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REDUCTIONS FOR SOME ORDINARY DIFFERENTIAL EQUATIONS THROUGH NONLOCAL SYMMETRIES

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In [19] we derive nonlocal symmetries for ordinary differential equations by embedding the given equation in an auxiliary system. Since the nonlocal symmetries of the ODE's are local symmetries of the associated auxiliary system this result provides an algorithmic method to derive this kind of nonlocal symmetries. In this work we show some classes of ordinary differential equations which do not admit any Lie symmetry unless some conditions are satisfied but for which we have derived nonlocal symmetries. These nonlocal symmetries allow us to reduce the order for these equations even if these equations do not admit point symmetries.

Keywords: Conditional symmetry; nonlocal symmetry; ordinary differential equation.

Mathematics Subject Classification 2000: 22E46, 53C35, 57S20

1. Introduction

In recent years, increasing attention has been devoted to the study and reduction of ordinary differential equations (ODE's) by using symmetries [1–4, 15, 29–31]. It is clear that the presence of symmetries is not always sufficient to ensure the integrability by quadratures of a differential equation. The converse, that every differential equation which is integrable by quadratures has a nontrivial symmetry group was proved false in [21] where an example of second order ODE integrable by quadratures but with a trivial symmetry group was presented.

An obvious limitation of group-theoretical methods based on local symmetries is that many ODE's do not have local symmetries. It turns out that ODE's can admit nonlocal symmetries whose infinitesimal generators depend on the integrals of the dependent variable in some specific manner.

Nonlocal symmetries were introduced by I.S. Krasil'shchik, A.M. Vinogradov in [25–28]. Potential symmetries constitute a particular case of general nonlocal symmetries and for

ODE's were introduced by G. Bluman in [10]. Potential symmetries for some ODE's have been derived in [16] and [17]. By using these symmetries we find that the order of the ODE can be reduced even if this equation does not admit point symmetries. Moreover, in the case for which the ODE admits a group of point symmetries, we find that the potential symmetries allow us to perform further reductions than its point symmetries.

A constructive method to derive nonlocal symmetries of differential equations based on the Lie-Bäcklund theory of groups was developed in [8]. In [32] the exponential vectors fields were introduced and it was proved that an ODE which is invariant under an exponential vector field can be reduced in order by one. In [29] Muriel and Romero introduced the λ -symmetries in order to reduce the order of ODE's. In [15, 30] the relationship between these symmetries and nonlocal symmetries has been established. Nonlocal symmetries have been considered in [2–4, 7, 11, 20] to integrate or to reduce the order of ODE's. In [14] Sundman symmetries were introduced. These are also a form of nonlocal symmetries for ODE's and can be used to reduce the order of the ODE's. In [13] the authors classified 2nd-order and 3rd-order ODE's with respect to this nonlocal symmetry. Usually these nonlocal symmetries are found as hidden symmetries of an ODE found by the transformation $u = f(x, w_x)$ which increases the order of the equation. The main difficulty is that there is not a method to construct these higher order equations. Moreover the computing programs such as `symmgrp.max` are not able to compute the Lie symmetries of the ODE in which we have made the transformation $u = f(x, w_x)$, so we need first to guess which is the function f and then to compute the symmetries.

In a recent paper [19] we propose for a given ODE to find useful nonlocal symmetries by embedding it in an auxiliary “covering” system with auxiliary dependent variables. A Lie symmetry of the auxiliary system, acting on the space consisting of the independent and dependent variables of the given ODE as well as the auxiliary variables, yields a nonlocal symmetry of the given ODE if it does not project on to a point symmetry acting in its space of the independent and dependent variables. We introduce an auxiliary nonlocal variable v and the auxiliary system

$$\Delta(x, u, u^{(1)}, \dots, u^{(n)}) = 0, \quad (1.1)$$

$$v_x = f(x, u). \quad (1.2)$$

Any Lie group of point transformations

$$\mathbf{v} = \xi(x, u, v) \frac{\partial}{\partial x} + \phi(x, u, v) \frac{\partial}{\partial u} + \psi(x, u, v) \frac{\partial}{\partial v}, \quad (1.3)$$

admitted by (1.1)–(1.2) yields a nonlocal symmetry of the given ODE (1.1) if any of the generators ξ or ϕ depend explicitly on the new variable v , i.e. if the following condition is satisfied

$$\xi_v^2 + \phi_v^2 \neq 0. \quad (1.4)$$

Since the nonlocal symmetries of (1.1) are local symmetries of (1.1)–(1.2) this result provides an algorithmic method to derive this kind of nonlocal symmetries. This enables one to use the computer to perform the usually lengthy calculations. In [19] we have considered some examples concerning canonical equations of type II [24].

In this paper we focus our attention on canonical equations of type III [24]. We show some new classes of ODE's that do not admit any Lie symmetry but for which we have derived nonlocal symmetries. These nonlocal symmetries allow us to reduce the order for these equations. We also show a Painlevé-type equation for which any Lie group of point transformations

$$\mathbf{v} = \xi(x, u, v, w) \frac{\partial}{\partial x} + \phi(x, u, v, w) \frac{\partial}{\partial u} + \psi(x, u, v, w) \frac{\partial}{\partial v} + \eta(x, u, v, w) \frac{\partial}{\partial w}, \quad (1.5)$$

admitted by

$$\begin{aligned} \Delta(x, u, u^{(1)}, \dots, u^{(n)}) &= 0, \\ u_x &= v, \\ w_x &= f(x, u). \end{aligned} \quad (1.6)$$

yields a nonlocal symmetry of the given ODE.

2. Nonlocal Symmetries

Equation 1. The second order differential equation

$$u_{xx} - \frac{(u_x)^2}{u} - \frac{(x^2 + x)}{u} u_x + 2x + 1 = 0 \quad (2.1)$$

has been recently proposed by Nail Ibragimov in [23] as an example of equation which has no point symmetries, and hence cannot be integrated by Lie's method. He applied the method of integrating factors for higher-order equations developed in [23] and he has expressed the solution in terms of elementary functions. We observe that although Eq. (2.1) has no point symmetries, by introducing a nonlocal variable v the associated system admits Lie symmetries which are nonlocal symmetries of (2.1).

Theorem 2.1. *System*

$$\begin{aligned} u_{xx} - \frac{(u_x)^2}{u} - \frac{(x^2 + x)}{u} u_x + 2x + 1 &= 0, \\ v_x &= f(x, u), \end{aligned} \quad (2.2)$$

has Lie symmetries with infinitesimal generator

$$\mathbf{v} = e^v u \frac{\partial}{\partial u} + (k_1 - e^v) \frac{\partial}{\partial v}, \quad (2.3)$$

for

$$f(x, u) = \frac{(x^2 + x)}{u}. \quad (2.4)$$

Proof. If a vector field $\mathbf{v} = \xi(x, u, v) \frac{\partial}{\partial x} + \phi(x, u, v) \frac{\partial}{\partial u} + \psi(x, u, v) \frac{\partial}{\partial v}$ is a Lie symmetry of system (2.2) its infinitesimals ξ , ϕ and ψ must satisfy the following determining

equations

$$\begin{aligned}
 \xi_{uu} u + \xi_u &= 0, \\
 \psi_u - \xi_u f &= 0, \\
 \psi_x + f \psi_v - f_u \phi - \xi f_x - \xi_v f^2 - \xi_x f &= 0, \\
 -2 \xi_u u x^2 - 2 \xi_u u x + \phi_{uu} u^2 - \xi_v f_u u^2 - 2 \xi_{uv} f u^2 - 2 \xi_{ux} u^2 - \phi_u u + \phi &= 0, \\
 -\phi_x x^2 - f \phi_v x^2 - 2 \phi_u u x + 4 \xi_v f u x + 4 \xi_x u x - \phi_x x - f \phi_v x + \phi_{xx} u \\
 + f^2 \phi_{vv} u + 2 f \phi_{vx} u + f_x \phi_v u - \phi_u u + 2 \xi_v f u + 2 \frac{d\eta_1}{dx} u + 2 \eta_1 u &= 0, \\
 -\xi_v f u x^2 - \xi_x u x^2 + \phi x^2 + 6 \xi_u u^2 x - \xi_v f u x - \xi_x u x - 2 \xi u x + \phi x \\
 + f_u \phi_v u^2 + 2 \phi_{ux} u^2 + 2 f \phi_{uv} u^2 - \xi_v f_x u^2 - \xi_{vv} f^2 u^2 \\
 - 2 \xi_{vx} f u^2 - \xi_{xx} u^2 + 3 \xi_u u^2 - 2 \phi_x u - 2 f \\
 \phi_v u - \xi u &= 0.
 \end{aligned}$$

By solving this system we obtain for $f(x, u) = \frac{(x^2+x)}{u}$ the infinitesimals $\xi = 0$, $\phi = e^v u$ and $\psi = k_1 - e^v$. □

By using the usual prolongation formula we obtain

$$\mathbf{v}^{(1)} = e^v u \frac{\partial}{\partial u} + (k_1 - e^v) \frac{\partial}{\partial v} + e^v (u v_x + u_x) \frac{\partial}{\partial u_x} - e^v v_x \frac{\partial}{\partial v_x}. \tag{2.5}$$

Two functionally independent invariants are

$$z = x, \quad \zeta = \frac{u_x}{u} + \frac{x^2 + x}{u}. \tag{2.6}$$

By derivating we obtain that the expression of (2.1) in terms of the invariants is the trivial first order equation

$$\zeta_z = 0$$

with general solution $\zeta = C$. In terms of the original variables we get

$$\frac{u_x + x^2 + x}{u} = C.$$

Equation 2. In the following, we propose the second order ODE of type II [24],

$$u_{xx} = a \frac{(u_x)^2}{u} - g(x, u), \tag{2.7}$$

where

$$g = -\frac{k^2 c^2 u^{4a-3}}{4(a-1)} - \frac{k c' u^{2a-1}}{2(a-1)} + e u, \tag{2.8}$$

$a = \text{constant}$, $a \neq 1$, $c = c(x)$ and $e = e(x)$ as an example of equation which does not have Lie point symmetries unless, $c = c(x)$ and $e = e(x)$ satisfy some conditions. Hence in general it cannot be integrated by Lie's method. Nevertheless we will prove that an associated system

admits Lie symmetries which are nonlocal symmetries of Eq. (2.7). These symmetries allow us to reduce the order of (2.7).

Theorem 2.2. *The second order differential equation (2.7) admits Lie symmetries with infinitesimal generators*

$$\xi = -\frac{2k_1(a-1)c}{c'}, \quad \phi = \left(k_1 - \frac{\xi'}{2(a-1)}\right) u \quad (2.9)$$

if $c = c(x)$, $c' \neq 0$, and $e = e(x)$ must be related by

$$2c(c')^3 e'(a-1) - 4c(c')^2 c'' e(a-1) + 4(c')^4 e(a-1) + c(c')^2 c'''' - 6cc'c''c''' + 2(c')^3 c''' + 6c(c'')^3 - 3(c')^2 (c'')^2 = 0.$$

This condition is not always satisfied for example if $c(x) = \exp(px)$ and $e = \exp(qx)$ with $pq \neq 0$ or if $c = x^p$, $e = x^q$ and $q \neq -2$.

Proof. If a vector field $\mathbf{v} = \xi \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial u}$ is a Lie symmetry of Eq. (2.7) its infinitesimals ξ , ϕ must satisfy the following determining equations

$$\begin{aligned} (a-1)(\xi_{uu}u + a\xi_u) &= 0, \\ (a-1)(\phi_{uu}u^2 - 2\xi_{ux}u^2 - a\phi_uu + a\phi) &= 0, \\ 3c^2\xi_uk^2u^{4a} - 12ae\xi_uu^4 + 12e\xi_uu^4 - 8a\phi_{ux}u^3 + 8\phi_{ux}u^3 + 4a\xi_{xx}u^3 \\ &\quad - 4\xi_{xx}u^3 + 8a^2\phi_xu^2 - 8a\phi_xu^2 = 0, \\ c^2k^2\phi_uu^{4a+1} - 2c^2\xi_xk^2u^{4a+1} - 2cc'\xi k^2u^{4a+1} - 4ac^2k^2\phi u^{4a} + 3c^2k^2\phi u^{4a} - 4ae\phi_uu^5 \\ &\quad + 4e\phi_uu^5 + 8ae\xi_xu^5 - 8e\xi_xu^5 + 4ae'\xi u^5 - 4e'\xi u^5 + 4a\phi_xu^4 \\ &\quad - 4\phi_{xx}u^4 + 4ae\phi u^4 - 4e\phi u^4 = 0. \end{aligned}$$

By solving this system we obtain (2.9). □

By introducing the variable v , we prove the following

Theorem 2.3. *System*

$$\begin{aligned} u_{xx} &= a\frac{(u_x)^2}{u} - g(x, u), \\ v_x &= f(x, u), \end{aligned} \quad (2.10)$$

where g is given by (2.8), has Lie symmetries with infinitesimal generator

$$\mathbf{v} = bu e^{kv} \frac{\partial}{\partial u} + \frac{2(a-1)b}{k} e^{kv} \frac{\partial}{\partial v} \quad (2.11)$$

when

$$f(x, u) = cu^{2(a-1)} - \frac{b'}{kb}.$$

Proof. Invariance of system (2.10) under a Lie group of point transformations with infinitesimal generator $\mathbf{v} = \xi(x, u, v) \frac{\partial}{\partial x} + \phi(x, u, v) \frac{\partial}{\partial u} + \psi(x, u, v) \frac{\partial}{\partial v}$ leads to a set of determining equations. Solving this system we obtain ξ , ϕ and ψ . □

By using the usual prolongation formula we obtain

$$\mathbf{v}^{(1)} = bu e^{kv} \frac{\partial}{\partial u} + \frac{2(a-1)b}{k} e^{kv} \frac{\partial}{\partial v} + b' e^{kv} u \frac{\partial}{\partial u_x} + \frac{2(a-1)b' e^{kv}}{k} \frac{\partial}{\partial v_x}. \tag{2.12}$$

Two functionally independent invariants are

$$z = x, \quad \zeta = \frac{u_x}{u} - \frac{cku^{2(a-1)}}{2(a-1)}. \tag{2.13}$$

By derivating we obtain that the expression of (2.7) in terms of the invariants is the first order equation

$$\zeta_z + (1-a)\zeta^2 + e = 0,$$

with $e = e(z)$, arbitrary function.

Equation 3. In the following, we propose the second order ODE of type II [24],

$$u_{xx} + \frac{u_x^2}{2u} - bu_x - g(x, u) = 0, \tag{2.14}$$

where

$$g(x, u) = \beta u + \frac{\alpha b k}{3u^2} - \frac{\alpha^2 k^2}{6u^5},$$

which has no Lie symmetries for $b \neq 0$, $\beta_x \neq 0$ and α arbitrary constant.

Theorem 2.4. *The second order differential Eq. (2.14) does not admit non-trivial Lie symmetries for $b \neq 0$, $\beta_x \neq 0$ and α arbitrary constant.*

Proof. If a vector field $\mathbf{v} = \xi \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial u}$ is a Lie symmetry of Eq. (2.14) its infinitesimals ξ , ϕ must satisfy the following determining equations

$$\begin{aligned} u \xi_{uu} - \frac{\xi_u}{2} &= 0, \\ -2u^2 \xi_{ux} + \phi_{uu} u^2 + \frac{\phi_u}{2} - \frac{\phi}{2} &= 0, \\ -2u^6 \xi_{xx} + 6\beta u^7 \xi_u + 2\alpha b k u^4 \xi_u - \alpha^2 k^2 u \xi_u + 4\phi_{ux} u^6 + 2\phi_x u^5 &= 0, \\ -12\beta u^6 \xi_x - 4\alpha b k u^3 \xi_x + 2\alpha^2 k^2 \xi_x - 6\beta_x u^6 \xi - 2\alpha b_x k u^3 \xi & \\ 6\beta \phi_u u^6 - 6\phi_{xx} u^5 + 2\alpha b k \phi_u u^3 - \alpha^2 k^2 \phi_u &= 0. \end{aligned}$$

From the two first conditions we get

$$\begin{aligned} \xi &= f_2 u^{3/2} - \frac{f_1}{3}, \\ \phi &= \left(2 \frac{df_2}{dx} + 2bf_2 \right) u^{5/2} + f_3 u + \frac{f_4}{\sqrt{u}}, \end{aligned}$$

where $f_i = f_i(x)$ $i = 1, \dots, 4$.

Substituting into the remaining conditions we get $f_2 = f_4 = 0$ $f_3 = -\frac{f_1}{9}$ and for $b \neq 0$ and $\beta_x \neq 0$ then $f_1 = 0$ and $f_3 = 0$. Consequently for $b \neq 0$, $\alpha = constant$ and $\beta_x \neq 0$ Eq. (2.14) does not admit any nontrivial Lie symmetry. \square

Theorem 2.5. *System*

$$\begin{aligned} u_{xx} + \frac{u_x^2}{2u} - bu_x - g(x, u) &= 0, \\ v_x &= f(x, u), \end{aligned} \tag{2.15}$$

with

$$g(x, u) = \beta u + \frac{\alpha bk}{3u^2} - \frac{\alpha^2 k^2}{6u^5},$$

has Lie symmetries with infinitesimal generator

$$\mathbf{v} = cu e^{kv} \frac{\partial}{\partial u} - \frac{3ce^{kv}}{k} \frac{\partial}{\partial v} \tag{2.16}$$

for

$$f(x, u) = \frac{\alpha}{u^3} - \frac{c'}{ck}. \tag{2.17}$$

Proof. If a vector field

$$\mathbf{v} = \xi \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial u} + \psi \frac{\partial}{\partial v}$$

is a Lie symmetry of system (2.15) its infinitesimals ξ , ϕ and ψ must satisfy the following determining equations

$$\begin{aligned} u \xi_{uu} - \frac{\xi_u}{2} &= 0, \\ \psi_u - f \xi_u &= 0, \\ -f \xi_x - f^2 \xi_v - f_x \xi + \psi_x + f \psi_v - f_u \phi &= 0, \\ -f_u u^2 \xi_v - 2u^2 \xi_{ux} - 2f u^2 \xi_{uv} \\ -2b u^2 \xi_u + \phi_{uu} u^2 + \phi_u \frac{u}{2} - \frac{\phi}{2} &= 0, \\ -2\beta u \xi_x - \frac{2\alpha bk \xi_x}{3u^2} + \frac{\alpha^2 k^2 \xi_x}{3u^5} \\ -2\beta f u \xi_v - \frac{2\alpha b f k \xi_v}{3u^2} + \frac{\alpha^2 f k^2 \xi_v}{3u^5} \\ + \beta \phi_u u + \frac{\alpha bk \phi_u}{3u^2} + \frac{2\alpha bk \phi}{3u^3} - \frac{\alpha^2 k^2 \phi_u}{6u^5} - \frac{5\alpha^2 k^2 \phi}{6u^6} \\ + \phi_{xx} - b \phi_x + f^2 \phi_{vv} + 2f \phi_{vx} + f_x \phi_v - b f \phi_v - \beta \phi &= 0, \\ -u \xi_{xx} - bu \xi_x - f^2 u \xi_{vv} \\ -2f u \xi_{vx} - f_x u \xi_v - b f u \xi_v \\ -3\beta u^2 \xi_u - \frac{\alpha bk \xi_u}{u} + \frac{\alpha^2 k^2 \xi_u}{2u^4} \\ + f_u \phi_v u + 2\phi_{ux} u + 2f \phi_{uv} u + \phi_x + f \phi_v &= 0. \end{aligned} \tag{2.18}$$

It can be checked that this system admits the solution $\xi = 0$, $\phi = cu e^{kv}$, $\psi = -\frac{3ce^{kv}}{k}$ and $f(x, u) = \frac{\alpha}{u^3} - \frac{c'}{ck}$. Consequently it has Lie symmetries with infinitesimal generator (2.16). \square

By using the usual prolongation formula we obtain

$$\mathbf{v}^{(1)} = ce^{kv}u \frac{\partial}{\partial u} - \frac{3c}{k}e^{kv} \frac{\partial}{\partial v} + e^{kv} \left(cu_x + \frac{k c \alpha}{u^2} \right) \frac{\partial}{\partial u_x} - 3e^{kv} \left(\frac{c_x}{k} + cv_x \right) \frac{\partial}{\partial v_x}. \tag{2.19}$$

Two functionally independent invariants are

$$z = x, \quad \zeta = \frac{u_x}{u} + \frac{k\alpha}{3u^3}. \tag{2.20}$$

By derivating we obtain that the expression of (2.14) in terms of the invariants is the first order equation

$$\zeta_z + \frac{3\zeta^2}{2} - b\zeta - \beta = 0,$$

with $b = b(z)$, $\beta = \beta(z)$ arbitrary functions.

Equation 4. In [31] the Painlevé-type second-order ordinary differential equation

$$u'' = \frac{u'^2}{u} + \left(u + \frac{x}{u}\right)u' - 1 \tag{2.21}$$

has been transformed, by using the equation for Jacobi last multiplier, into a system of three equations. One of these equations is a second order equation and the other two equations are first order equations. This system admits a three-dimensional solvable Lie point symmetry algebra which allow us to reduce the system to a single first-order ODE which can be integrated in terms of Airy functions. Equation (2.21) is a particular case of the Painlevé XIV Eq. [24]

$$u'' = \frac{u'^2}{u} + \left(a(x)u + \frac{b(x)}{u}\right)u' + a'(x)u^2 - b'(x), \tag{2.22}$$

which it is known to possess a first integral of the Riccati type, i.e.:

$$u' = a(x)u^2 + ku - b(x), \tag{2.23}$$

where k is an arbitrary constant. Equations (2.21) and (2.22) for arbitrary $a(x)$, $b(x)$ do not possess any Lie point symmetries.

In the following, we consider the second order ODE (2.22), we will prove that an associated system admits Lie symmetries which are nonlocal symmetries of Eq. (2.22). These symmetries allow us to reduce the order of (2.22) and the solutions of (2.22) are described by the generalized Riccati equation.

By introducing the variable v , we get the following

Theorem 2.6. *System*

$$\begin{aligned} u_{xx} &= \frac{u_x^2}{u} + \left(a(x)u + \frac{b(x)}{u} \right) u_x + a'(x)u^2 - b'(x), \\ u_x &= v, \\ w_x &= f(x, u), \end{aligned} \tag{2.24}$$

has Lie symmetries with infinitesimal generator

$$\mathbf{v} = ue^w \frac{\partial}{\partial u} + (v + a(x)u^2 + b(x))e^w \frac{\partial}{\partial v} + \beta(x, u, v)e^w \frac{\partial}{\partial w} \tag{2.25}$$

when

$$f(x, u) = a(x)u + \frac{b(x)}{u}$$

and β satisfies

$$uv\beta_u + (v^2 + au^2v + bv + a'u^3 - b'u)\beta_v + u\beta_x + (au^2 + b)\beta + b - au^2 = 0. \tag{2.26}$$

Proof. Invariance of system (2.24) under a Lie group of point transformations with infinitesimal generator $\mathbf{v} = \xi(x, u, v, w) \frac{\partial}{\partial x} + \phi(x, u, v, w) \frac{\partial}{\partial u} + \psi(x, u, v, w) \frac{\partial}{\partial v} + \eta(x, u, v, w) \frac{\partial}{\partial w}$ leads to a set of determining equations. Solving this system we obtain ξ, ϕ, ψ and η . \square

By using the prolongation formula two functionally independent invariants are

$$z = x, \quad \zeta = \frac{u_x}{u} - a(x)u + \frac{b(x)}{u}. \tag{2.27}$$

By derivating we obtain that the expression of (2.22) in terms of the invariants is the first order equation

$$\zeta_z = 0.$$

It follows that $\zeta = k$ and the solutions of (2.22) are described by the generalized Riccati equation

$$u_x = a(x)u^2 + ku - b(x). \tag{2.28}$$

3. Conclusions

In this paper we propose for a given ODE to find nonlocal symmetries by embedding it in an auxiliary “covering” system with auxiliary dependent variables. Since the nonlocal symmetries of the ODE’s are local symmetries of the associated auxiliary system this result provides an algorithmic method to derive this kind of nonlocal symmetries. This enables one to use the computer to perform the usually lengthy calculations.

We have shown some classes of ODE’s (2.1), (2.7), (2.14) and (2.22) which do not admit any Lie symmetry unless some conditions are satisfied but for which we have derived nonlocal symmetries. These nonlocal symmetries allow us to reduce the order for these equations.

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