_{m=0}^{N-1} [\alpha_{m+1} Y_m + \beta_{m+1} X_m] - X_{N-1} Y_0 - Y_{N-1} X_0, \tag{16b}$$

$$\dot{X}_m = X_{m+1} - X_m X_0 + Y_m Y_0, \quad m = 0, \ldots, N-2, \tag{16c}$$

$$\dot{Y}_m = Y_{m+1} - X_m Y_0 - Y_m X_0, \quad m = 0, \ldots, N-2. \tag{16d}$$

This system is the main protagonist of the following developments: and let us re-emphasize that hereafter we assume the numbers α_n and β_n to be determined — in terms of the corresponding numbers k_n, assigned on a case-by-case manner, see below — via (2c) (see in particular (4)) with (14).

4. Isochronous Nonlinear Dynamical Systems

In this section we focus on systems of nonlinear first-order *real* ODEs of type (16) — hence *solvable* — such that the solution of their initial-value problem with *generic* initial data is *isochronous*.

To arrive at this class one starts by assigning arbitrarily the *real* parameter ω as well as N different *real integers* k_n, $n = 2, \ldots, N+1$ (with $k_1 = 1$), and then — by solving, for the $N+1$ unknowns a_m, $m = 0, 1, \ldots, N$, the system of $N+1$ linear algebraic equations that obtains by inserting k_n, $n = 1, \ldots, N+1$ in place of k in (2b) — one computes the corresponding $N+1$ coefficients a_m, which will then generally turn out (except in two special cases, as tersely discussed at the end of this section) to be *complex*, see (14). The (*solvable*, *isochronous*) dynamical system is then provided by the $2N$ first-order nonlinear matrix ODEs (16).

This system is of course also *isochronous* if the $N+1$ different *real* numbers k_n, $n = 1, \ldots, N+1$ (with $k_1 = 1$) are *rational* numbers rather than *integers*: in the following we omit for simplicity to elaborate explicitly on this possibility.

In particular for $N = 1$ the system (16) yields (via (4)) the following *solvable* system of 2 first-order *real matrix* ODEs:

$$\dot{X} = k\omega^2 - (1+k)\omega Y - X^2 + Y^2, \quad \dot{Y} = (1+k)\omega X - XY - YX, \tag{17}$$

where we of course set $k_2 \equiv k$. It is easily seen (from (8) with (2d) and (12)) that the solution of the initial-value problem for this system reads, in the *scalar* case, as follows:

$$X(t) = -\frac{(k-1)\omega A \sin(\tau + \varphi)}{D(t)}, \tag{18a}$$

$$Y(t) = \omega \left[\frac{1 + (k+1) A \cos(\tau + \varphi) + k A^2}{D(t)} \right], \tag{18b}$$

$$D(t) = 1 + 2A\cos(\tau + \varphi) + A^2, \tag{18c}$$

$$A = \sqrt{\frac{[X(0)]^2 + [Y(0) - \omega]^2}{[X(0)]^2 + [Y(0) - k\omega]^2}}, \tag{18d}$$

$$\varphi = \arctan \left\{ \frac{(k-1)\omega X(0)}{[X(0)]^2 + [Y(0) - \omega][Y(0) - k\omega]} \right\}, \tag{18e}$$

$$\tau = (k-1)\omega t, \tag{18f}$$

confirming its *isochrony* with period $T/|k-1|$, see (8) and (5a), for arbitrary real ω and provided k is an arbitrary *integer* different from unity, $k = 0, -1, \pm 2, \pm 3, \ldots$. (In fact, if k is an arbitrary *real* number rather than an *integer*, clearly this same conclusion holds.) Note that this solution is *nonsingular* for all time, except for the special (non generic) initial data such that

$$\cos\varphi = -(k-1)/(2A), \quad \text{i.e. } Y(0) = \frac{k+1}{2}\omega. \tag{19}$$

It is not difficult to obtain the solution in the, more general, *matrix* case in terms of the polar decomposition of the matrix $[k\omega - Y(0) + iX(0)]^{-1}[\omega - Y(0) + iX(0)]$ — but notationally it is a bit more cumbersome, so we do not display it.

Let us on the other hand exhibit the form taken by this (*solvable, isochronous*) system when the matrices X and Y have order 2 and are symmetrical:

$$X = \begin{pmatrix} x_1 & \xi \\ \xi & x_2 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & \eta \\ \eta & y_2 \end{pmatrix}. \tag{20}$$

Then one gets the following system of 6 first-order nonlinear (scalar) ODEs:

$$\begin{aligned}
\dot{x}_1 &= k\omega^2 - (1+k)\omega y_1 - x_1^2 + y_1^2 - \xi^2 + \eta^2, \\
\dot{x}_2 &= k\omega^2 - (1+k)\omega y_2 - x_2^2 + y_2^2 - \xi^2 + \eta^2, \\
\dot{y}_1 &= (1+k)\omega x_1 - 2(x_1 y_1 + \xi\eta), \\
\dot{y}_2 &= (1+k)\omega x_2 - 2(x_2 y_2 + \xi\eta), \\
\dot{\xi} &= -(1+k)\omega\eta - (x_1 + x_2)\xi + (y_1 + y_2)\eta, \\
\dot{\eta} &= (1+k)\omega\xi - (x_1 + x_2)\eta - (y_1 + y_2)\xi.
\end{aligned} \tag{21}$$

We leave to the diligent reader the easy task to display the system of 8 first-order nonlinear scalar ODEs that obtains from (17) when X and Y are generic (i.e., not symmetrical) matrices of order 2; while we note that the system we just displayed can be reduced to a system of 4 (rather than 6) first-order nonlinear ODEs via the (obviously compatible) reduction

$$x_1(t) = x_2(t) = x(t), \quad y_1(t) = y_2(t) = y(t). \tag{22}$$

Likewise, for $N = 2$, we can assert that the following *solvable* system of 4 first-order *real* matrix ODEs is *isochronous* with period T (more precisely, the period is the largest common multiple among $T/|k_2 - 1|$ and $T/|k_3 - 1|$: see (8)):

$$\dot{X}_0 = X_1 - X_0^2 + Y_0^2, \quad \dot{Y}_0 = Y_1 - X_0 Y_0 - Y_0 X_0, \tag{23a}$$

$$\dot{X}_1 = (k_2 + k_3 + k_2 k_3)\omega^2 X_0 - (1 + k_2 + k_3)\omega Y_1 - X_1 X_0 + Y_1 Y_0, \tag{23b}$$

$$\dot{Y}_1 = -k_2 k_3 \omega^3 + (k_2 + k_3 + k_2 k_3)\omega^2 Y_0 + (1 + k_2 + k_3)\omega X_1 - X_1 Y_0 - Y_1 X_0, \tag{23c}$$

provided ω is real and k_2, k_3 are two arbitrary different *integers* both different from unity, $k_2 \neq k_3$, $k_2 \neq 1$, $k_3 \neq 1$.

Many more *isochronous* systems can obviously be manufactured in this manner by making other choices of the number N and/or of the order of the two matrices X and Y.

Finally let us note that a reduction of this system, (23), obtains if $k_2 = -1$ and $k_3 = 0$ (or, equivalently, viceversa), since one can then set $Y_0 = Y_1 = 0$, thereby getting the following system of 2 matrix ODEs:

$$\dot{X}_0 = X_1 - X_0^2, \quad \dot{X}_1 = -\omega^2 X_0 - X_1 X_0. \tag{24}$$

This system, however, is only isochronous in a sector of its phase space, indeed its general solution reads (see (8)):

$$X_0(t) = \left[X_0(0)\cos(\omega t) + X_1(0)\frac{\sin(\omega t)}{\omega}\right] \cdot \left[\mathbf{1} + X_0(0)\frac{\sin(\omega t)}{\omega} + X_1(0)\frac{1-\cos(\omega t)}{\omega^2}\right]^{-1}, \tag{25a}$$

$$X_1(t) = \left[-\omega X_0(0)\sin(\omega t) + X_1(0)\cos(\omega t)\right] \cdot \left[\mathbf{1} + X_0(0)\frac{\sin(\omega t)}{\omega} + X_1(0)\frac{1-\cos(\omega t)}{\omega^2}\right]^{-1}. \tag{25b}$$

And this is nonsingular for all time only if the matrices $X_0(0)$ and $X_1(0)$ have sufficiently small norm.

Note that this reduction corresponds to the case in which the *real* dynamical system under consideration is (11) rather than (16), with the coefficients a_n being *all real* and yielding roots k_n of the Eq. (2b) which are also *all real integers* (all different among themselves), which are then necessarily characterized, for *even* N, by the property $k_{2j-1} = -k_{2j} \neq 0$ for $j = 1, \ldots, N/2$, with $k_{N+1} = 0$: the case we just treated corresponds to $N = 2$, but it is easily seen that the situation is in fact analogous for larger *even* values of N. For *odd* N and *real* coefficients a_n all the (different, integer) roots k_n are characterized by the rule $k_{2j-1} = -k_{2j} \neq 0$ for $j = 1, \ldots (N+1)/2$, and it is then easily seen that in this case all solutions of the dynamical system (11) eventually run into singularities.

5. Multi-Periodic Nonlinear Dynamical Systems

In this terse section we focus on systems of nonlinear first-order *real* matrix ODEs of type (16) — hence *solvable* — such that the solution of their initial-value problem with *generic* initial data is *multi-periodic*. These systems are given by (16) with the simple requirement that $N \geq 2$, the N numbers k_n, $n = 2, \ldots, N+1$ are all *real*, different among each other and different from $k_1 = 1$, and that the numbers $k_n - 1$ are not congruent. For instance the system (23) is such a system, provided k_2 and k_3 satisfy these requirements. The more diligent reader — especially if capable to employ computer algebra — will enjoy producing in explicit form the solution of this system. We only note here that the *generic* solution thereby obtained shall never become singular, although it features a denominator which, in the remote future, shall at times come closer and closer to zero: a behavior qualitatively analogous to that of the *real* function $1/\{[A_1 \sin(\omega_1 t)]^2 + [A_2 \sin(\omega_2 t)]^2\}$ with the 4 *real* constants A_1, A_2, ω_1 and ω_2 arbitrary except for the requirement that ω_1 and ω_2 not be congruent, namely that the ratio ω_1/ω_2 be *irrational*.

6. Asymptotically Isochronous Nonlinear Dynamical Systems

In this terse section we focus on systems of nonlinear first-order *real* matrix ODEs of type (16) — hence *solvable* — such that the solution of their initial-value problem with *generic* initial data is *asymptotically isochronous.* These systems are given by (16) with the requirements on the real parameter ω and the $N+1$ numbers k_n (with $k_1 = 1$) specified at the end of Sec. 2 (but see below). Let us exhibit 2 cases, which constitute rather obvious extensions of those treated above, so that the following treatment can be quite terse.

The *first* case is identified by the assignment

$$N = 1, \quad k_1 = 1, \quad k_2 = q + i\frac{p}{\omega}, \tag{26}$$

where q and ω are two arbitrary *real* numbers (of course $\omega \neq 0$) and p is an arbitrary *positive* constant (with the dimension of inverse time), $p > 0$. The corresponding system of two first-order *real* ODEs reads

$$\dot{X} = q\omega^2 - pX - (1+q)\omega Y - X^2 + Y^2, \quad \dot{Y} = p\omega - pY + (1+q)\omega X - XY - YX, \tag{27a}$$

and reduces of course to the *isochronous* system (17) for $p = 0$ and $q = k$. But of course the solutions of this system, (27a), are *not* asymptotically isochronous: they converge asymptotically to the equilibrium configuration

$$X(\infty) = 0, \quad Y(\infty) = \omega, \tag{27b}$$

as implied by (15) and (7) with (2a) and (26).

The *second* case is identified by the assignment

$$N = 2, \quad k_1 = 1, \quad k_2 = q_1 + i\frac{p}{\omega}, \quad k_3 = q_2, \tag{28}$$

where q_1, q_2 and ω are 3 arbitrary *real* numbers (of course $\omega \neq 0$ and $q_2 \neq 1$) and p is an arbitrary *positive* number, $p > 0$. The corresponding system of 4 first-order *real* matrix ODEs reads

$$\dot{X}_0 = X_1 - X_0^2 + Y_0^2, \quad \dot{Y}_0 = Y_1 - X_0Y_0 - Y_0X_0, \tag{29a}$$

$$\begin{aligned}\dot{X}_1 &= q_2 p\omega^2 + (q_1 + q_2 + q_1q_2)\omega^2 X_0 - (1+q_2)p\omega Y_0 \\ &\quad - pX_1 - (1 + q_1 + q_2)\omega Y_1 - X_1X_0 + Y_1Y_0,\end{aligned} \tag{29b}$$

$$\begin{aligned}\dot{Y}_1 &= -q_1q_2\omega^3 + (q_1 + q_2 + q_1q_2)\omega^2 Y_0 + (1+q_2)p\omega X_0 \\ &\quad - pY_1 + (1 + q_1 + q_2)\omega X_1 - X_1Y_0 - Y_1X_0,\end{aligned} \tag{29c}$$

and reduces of course to the *isochronous* system (23) for $p = 0$ and $q_1 = k_2, q_2 = k_3$. This model is clearly *asymptotically isochronous*. Note however that, if k_3 — rather than being *real* — were also to feature, as k_2, a positive imaginary part, then the corresponding system, rather than being *asymptotically isochronous*, would converge asymptotically to *equilibrium*, just as the system (27a), see above.

The task to obtain the general solution of these systems is again left to the diligent reader.

7. Outlook

Throughout this paper we have assumed that the dependent variables of the dynamical systems under consideration are *matrices* of *arbitrary* rank. By appropriate parametrizations of these matrices — see for instance [2] (in particular Subsecs. 5.5 and 5.6.5 of this book), [4] and [5] — it is then possible to introduce interesting reformulations of these systems, also in the guise of nonlinear systems of *covariant* vector ODEs. We leave for the moment these developments to researchers interested in their applications. But let us end this paper by emphasizing that the results presented above have an interesting potential for application even when attention is restricted to matrices of *unit* order, namely to scalar dependent variables — in which cases some of the formulas reported above simplify marginally (since matrices of unit order of course commute).

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