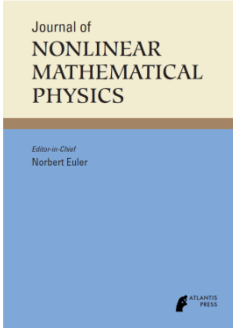

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1776-0852**
Journal Home Page: https://www.tandfonline.com/loi/tnmp20

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To cite this article: Norbert Euler, Marianna Euler (2011) The Converse Problem for the Multipotentialisation of Evolution Equations and Systems, Journal of Nonlinear Mathematical Physics 18:Supplement 1, 77–105, DOI: https://doi.org/10.1142/S1402925111001295

To link to this article: https://doi.org/10.1142/S1402925111001295

Published online: 04 January 2021

Article

Journal of Nonlinear Mathematical Physics, Vol. 18, Suppl. 1 (2011) 77–105

DOI: 10.1142/S1402925111001295

THE CONVERSE PROBLEM FOR THE MULTIPOTENTIALISATION OF EVOLUTION EQUATIONS AND SYSTEMS

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Received 29 September 2010
Accepted 5 January 2011

We propose a method to identify and classify evolution equations and systems that can be multi-potentialised in given target equations or target systems. We refer to this as the *converse problem*. Although we mainly study a method for $(1+1)$-dimensional equations/system, we do also propose an extension of the methodology to higher-dimensional evolution equations. An important point is that the proposed converse method allows one to identify certain types of auto-Bäcklund transformations for the equations/systems. In this respect we define the *triangular-auto-Bäcklund transformation* and derive its connections to the converse problem. Several explicit examples are given. In particular, we investigate a class of linearisable third-order evolution equations, a fifth-order symmetry-integrable evolution equation as well as linearisable systems.

Keywords: Nonlinear evolution equations; potentialisation; auto-Bäcklund transformations; linearisation; the converse problem.

Mathematics Subject Classification 2010: 35B06, 35K55, 58J55

Contents

1. Introduction

Potentialisations of evolution equations provides a natural way to study special types of nonlocal symmetries for partial differential equations and systems, known as potential symmetries [1]. In some cases it is possible to apply the potentialisation process again on the derived potential equations themselves, which is known as the mulipotentialisation process. This procedure of multipotentialisation was applied in [5, 6] to investigate higher-degree potential symmetries, nonlocal transformations, nonlocal conservation laws, as well as iterating-solution formulae; all of which were derived as a direct consequence of a systematic multipotentialisation of the equations. In [5] we introduced higher-degree potential symmetries for the Burgers' [7] and Calogero–Degasperis–Ibragimov–Shabat hierarchies [9] and derived the nonlocal linearisation transformations by means of a multipotentialisation of these hierarchies.

In the current paper we turn this question around: The aim is to identify and classify those evolution equations/systems which can be multipotentialised into some given target potential equation/system. This is the *converse problem*. In principle, the converse problem consists of a "*backwards-calculation-technique*" that identifies both the equations and the potential variables that relates the equations to a given potential equation. It is important to point out that the method proposed here does not require the calculation of integrating factors for the equations/systems (see Proposition 1).

To set the stage, we give an example of the *usual* (not *converse*) potentialisation of a linear equation. Consider the following problem: Find all third-order evolution equations of the form

$$u_t = F(u, u_x, u_{xx}, u_{xxx}) \tag{1.1}$$

that can be derived by the potentialisation of the linear equation

$$E := v_t - v_{xxx} = 0. \tag{1.2}$$

The corresponding auxiliary system for (1.2) is

$$\begin{aligned} u_x &= \Phi^t(x, v, v_x, \dots) \\ u_t &= -\Phi^x(x, v, v_x, \dots) \end{aligned}$$

where

$$D_t\Phi^t(x, v, \dots) + D_x\Phi^x(x, v, \dots)|_{v_t=v_{xxx}} = 0.$$

Clearly F in (1.1) is not arbitrary but is constrained by (1.2) and its corresponding Φ^t and Φ^x. In order to derive Eq. (1.1), we need to find all integrating factors, $\Lambda(t, x, v, v_x, v_{xx}, \dots)$, for (1.2). Those can be calculated by the conditions (see e.g. [5])

$$\hat{E}[v]\,(\Lambda E) = 0 \iff L_E^*[v]\Lambda = 0, \quad L_\Lambda[v]E = L_\Lambda^*[v]E,$$

where $\hat{E}[v]$ is the Euler operator

$$\hat{E}[v] = \frac{\partial}{\partial v} - D_x \circ \frac{\partial}{\partial v_x} - D_t \circ \frac{\partial}{\partial v_t} + D_x^2 \circ \frac{\partial}{\partial v_{2x}} - D_x^3 \circ \frac{\partial}{\partial v_{3x}} + \cdots$$

and $L^*[v]$ is the adjoint of the linear operator $L[v]$,

$$L[v] = \frac{\partial E}{\partial v} + \frac{\partial E}{\partial v_t} D_t + \frac{\partial E}{\partial v_x} D_x + \frac{\partial E}{\partial v_{xx}} D_x^2 + \frac{\partial E}{\partial v_{xxx}} D_x^3,$$

$$L^*[v] = \frac{\partial E}{\partial v} - D_t \circ \left(\frac{\partial E}{\partial v_t}\right) - D_x \circ \left(\frac{\partial E}{\partial v_x}\right) + D_x^2 \circ \left(\frac{\partial E}{\partial v_{xx}}\right) - D_x^3 \circ \left(\frac{\partial E}{\partial v_{xxx}}\right).$$

The relation of Λ to the conserved currents, Φ^t, for (1.2) is

$$\Lambda = \hat{E}[v]\,\Phi^t.$$

Following the above method, the only nonlinear equation of the form (1.1), so obtained, is [6]

$$u_t = u_{xxx} - \frac{3}{4}\frac{u_{xx}^2}{u_x}. \tag{1.4}$$

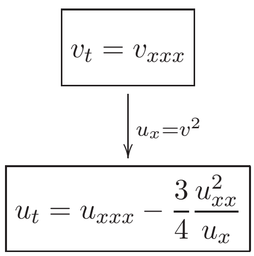

Fig. 1. Potentialisation of $v_t = v_{xxx}$.

In Sec. 3 we consider the converse problem of the above, i.e. we seek the equations of the form (1.1) for which (1.2) is the potential equation. The results of the converse potentialisation are listed as Case I in Sec. 3 and the results of the converse multipotentialisations of (1.2) are listed in Cases II and III (see Fig. 6).

The paper is organized as follows: In Sec. 2 we give the main propositions that describe the methodology of the proposed problem and introduce triangular-auto-Bäcklund transformations. These transformations act as solution generators for the equations. In Sec. 3 we classify third-order evolution equations which can be linearise by a suitable multipotentialisation. For example, in this section we shown that the Calogero–Degasperis–Ibragimov–Shabat equation and the third-order Burgers' equations, are just special cases of a class of third-order evolution equations which possess this type of linearisation property. In Sec. 4 we study a fifth-order evolution equation and show that the converse multipotentialisation leads in a natural way to an interesting triangular-auto-Bäcklund transformation for the equation. In Sec. 5 we propose the converse problem for systems of evolution in $(1 + 1)$ dimensions and in Sec. 6 we extend our methodology to evolution equations in higher dimensions. Some conluding remarks are made in Sec. 7.

2. The Converse Problem for the Multipotentialisation of (1+1)-dimensional Evolution Equations

In this section we consider $(1 + 1)$-dimensional evolution equations and propose a method to study the converse problem that aims to identify equations that can be potentialised in a target potential equation. This addresses the problem of deriving auto-Bäcklund transformations for evolution equations.

2.1. *Definitions and propositions*

Consider the following general x- and t-independent evolution equation of order p in the form

$$u_t = F(u, u_x, u_{xx}, u_{3x}, \dots, u_{px}). \tag{2.1}$$

We now define the converse problem and state conditions by which it can be studied.

Definition 1. The **converse problem** of the potentialisation of (2.1) aims to determine the functional form(s) of F in (2.1) for which (2.1) potentialises in a target equation of order p, given by

$$v_t = H(v_x, v_{xx}, \dots, v_{px}) + \alpha_0 v, \quad \alpha_0 : \text{constant}, \tag{2.2}$$

with potential variable, v, and auxiliary system

$$v_x = \Phi^t(x, u, u_x, \dots) \tag{2.3a}$$

$$v_t = -\Phi^x(x, u, u_x, \dots), \tag{2.3b}$$

where

$$D_t\Phi^t(x, u, u_x, \dots) + D_x\Phi^x(x, u, u_x, \dots)|_{u_t=F(u,u_x,u_{xx},\dots,u_{px})} = 0 \tag{2.4}$$

holds.

Following Definition 1 we replace v_t from (2.2) in (2.3b), differentiate (2.3b) with respect to x, and use (2.3a) and (2.4) to express the resulting relation in terms of Φ^t. This leads to

Proposition 1. *The condition on* Φ^t, *such that*

$$u_t = F(u, u_x, u_{xx}, u_{3x}, \dots, u_{px}),$$

potentialises in

$$v_t = H(v_x, v_{xx}, \dots, v_{px}) + \alpha_0 v,$$

is

$$D_x H(\Phi^t,\, D_x\Phi^t, D_x^2\Phi^t, \dots, D_x^{p-1}\Phi^t) + \alpha_0\Phi^t = D_t\Phi^t|_{u_t=F(u,u_x,\dots,u_{px})}, \tag{2.5}$$

where H *is a given function and* α_0 *a given constant.*

Note that condition (2.5) places a constrained on both Φ^t and F for a given H, which assures that (2.1) potentialises in (2.2). Note that, in order to solve condition (2.5) for both F and Φ^t, we need to make an assumption regarding the functional dependence of Φ^t. That is, we have to make a choice for the number of derivatives, q, allowed for Φ^t:

$$\Phi^t = \Phi^t(x, u, u_x, \ldots, u_{qx}).$$

Next we describe the *converse multipotentialisation process*. Consider again the general equation, (2.1), viz.

$$u_t = F(u, u_x, u_{xx}, u_{3x}, \ldots, u_{px}),$$

and assume that it can be potentialised in some given evolution equation of order p, say

$$v_t = G(v_x, v_{xx}, v_{3x}, \ldots, v_{px}), \tag{2.6}$$

where (2.1) admits the auxiliary system

$$v_x = \Phi_1^t(x, u, u_x, \ldots) \tag{2.7a}$$

$$v_t = -\Phi_1^x(x, u, u_x, \ldots) \tag{2.7b}$$

and

$$D_t\Phi_1^t(x, u, \ldots) + D_x\Phi_1^x(x, u, \ldots)|_{u_t=F} = 0. \tag{2.8}$$

Introduce now a second auxiliary system, namely for (2.6), of the form

$$w_x = \Phi_2^t(x, v, v_x, \ldots) \tag{2.9a}$$

$$w_t = -\Phi_2^x(x, v, v_x, \ldots), \tag{2.9b}$$

such that w is the dependent variable for yet another evolution equation, say

$$w_t = H(w_x, w_{xx}, \ldots, w_{px}) \tag{2.10}$$

and

$$D_t\Phi_2^t(x, v, \ldots) + D_x\Phi_2^x(x, v, \ldots)|_{v_t=G} = 0. \tag{2.11}$$

The above procedure provides a method to identify all equations of the form (2.1) that can be potentialise in (2.6) under the first potential variable, v, with corresponding auxiliary system (2.7a) and (2.7b), and which furthermore potentialises into (2.10) under the second potential variable, w, with auxiliary system (2.9a) and (2.9b). Hence this multipotentialisation procedure identifies the family of equations, (2.1), that are related to (2.10) with a transformation that can be obtain by composing

$$v_x = \Phi_1^t(x, u, u_x, \ldots) \tag{2.12a}$$

$$w_x = \Phi_2^t(x, v, v_x, \ldots). \tag{2.12b}$$

We call this the *second-degree converse multipotentialisation* of (2.10). The nth-degree converse multipotentialisations with potential variables, $\{v_1,\ v_2, \ldots, v_{n-1}, w\}$ can then be introduced in an obvious manner, where (2.12a) and (2.12b) extends to

$$
\begin{aligned}
v_{1,x} &= \Phi_1^t(x, u, u_x, \ldots)\\
v_{2,x} &= \Phi_2^t(x, v_1, v_{1,x}, \ldots)\\
v_{3,x} &= \Phi_3^t(x, v_2, v_{2,x}, \ldots)\\
&\ \vdots\\
v_{(n-1),x} &= \Phi_{n-1}^t(x, v_{n-2}, v_{n-2,x}, \ldots)\\
w_x &= \Phi_n^t(x, v_{n-1}, v_{n-1,x}, \ldots).
\end{aligned}
\tag{2.13}
$$

Figure 2 describes the nth degree converse multipotentialisation of (2.10):

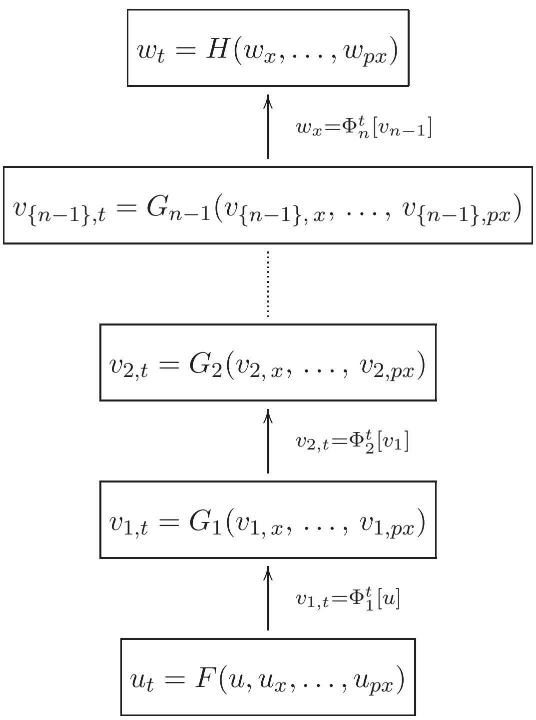

Fig. 2. Converse multipotentialisation of $w_t = H$ of degree n

2.2. *Triangular auto-Bäcklund transformations*

In some cases we can combine and compose several conserved currents, Φ^t, to form nonpoint mappings of the dependent variable of an equation to the same equation. This maps solutions to solutions and can hence be applied to generate nontivial new solutions. We name such transformations **triangular auto-Bäcklund transformation**, or **△-auto-Bäcklund transformation**. There are essentially three types of △-auto-Bäcklund transformations. This is demonstrated in Fig. 3. Note that "Equation A $[V]$" represents an evolution equation with V as its dependent variable and $\Phi^t[V]$ denotes the equation's conserved current, which is a function of $x, V,\ V_x,\ V_{xx}$, etc.

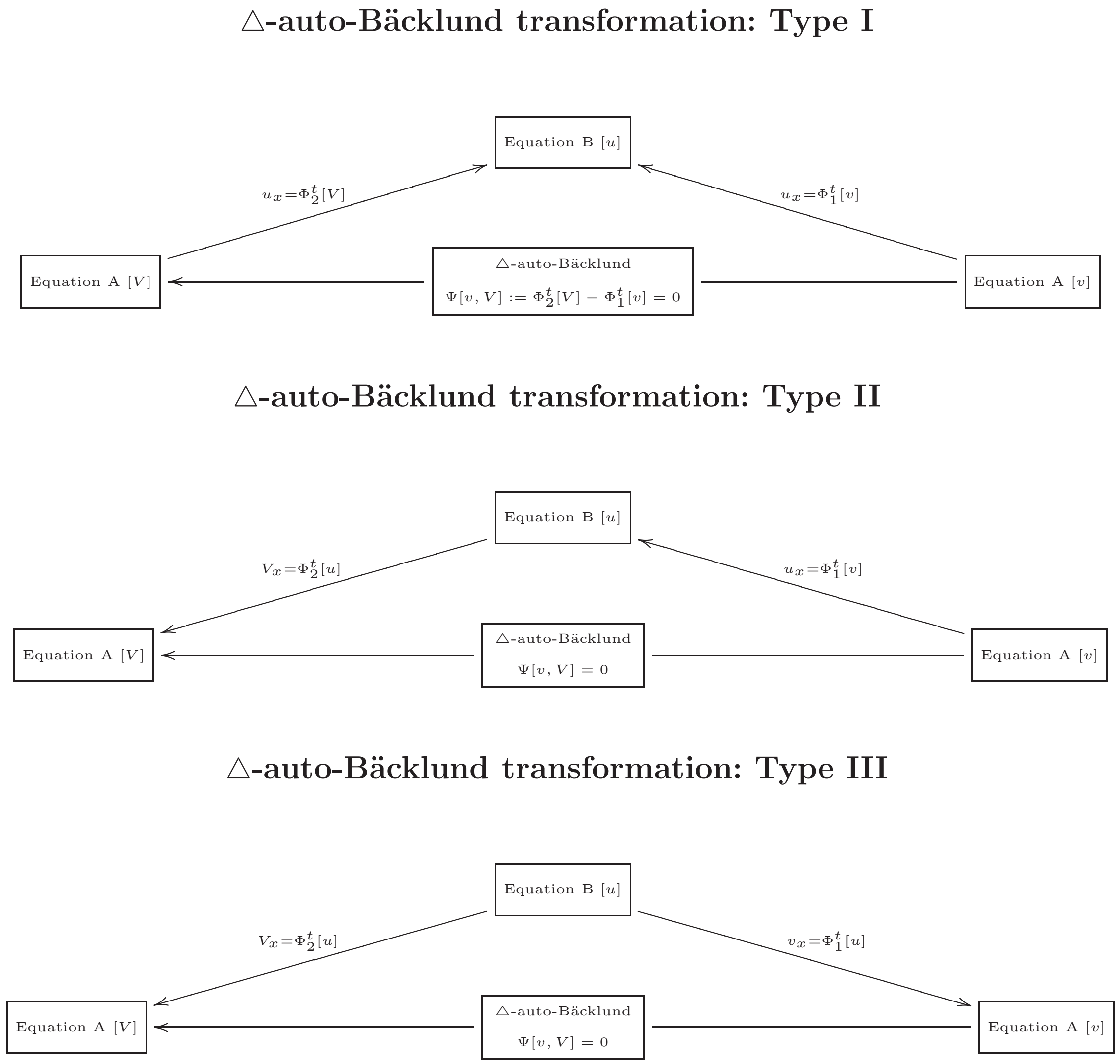

Fig. 3.

Several △-auto-Bäcklund transformations are reported in Propositions 2–6.

3. Third-Order Linearisable Equations in (1+1) Dimensions

3.1. *First-degree converse potentialisation*

For an application of Proposition 1, we now discuss the converse problem of linearisable evolution equations, i.e. the problem by which to determine the functional form(s) of F

in (2.1), *viz.*

$$u_t = F(u, u_x, u_{xx}, u_{3x}, \dots, u_{px}).$$

for which (2.1) potentialises in the linear evolution equation of order p,

$$v_t = \mathcal{L}^{(p)}[\alpha]v, \tag{3.1}$$

under the first potential variable, v, with auxiliary system (2.3a) and (2.3b). Here $\mathcal{L}^{(p)}$ is the general linear operator with parameters $\{\alpha_0, \alpha_1, \dots, \alpha_p\}$ defined by

$$\mathcal{L}^{(p)}[\alpha] := \sum_{j=0}^{p} \alpha_j D_x^j. \tag{3.2}$$

Note that

$$D_x \mathcal{L}^{(p)}[\alpha]\, v|_{v_x=\Phi^t} = \mathcal{L}^{(p)}[\alpha]\, \Phi^t. \tag{3.3}$$

Following Proposition 1, the condition on Φ^t and F for potentialisation the (2.1) in the linear equation (3.1), then becomes

$$D_t \Phi^t|_{u_t=F} = \mathcal{L}^{(p)}[\alpha]\, \Phi^t. \tag{3.4}$$

As a special case we study third-order evolution equations with potentialisations in

$$v_t = v_{xxx} \tag{3.5}$$

in detail. Consider the third-order evolution equations in the form

$$u_t = F(u, u_x, u_{xx}, u_{xxx}) \tag{3.6}$$

and assume that (3.6) admits a conserved current of the form

$$\Phi^t = \Phi^t(u, u_x, u_{xx}). \tag{3.7}$$

Solving condition (3.4), with the assumption of (3.7), we find that the most general form of (3.6) which potentialises in the linear equation (3.5) is given by the following two cases:

Case I a. The conserved current

$$\Phi^t(u, u_x) = \frac{1}{\sqrt{2}} \left(\frac{u_x}{h} + c_1 \right)^{1/2}, \tag{3.8}$$

leads to the equation

$$\begin{aligned} u_t = u_{xxx} - \frac{3}{4}\left(\frac{u_{xx}^2}{u_x + c_1 h}\right) - \frac{3}{2}\frac{h'}{h}\left(\frac{u_x + 2c_1 h}{u_x + c_1 h}\right) u_x u_{xx} + \left(\frac{5}{4}\left(\frac{h'}{h}\right)^2 - \frac{h''}{h}\right) u_x^3 \\ + \frac{3}{4}\frac{c_1 (h')^2}{h}\, u_x^2 - \frac{3}{4} c_1^2 (h')^2\, u_x - \frac{3}{4}\left(\frac{c_1^4 h^2 (h')^2}{u_x + c_1 h}\right) + \frac{3}{4} c_1^3 (h')^2 h + c_2 h, \end{aligned} \tag{3.9}$$

where h is an arbitrary but nonzero differentiable function of u and c_1, c_2 are arbitrary constants.

Case I b. The conserved current

$$\Phi^t(u, u_x, u_{xx}) = \frac{1}{\sqrt{2hu_x + 2c_1h^2}}\left(u_{xx} - \left(\frac{h'}{h}\right)u_x^2\right) \tag{3.10}$$

leads to the equation

$$\begin{aligned} u_t = u_{xxx} - \frac{3}{4}\frac{u_{xx}^2}{u_x + c_1h} - \frac{3}{2}\frac{h'}{h}\left(\frac{u_x + 2c_1h}{u_x + c_1h}\right)u_xu_{xx} + \frac{1}{4}\left(\frac{5(h')^2 - 4hh''}{h^2(u_x + c_1h)}\right)u_x^4 \\ + c_1\left(\frac{2(h')^2 - hh''}{h(u_x + c_1h)}\right)u_x^3 + c_2\left(\frac{h}{u_x + c_1h}\right)u_x + c_1c_2\left(\frac{h^2}{u_x + c_1h}\right), \end{aligned} \tag{3.11}$$

where h is an arbitrary but nonzero differentiable function of u and c_1, c_2 are arbitrary constants.

Remark 1. The case, $\Phi^t = f_1(u)u_x + f_2(u)$ for any differentiable functions $f_1(u)$ and $f_2(u)$, result in linear equations for (3.6) under the point transformation $u \mapsto h(u)$ and are therefore not listed here.

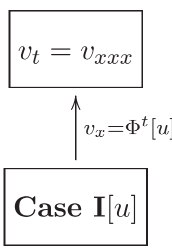

Fig. 4. Converse potentialisation of $v_t = v_{xxx}$.

The above Case Ia and Case Ib lead to

Proposition 2. *An $\triangle$-auto-Bäcklund transformation of type I for*

$$u_t = u_{xxx} - \frac{3}{4}\left(\frac{u_{xx}^2}{u_x}\right) - \frac{3}{2}\frac{h'}{h}u_xu_{xx} + \left(\frac{5}{4}\left(\frac{h'}{h}\right)^2 - \frac{h''}{h}\right)u_x^3 \tag{3.12}$$

is given by the relation

$$\frac{U_x}{h(U)} = \frac{1}{h(u)u_x}\left(u_{xx} - \left(\frac{h'(u)}{h(u)}\right)u_x^2\right)^2, \tag{3.13}$$

where u and U satisfy (3.12) *for any nonzero arbitrary differentiable function h.*

Proof. Equations (3.9) and (3.11) with

$$c_1 = c_2 = 0 \tag{3.14}$$

reduce to the same equation, namely (3.12). Consider now (3.9) with (3.14) in terms of the dependent variable U, i.e.,

$$U_t = U_{xxx} - \frac{3}{4}\left(\frac{U_{xx}^2}{U_x}\right) - \frac{3}{2}\frac{h'(U)}{h(U)}U_xU_{xx} + \left(\frac{5}{4}\left(\frac{h'(U)}{h(U)}\right)^2 - \frac{h''(U)}{h(U)}\right)U_x^3 \tag{3.15}$$

with the conserved current, (3.10), and its relation to the potential variable v,

$$v_x = \frac{1}{\sqrt{2}}\left(\frac{U_x}{h(U)}\right)^{1/2}. \tag{3.16}$$

Moreover, (3.12) has the following relation to the same potential variable, v, namely

$$v_x = \frac{1}{\sqrt{2h(u)u_x}}\left(u_{xx} - \left(\frac{h'(u)}{h(u)}\right)u_x^2\right). \tag{3.17}$$

Relation (3.13), then follows by (3.16) and (3.17). □

Remark 2. Equation (3.12) with $h(u) = 1$, reduces to

$$u_t = u_{xxx} - \frac{3}{4}\frac{u_{xx}^2}{u_x} \tag{3.18}$$

and the $\triangle$-auto-Bäcklund transformation, (3.13), takes the form

$$s_x = \frac{u_{xx}^2}{u_x}. \tag{3.19}$$

This special case, (3.18), and its auto-Bäcklund transformation, (3.19), has been reported in [6].

3.2. *Converse multipotentialisation*

For second degree converse multipotentialisations of the linear evolution equation

$$w_t = w_{xxx} \tag{3.20}$$

we consider (3.12) with

$$h(u) = \exp(\alpha u), \quad \alpha : \text{arbitrary constant}, \tag{3.21}$$

that is

$$u_t = u_{xxx} - \frac{3}{4}\frac{u_{xx}^2}{u_x} - \frac{3}{2}\alpha u_x u_{xx} + \frac{1}{4}\alpha^2 u_x^3. \tag{3.22}$$

We now construct the most general equation of the form (3.6), now written in terms of the variable v,

$$v_t = F(v, v_x, v_{xx}, v_{xxx}), \tag{3.23}$$

which admits (3.22) as its potential equation with auxiliary system

$$u_x = \Phi^t(v, v_x, v_{xx}) \tag{3.24a}$$

$$u_t = -\Phi^x(v, v_x, \ldots). \tag{3.24b}$$

Applying Proposition 1 we obtain the following constraint on Φ^t:

$$D_x^3\Phi^t - \frac{3}{2}(\Phi^t)^{-1}D_x\Phi^t\, D_x^2\Phi^t + \frac{3}{4}(\Phi^t)^{-2}(D_x\Phi^t)^3 - \frac{3\alpha}{2}(D_x\Phi^t)^2 - \frac{3\alpha}{2}\Phi^t D_x^2\Phi^t + \frac{3\alpha^2}{4}(\Phi^t)^2 D_x\Phi^t = D_t\Phi^t|_{v_t=F}. \tag{3.25}$$

By condition (3.25), the most general form of (3.23) for which (3.22) is the potential form of (3.23) with the conserved current $\Phi^t = \Phi^t(v, v_x, v_{xx})$, is given by the following cases:

Case II a. The conserved current

$$\Phi^t(v, v_x) = \frac{v_x}{f(v)} - c_1 \tag{3.26}$$

leads to the equation

$$\begin{aligned} v_t = v_{xxx} &+ \frac{3}{4}\left(\frac{v_{xx}^2}{c_1 f - v_x}\right) - \frac{3}{2}\left(\frac{c_1 f(\alpha + 2f') - (f' + \alpha)v_x}{f(c_1 f - v_x)}\right) v_x v_{xx} + \frac{3}{2}\alpha c_1 v_{xx} \\ &- \frac{1}{4}\left(\frac{4f'' f - 5(f')^2 - 6\alpha f' - \alpha^2}{f^2}\right) v_x^3 - \frac{3}{4}\frac{c_1(f' + \alpha)^2}{f}\, v_x^2 \\ &- \frac{3}{4}c_1^2\left((f')^2 - \alpha^2\right) v_x + \frac{3}{4}\left(\frac{(f' f)^2 c_1^4}{c_1 f - v_x}\right) - \frac{3}{4}c_1^3 f' f + c_2 f, \end{aligned} \tag{3.27}$$

where f is a nonzero arbitrary differentiable function of v and α, c_1, c_2 are arbitrary constants.

Case II b. The conserved current

$$\Phi^t(v) = f(v), \tag{3.28}$$

leads to the equation

$$\begin{aligned} v_t = v_{xxx} &+ \left(\frac{3f''}{f'} - \frac{3f'}{2f}\right) v_x v_{xx} + \left(\frac{f'''}{f'} - \frac{3f''}{2f} + \frac{3}{4}\left(\frac{f'}{f}\right)^2\right) v_x^3 - \frac{3}{2}\alpha f v_{xx} \\ &+ \frac{3}{4}\alpha^2 f^2 v_x - \frac{3}{2}\alpha\left(f' + \frac{f f''}{f'}\right) v_x^2, \end{aligned} \tag{3.29}$$

where f is a nonconstant arbitrary differentiable function of v and α is an arbitrary constant.

Case II c. For $\alpha = 0$, the conserved current

$$\Phi^t(v, v_x) = \frac{(f'(v))^2}{f(v)} v_x^2, \quad f'(v) \neq 0, \tag{3.30}$$

leads to the equation

$$v_t = v_{xxx} + \left(\frac{3f''}{f'} - \frac{3f'}{2f}\right) v_x v_{xx} + \left(\frac{f'''}{f'} - \frac{3f''}{2f} + \frac{3}{4}\left(\frac{f'}{f}\right)^2\right) v_x^3, \tag{3.31}$$

where f is a nonconstant arbitrary differentiable function of v.

Remark 3. It is interesting to note that (3.29) contains, for special values of α and special functions f, two well-known equations, namely the following:

With $\alpha = -2$ and $f(v) = v^2$ Eq. (3.29) is the Calogero–Degasperis–Ibragimov–Shabat equation (CDIS) [2,8]

$$v_t = v_{xxx} + 3v^2 v_{xx} + 9v v_x^2 + 3v^4 v_x \tag{3.32}$$

and with $\alpha = 0$ and $f(v) = \exp(2v)$ Eq. (3.29) is the third-order potential Burgers' equation [5]

$$v_t = v_{xxx} + 3v_x v_{xx} + v_x^3. \tag{3.33}$$

In [5] we showed that both (3.32) and (3.33) linearise under a suitable multipotentialisation. Hence the Eq. (3.29) can be viewed as a generalisation of the Calogero–Degasperis–Ibragimov–Shabat equation, (3.32), and the third-order Burgers' equation, (3.33), as (3.29) combines both of these interesting equations into a single equation with arbitrary function, $f(v)$. See also Fig. 6.

A closer look at Cases IIb and IIc reveals a $\triangle$-auto-Bäcklund transformation for (3.31).

Proposition 3. *An $\triangle$-auto-Bäcklund transformation of type I for (3.31), viz.*

$$v_t = v_{xxx} + \left(\frac{3f''}{f'} - \frac{3f'}{2f}\right) v_x v_{xx} + \left(\frac{f'''}{f'} - \frac{3f''}{2f} + \frac{3}{4}\left(\frac{f'}{f}\right)^2\right) v_x^3,$$

is given by the relation

$$f(V) = \frac{(f'(v))^2}{f(v)}\, v_x^2, \tag{3.34}$$

where v and V satisfy (3.31) for any nonconstant differentiable function f.

Applying Proposition 3 with $f(v) = e^{2v}$ on the third-order potential Burgers' equation, (3.33), *viz.*

$$v_t = v_{xxx} + 3v_x v_{xx} + v_x^3,$$

we obtain the $\triangle$-auto-Bäcklund transformation of type I for (3.33) in the form

$$e^{2V} = 4e^{2v} v_x^2. \tag{3.35}$$

By differentiating (3.35) we arrive at the relation

$$V_x = v_x + D_x \ln |v_x| \tag{3.36}$$

which can be applied to gain auto-Bäcklund transformations for those equations which can be potentialised in (3.33). Note that (3.22) with $\alpha = 0$, i.e.,

$$u_t = u_{xxx} - \frac{3}{4}\frac{u_{xx}^2}{u_x}, \tag{3.37}$$

and (3.33), both admit linear integro-differential recursion operators: Equation (3.37) admits the second-order recursion operator, $R_1[u]$, given by (e.g. [4])

$$R_1[u] = D_x^2 - \frac{u_{xx}}{u_x} D_x + \frac{1}{2}\frac{u_{xxx}}{u_x} - \frac{1}{4}\left(\frac{u_{xx}}{u_x}\right)^2 \tag{3.38}$$

$$-\frac{1}{2} D_x^{-1} \circ \left(\frac{u_{xxxx}}{u_x} - \frac{2u_{xx}u_{xxx}}{u_x^2} + \left(\frac{u_{xx}}{u_x}\right)^3\right), \tag{3.39}$$

whereas (3.33) admits the first-order recursion operator, $R_2[v]$, given by (e.g. [7])

$$R_2[v] = D_x + v_x. \tag{3.40}$$

Equations (3.37) and (3.36) can now be written, respectively, in the form

$$u_t = R_1[u]u_x, \quad v_t = R_2^2[v]v_x, \tag{3.41}$$

and the hierarchies of n equations are

$$u_t = R_1^n[u]u_x \tag{3.42a}$$

$$v_t = R_2^n[v]v_x, \quad n \in \mathcal{N}. \tag{3.42b}$$

Hierarchy (3.42b) is known as the potential Burger' hierarchy [5]. Since all equations in a given hierarchy of evolution equations admit the same conserved currents, the $\triangle$-auto-Bäcklund transformation for (3.36) is valid for the entire potential Burgers' hierarchy (3.42b). The transformations between the two hierarchies and their $\triangle$-auto-Bäcklund transformation are illustrated in Fig. 5.

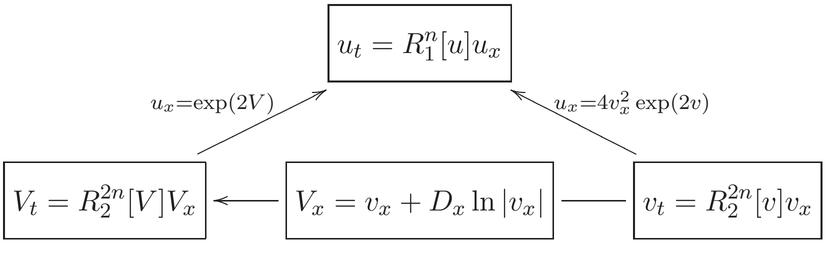

Fig. 5.

We now consider (3.33), *viz.*

$$v_t = v_{xxx} + 3v_xv_{xx} + v_x^3,$$

for the third degree converse potentialisation of (3.20). Applying Proposition 1, we obtain the constraint

$$D_x^3\Phi^t + 3(D_x\Phi^t)^2 + 3\Phi^tD_x^2\Phi^t + 3(\Phi^t)^2D_x\Phi^t = D_t\Phi^t|_{q_t=F(q,q_x,q_{xx},q_{xxx})}, \tag{3.43}$$

which allows

$$q_t = F(q, q_x, q_{xx}, q_{xxx}) \tag{3.44}$$

to be potentialised in (3.33) with the auxiliary system

$$v_x = \Phi^t(q, q_x) \tag{3.45a}$$

$$v_t = -\Phi^x(q, q_x, \ldots). \tag{3.45b}$$

This identifies five cases:

Case III a. The conserved current

$$\Phi^t(q) = g(q), \quad g'(q) \neq 0 \tag{3.46}$$

leads to the equation

$$q_t = q_{xxx} + 3\left(\frac{g''}{g'}\right) q_x q_{xx} + 3g q_{xx} + \left(\frac{g'''}{g'}\right) q_x^3 + 3\left(g' + \frac{gg''}{g'}\right) q_x^2 + 3g^2 q_x, \tag{3.47}$$

where g is an arbitrary nonconstant differentiable function of q.

Note. With $g = q$, (3.47) is the third-order Burgers' equation [3, 7],

$$q_t = q_{xxx} + 3q_x^2 + 3q q_{xx} + 3q^2 q_x. \tag{3.48}$$

Case III b. The conserved current

$$\Phi^t(q, q_x) = g(q) q_x + c_1, \quad g(q) \neq 0 \tag{3.49}$$

leads to the equation

$$\begin{aligned} q_t = q_{xxx} &+ 3\left(\frac{g'}{g} + g\right) q_x q_{xx} + 2c_1 q_{xx} + \left(3g' + g^2 + \frac{g''}{g}\right) q_x^3 \\ &+ 3c_1\left(g + \frac{g'}{g}\right) q_x^2 + 3c_1^2 q_x + \frac{c_2}{g}, \end{aligned} \tag{3.50}$$

where g is an arbitrary nonzero differentiable function of q and $c_1,\ c_2$ are arbitrary constants.

Case III c. The conserved current

$$\Phi^t(q, q_x) = \left(\frac{g'(q)}{g(q) + c_1}\right) q_x + g(q), \quad g'(q) \neq 0 \tag{3.51}$$

leads to the equation

$$\begin{aligned} q_t = q_{xxx} &+ 3\left(\frac{gg'' + c_1 g'' + (g')^2}{g'(g + c_1)}\right) q_x q_{xx} + 3\left(\frac{g'}{g + c_1}\right) frm[o]{-}{-} q_x q_{xx} + 3g q_{xx} \\ &+ \left(\frac{g'''}{g'}\right) q_x^3 + 3\left(\frac{gg''}{g'} + g'\right) q_x^2 + 3g^2 q_x, \end{aligned} \tag{3.52}$$

where g is an arbitrary nonconstant differentiable function of q and c_1 is an arbitrary constant.

Case III d. The conserved current

$$\Phi^t(q, q_x) = \left(\frac{1}{2}\frac{g'(q)}{g(q)}\right) q_x + g(q), \quad g'(q) \neq 0 \tag{3.53}$$

leads to the equation

$$\begin{aligned} q_t = {} & q_{xxx} + \frac{3}{2}\left(\frac{2gg'' - (g')^2}{gg'}\right) q_x q_{xx} + 3g q_{xx} \\ & + \frac{1}{4}\left(\frac{3(g')^3 - 6gg'g'' + 4g^2 g'''}{g^2 g'}\right) q_x^3 + 3\left(g' + \frac{gg''}{g'}\right) q_x^2 + 3g^2 q_x, \end{aligned} \tag{3.54}$$

where g is an arbitrary nonconstant differentiable function of q.

Case III e. The conserved current

$$\Phi^t(q, q_x) = \sqrt{Q} + g(q), \tag{3.55}$$

where

$$Q := g' q_x + g^2 + c_1, \quad g'(q) \neq 0, \tag{3.56}$$

leads to the equation

$$\begin{aligned} q_t = {} & q_{xxx} + \left(\frac{g'}{Q}\right) q_{xx}^2 + \frac{3}{2}\left(\frac{g''}{Q}\right) q_x^2 q_{xx} + \left(\frac{3}{Qg'}\right)(g^2 g'' + c_1 g'' - \sqrt{Q}(g')^2) q_x q_{xx} \\ & + \frac{3}{Q}(g^3 + c_1 g - \sqrt{Q} g^2 - c_1\sqrt{Q}) q_{xx} + \frac{1}{Qg'}\left(g' g''' - \frac{3}{4}(g'')^2\right) q_x^4 \\ & + \frac{1}{Qg'}(g^2 g''' + c_1 g''' + 6(g')^3 - 3\sqrt{Q} g' g'') q_x^3 \\ & + \frac{3}{Qg'}[gg''(c_1 - \sqrt{Q} g + g^2) + (g')^2(3g^2 + 3c_1 - 2\sqrt{Q} g) - c_1\sqrt{Q} g''] q_x^2 \\ & + \frac{3}{Q}(c_1^2 - 2c_1\sqrt{Q} g + 3c_1 g^2 - 2\sqrt{Q} g^3 + 2g^4) q_x. \end{aligned} \tag{3.57}$$

Here g is an arbitrary nonconstant differentiable function of q and c_1 is an arbitrary constant. A detailed graphical description of the converse multipotentialisation of (3.20) is given in Fig. 6.

As described in Sec. 2, the linearisation transformations of all the equations listed above can now be determined by composing the corresponding conserved currents. For example, Eq. (3.29) of Case IIb linearises in (3.20) under the nonlocal transformation

$$w_x = \frac{1}{\sqrt{2}} f(v)^{1/2} \exp\left(-\frac{\alpha}{2}\int f(v)\, dx\right), \tag{3.58}$$

which is obtained by composing

$$w_x = \frac{1}{\sqrt{2}} u_x^{1/2} \exp\left(-\frac{\alpha}{2} u\right) \tag{3.59a}$$

$$u_x = f(v). \tag{3.59b}$$

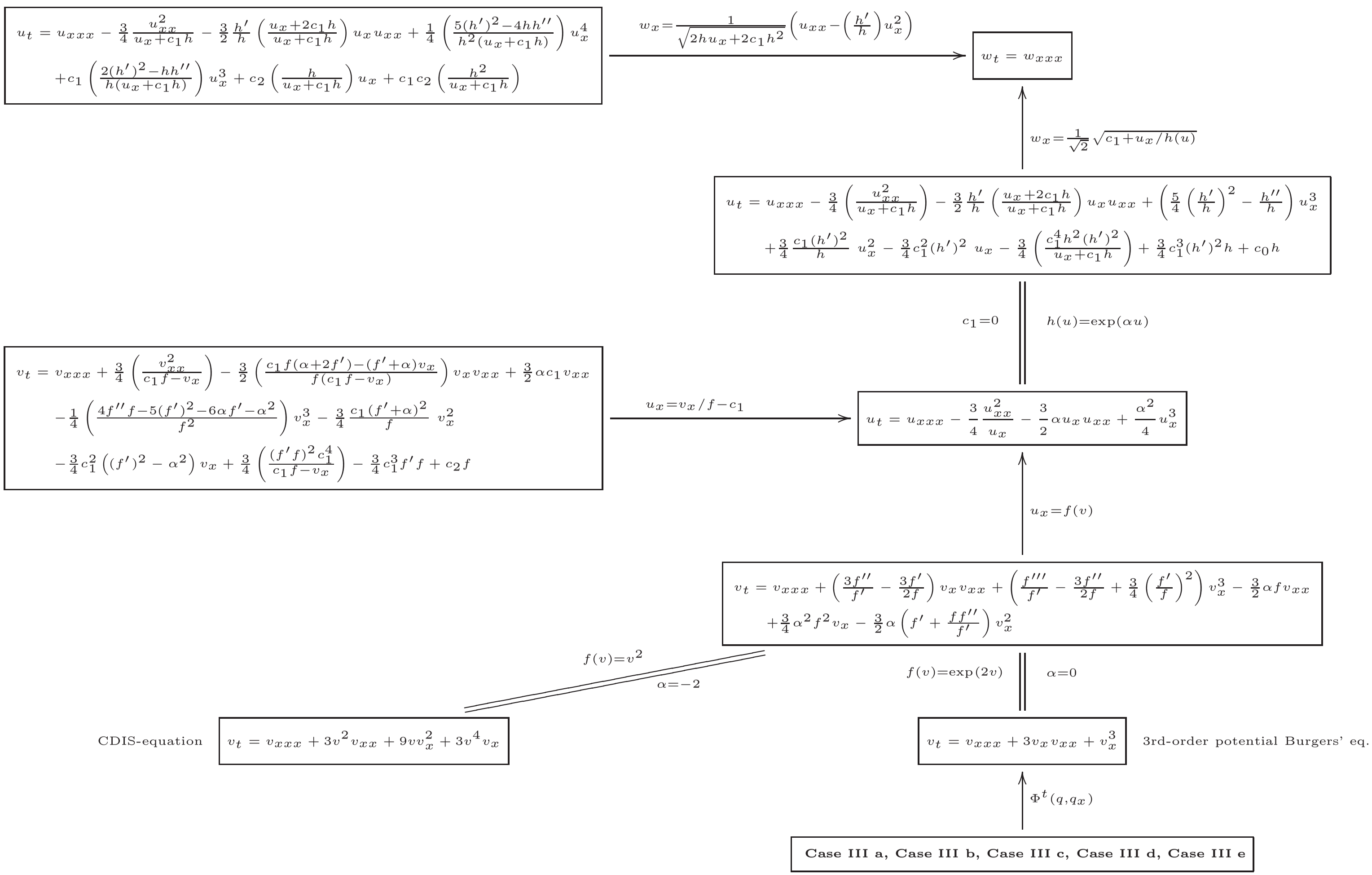
$u_t = u_{xxx} - \frac{3}{4}\frac{u_{xx}^2}{u_x+c_1h} - \frac{3}{2}\frac{h'}{h}\left(\frac{u_x+2c_1h}{u_x+c_1h}\right)u_xu_{xx} + \frac{1}{4}\left(\frac{5(h')^2-4hh''}{h^2(u_x+c_1h)}\right)u_x^4$
$+c_1\left(\frac{2(h')^2-hh''}{h(u_x+c_1h)}\right)u_x^3 + c_2\left(\frac{h}{u_x+c_1h}\right)u_x + c_1c_2\left(\frac{h^2}{u_x+c_1h}\right)$
$w_x = \frac{1}{\sqrt{2hu_x+2c_1h^2}}\left(u_{xx} - \left(\frac{h'}{h}\right)u_x^2\right)$
$w_t = w_{xxx}$
$w_x = \frac{1}{\sqrt{2}}\sqrt{c_1+u_x/h(u)}$
$u_t = u_{xxx} - \frac{3}{4}\left(\frac{u_{xx}^2}{u_x+c_1h}\right) - \frac{3}{2}\frac{h'}{h}\left(\frac{u_x+2c_1h}{u_x+c_1h}\right)u_xu_{xx} + \left(\frac{5}{4}\left(\frac{h'}{h}\right)^2 - \frac{h''}{h}\right)u_x^3$
$+\frac{3}{4}\frac{c_1(h')^2}{h}u_x^2 - \frac{3}{4}c_1^2(h')^2u_x - \frac{3}{4}\left(\frac{c_1^4h^2(h')^2}{u_x+c_1h}\right) + \frac{3}{4}c_1^3(h')^2h + c_0h$
$c_1=0$
$h(u)=\exp(\alpha u)$
$v_t = v_{xxx} + \frac{3}{4}\left(\frac{v_{xx}^2}{c_1f-v_x}\right) - \frac{3}{2}\left(\frac{c_1f(\alpha+2f')-(f'+\alpha)v_x}{f(c_1f-v_x)}\right)v_xv_{xx} + \frac{3}{2}\alpha c_1v_{xx}$
$-\frac{1}{4}\left(\frac{4f''f-5(f')^2-6\alpha f'-\alpha^2}{f^2}\right)v_x^3 - \frac{3}{4}\frac{c_1(f'+\alpha)^2}{f}v_x^2$
$-\frac{3}{4}c_1^2\left((f')^2-\alpha^2\right)v_x + \frac{3}{4}\left(\frac{(f'f)^2c_1^4}{c_1f-v_x}\right) - \frac{3}{4}c_1^3f'f + c_2f$
$u_x=v_x/f-c_1$
$u_t = u_{xxx} - \frac{3}{4}\frac{u_{xx}^2}{u_x} - \frac{3}{2}\alpha u_xu_{xx} + \frac{\alpha^2}{4}u_x^3$
$u_x=f(v)$
$v_t = v_{xxx} + \left(\frac{3f''}{f'} - \frac{3f'}{2f}\right)v_xv_{xx} + \left(\frac{f'''}{f'} - \frac{3f''}{2f} + \frac{3}{4}\left(\frac{f'}{f}\right)^2\right)v_x^3 - \frac{3}{2}\alpha fv_{xx}$
$+\frac{3}{4}\alpha^2f^2v_x - \frac{3}{2}\alpha\left(f' + \frac{ff''}{f'}\right)v_x^2$
$f(v)=v^2$
$\alpha=-2$
$f(v)=\exp(2v)$
$\alpha=0$
CDIS-equation
$v_t = v_{xxx} + 3v^2v_{xx} + 9vv_x^2 + 3v^4v_x$
$v_t = v_{xxx} + 3v_xv_{xx} + v_x^3$
3rd-order potential Burgers' eq.
$\Phi^t(q,q_x)$
Case III a, Case III b, Case III c, Case III d, Case III e

Fig. 6.

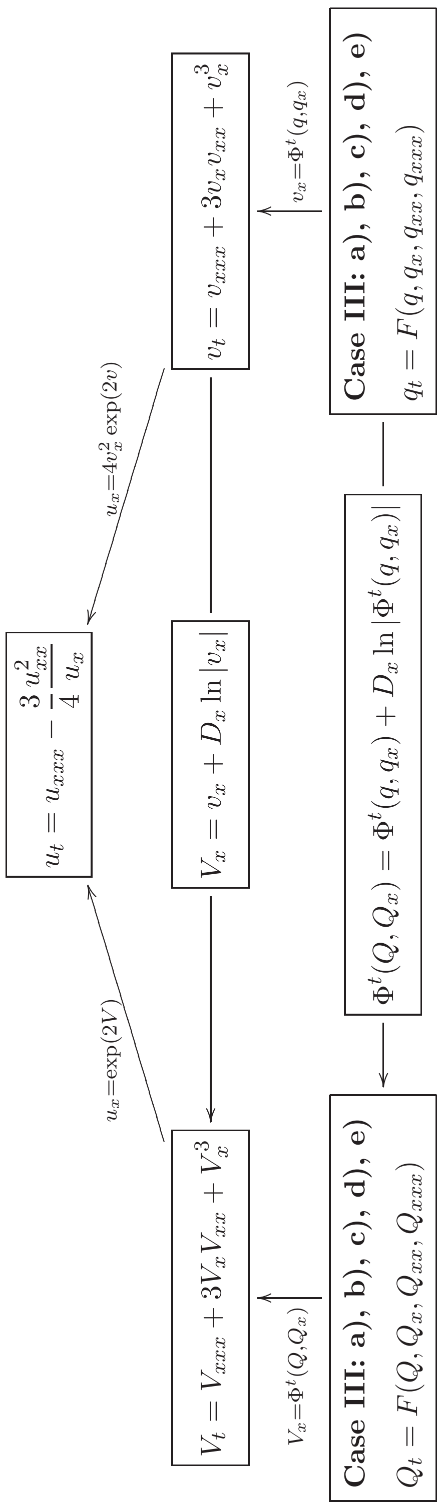
$u_t = u_{xxx} - \frac{3}{4}\frac{u_{xx}^2}{u_x}$
$u_x = 4v_x^2 \exp(2v)$
$u_x = \exp(2V)$
$v_t = v_{xxx} + 3v_x v_{xx} + v_x^3$
$V_x = v_x + D_x \ln|v_x|$
$V_t = V_{xxx} + 3V_x V_{xx} + V_x^3$
$v_x = \Phi^t(q, q_x)$
$V_x = \Phi^t(Q, Q_x)$
Case III: a), b), c), d), e)
$q_t = F(q, q_x, q_{xx}, q_{xxx})$
$\Phi^t(Q, Q_x) = \Phi^t(q, q_x) + D_x \ln|\Phi^t(q, q_x)|$
Case III: a), b), c), d), e)
$Q_t = F(Q, Q_x, Q_{xx}, Q_{xxx})$

Fig. 7.

By Proposition 3 and the auto-Bäcklund transformation (3.36) for the potential Burgers equation (3.33), we have auto-Bäcklund transformations for all equations listed in Case III above (see Fig. 7). These auto-Bäcklund transformation are of the form

$$\Phi^t(Q, Q_x) = \Phi^t(q, q_x) + D_x \ln |\Phi^t(q, q_x)|, \tag{3.60}$$

where Φ^t are the conserved currents of the equations in Case III. For example, the equation given in Case IIIa, namely (3.47),

$$q_t = q_{xxx} + 3\left(\frac{g''}{g'}\right) q_x q_{xx} + 3g q_{xx} + \left(\frac{g'''}{g'}\right) q_x^3 + 3\left(g' + \frac{gg''}{g'}\right) q_x^2 + 3g^2 q_x,$$

admits the auto-Bäcklund transformation

$$g(Q) = g(q) + \frac{g'(q)}{g(q)}\, q_x \tag{3.61}$$

where $\Phi^t = g(q)$. For $g(q) = q$, (3.47) is the well-known third-order Burgers' equation [5]

$$q_t = q_{xxx} + 3q q_{xx} + 3q_x^2 + 3q^2 q_x, \tag{3.62}$$

and (3.61) reduces to the well-known auto-Bäcklund transformation

$$Q = q + \frac{q_x}{q} \tag{3.63}$$

which can be derived by a truncated Painlevé expansion for the Burgers' equation (see e.g. [12]). As a second example, consider Case III(b). It follows that

$$g(Q) Q_x = g(q) q_x + \frac{g'(q) q_x^2 + g(q) q_{xx}}{g(q) q_x + c_1} \tag{3.64}$$

is an auto-Bäcklund transformation for (3.50).

4. Converse Multipotentialisation of a Fifth-Order Integrable Evolution Equation

In this section we apply the converse multipotentialisation methodology on the following fifth-order equation:

$$u_t = u_{5x} - \frac{5u_{xx}u_{4x}}{u_x} - \frac{15}{4}\frac{u_{xxx}^2}{u_x} + \frac{65}{4}\frac{u_{xx}^2 u_{xxx}}{u_x^2} - \frac{135}{16}\frac{u_{xx}^4}{u_x^3}. \tag{4.1}$$

Equation (4.1) plays a central role in the nonlocal invariance of the Kaup–Kupershmidt equation [10]. We show that a converse multipotentialisation of (4.1) leads to a $\triangle$-auto-Bäcklund transformation of type II.

Our aim is to find 5th-order equations of the form

$$v_t = F(v, v_x, v_{xx}, \ldots, v_{5x}), \tag{4.2}$$

such that (4.2) potentialises in (4.1). The auxiliary system for (4.2) is

$$u_x = \Phi_1^t(v, v_x, \ldots) \tag{4.3a}$$

$$u_t = -\Phi_1^x(v, v_x, \ldots). \tag{4.3b}$$

By Proposition 1 we obtain the following condition on Φ_1^t:

$$\begin{aligned} &D_x^5\Phi_1^t - \frac{25}{2}(\Phi_1^t)^{-1}D_x^2\Phi^t D_x^3\Phi^t - 5(\Phi_1^t)^{-1}D_x\Phi_1^t D_x^4\Phi_1^t + \frac{85}{4}(\Phi_1^t)^{-2}(D_x\Phi_1^t)^2 D_x^3\Phi_1^t \\ &\quad + \frac{145}{4}(\Phi_1^t)^{-2}D_x\Phi_1^t(D_x^2\Phi_1^t)^2 - \frac{265}{4}(\Phi_1^t)^{-3}(D_x\Phi_1^t)^3 D_x^2\Phi_1^t \\ &\quad + \frac{405}{16}(\Phi_1^t)^{-4}(D_x\Phi_1^t)^5 = D_t\Phi_1^t\Big|_{v_t=F(v,v_x,\ldots,v_{5x})}. \end{aligned} \tag{4.4}$$

A first-degree converse potentialisation of (4.1) is then obtained by solving (4.4) for Φ_1^t and F. One of the solutions is

$$\Phi_1^t = v^4 v_x^{-2} \tag{4.5}$$

for the equation

$$v_t = v_{5x} - \frac{5v_{xx}v_{4x}}{v_x} + \frac{5v_{xx}v_{xxx}}{v_x^2}. \tag{4.6}$$

For a second-degree converse multipotentialisation of (4.1), we apply a first-degree converse potentialisation on (4.6). That is, we seek an equation of the form

$$V_t = G(V, V_x, V_{xx}, \ldots, V_{5x}) \tag{4.7}$$

that would potentialise in (4.6). The auxiliary system for (4.7) is

$$v_x = \Phi_2^t(V, V_x, \ldots) \tag{4.8a}$$

$$v_t = -\Phi_2^x(V, V_x, \ldots) \tag{4.8b}$$

and by Proposition 1 we obtain the following condition on Φ_2^t:

$$\begin{aligned} &D_x^5\Phi_2^t + 5(\Phi_2^t)^{-2}(D_x\Phi_2^t)^2 D_x^3\Phi_2^t - 5(\Phi_2^t)^{-1}D_x^2\Phi_2^t D_x^3\Phi^t - 5(\Phi_2^t)^{-1}D_x\Phi_2^t D_x^4\Phi^t \\ &\quad - 10(\Phi_2^t)^{-3}(D_x\Phi_2^t)^2 D_x^2\Phi_2^t + 5(\Phi_2^t)^{-2}(D_x^2\Phi_2^t)^2 \\ &\quad + 5(\Phi_2^t)^{-2}D_x\Phi_2^t D_x^3\Phi_2^t = D_t\Phi_2^t|_{V_t=G(V,V_x,\ldots,V_{5x})}. \end{aligned} \tag{4.9}$$

A solution of (4.9) is

$$\Phi_2^t = V\,V_x^{-1/2}, \tag{4.10}$$

for the equation

$$V_t = V_{5x} - \frac{5V_{xx}V_{4x}}{V_x} - \frac{15}{4}\frac{V_{xxx}^2}{V_x} + \frac{65}{4}\frac{V_{xx}^2 V_{xxx}}{V_x^2} - \frac{135}{16}\frac{V_{xx}^4}{V_x^3}. \tag{4.11}$$

We note in passing that the Eqs. (4.11) and (4.1) are identical equations. Hence we have a $\triangle$-auto-Bäcklund transformation of type II for equation (4.1). This auto-Bäcklund transformation is given by the composition of

$$u_x = v^4 v_x^{-1}, \quad v_x = V\, V_x^{-1/2} \tag{4.12}$$

that leads to

Proposition 4. *A $\triangle$-auto-Bäcklund transformation of type II for (4.1), viz.*

$$u_t = u_{5x} - \frac{5u_{xx}u_{4x}}{u_x} - \frac{15}{4}\frac{u_{xxx}^2}{u_x} + \frac{65}{4}\frac{u_{xx}^2 u_{xxx}}{u_x^2} - \frac{135}{16}\frac{u_{xx}^4}{u_x^3},$$

is given by the relation

$$W_x = \frac{1}{4}\left(\frac{V_{xx}}{V_x} - \frac{2V_x}{V}\right) W + \frac{V^{1/2}}{V_x^{1/4}} \tag{4.13}$$

with

$$W^4(x,t) = \frac{\partial u(x,t)}{\partial x}, \tag{4.14}$$

where V and u satisfy Eq. (4.1).

5. Systems of Evolution Equations in (1+1) Dimensions

We consider a system of m evolution equations of order p in the form

$$u_{j,t} = F_j(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}_{px}), \quad j = 1, 2, \dots, m, \tag{5.1}$$

where

$$\mathbf{u} := (u_1, u_2, \dots, u_m), \quad \mathbf{u}_x := (u_{1,x}, u_{2,x}, \dots, u_{m,x}), \dots,$$
$$\mathbf{u}_{px} := (u_{1,px}, u_{2,px}, \dots, u_{m,px})$$
$$u_{j,kx} := \frac{\partial^k u_j}{\partial x^k}.$$

Assume that (5.1) admits m conserved currents, $\{\Phi_1^t, \Phi_2^t, \dots, \Phi_m^t\}$, with corresponding flux, $\{\Phi_1^x, \Phi_2^x, \dots, \Phi_m^x\}$, and the notation

$$\mathbf{\Phi}^t := (\Phi_1^t, \dots, \Phi_m^t), \quad \mathbf{\Phi}^x := (\Phi_1^x, \dots, \Phi_m^x).$$

That is

$$D_t \Phi_j^t(x, \mathbf{u}, \mathbf{u}_x, \dots) + D_x \Phi_j^x(x, \mathbf{u}, \mathbf{u}_x, \dots)|_{\mathbf{u}_t = \mathbf{F}(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}_{px})} = 0$$
$$j = 1, 2, \dots, m. \tag{5.2}$$

We now introduce m potential variables $\{v_1, v_2, \dots, v_m\}$, such that

$$v_{j,x} = \Phi_j^t(x, \mathbf{u}, \mathbf{u}_x, \dots) \tag{5.3a}$$
$$v_{j,t} = -\Phi_j^x(x, \mathbf{u}, \mathbf{u}_x, \dots), \tag{5.3b}$$

with corresponding potential system

$$v_{j,t} = H_j(\mathbf{v}_x, \mathbf{v}_{xx}, \ldots, \mathbf{v}_{px}) + \sum_{i=1}^{m} \gamma_{ij} v_i, \quad j = 1, 2, \ldots, m. \tag{5.4}$$

Analogue to Proposition 1, we now have

Proposition 5. *The condition on Φ_j^t which allows system (5.1) to be potentialised in system (5.4) is given by the following conditions*:

$$D_x H_j(\mathbf{\Phi}^t, D_x \mathbf{\Phi}^t, \ldots, D_x^{p-1} \mathbf{\Phi}^t) + \sum_{i=1}^{m} \gamma_{ij} \Phi_i^t = D_t \Phi_j^t \Big|_{\mathbf{u}_t = \mathbf{F}(\mathbf{u}, \mathbf{u}_x, \ldots, \mathbf{u}_{px})}$$

$$j = 1, 2, \ldots, m, \tag{5.5}$$

where $H_1, H_2, \ldots, H_m$ are given functions and γ_{ij} are given constants.

Similar to the case of scalar equations, we can define $\triangle$-auto-Bäcklund transformations of type I, II and III for systems of the form (5.1). An example of a $\triangle$-auto-Bäcklund transformation of type I is given below.

We now consider systems of the form (5.1) that can be potentialised or mutipotentialised in a linear system of m evolution equations of order p,

$$v_{j,t} = (\mathcal{L}_j^{(p)}[\boldsymbol{\alpha}_1^j], \mathcal{L}_j^{(p)}[\boldsymbol{\alpha}_2^j], \ldots, \mathcal{L}_j^{(p)}[\boldsymbol{\alpha}_m^j]) \cdot (v_1, v_2, \ldots, v_m)$$

$$\equiv \sum_{k=0}^{m} \mathcal{L}_j^{(p)}[\boldsymbol{\alpha}_k^j]\, v_k \quad j = 1, 2, \ldots, m, \tag{5.6}$$

where $\mathcal{L}_j^{(p)}$ is the linear operator of order p and $\boldsymbol{\alpha}_k^j = (\alpha_{k0}^j, \alpha_{k1}^j, \ldots, \alpha_{kp}^j)$ are constants. Here the linear operator $\mathcal{L}_j^{(p)}[\boldsymbol{\alpha}_k^j]$ is defined as follows:

$$\mathcal{L}_j^{(p)}[\boldsymbol{\alpha}_k^j] := \alpha_{k0}^j\, D_x^0 + \alpha_{k1}^j\, D_x^1 + \cdots + \alpha_{kp}^j\, D_x^p. \tag{5.7}$$

We consider an example of such linearisable systems.

Example. Let

$$v_{1,t} = v_{1,xx} \tag{5.8a}$$

$$v_{2,t} = v_{2,xx} \tag{5.8b}$$

and find F_1 and F_2 such that

$$u_{1,t} = F_1(u_1, u_2, u_{1,x}, u_{2,x}, u_{1,xx}, u_{2,xx}) \tag{5.9a}$$

$$u_{2,t} = F_2(u_1, u_2, u_{1,x}, u_{2,x}, u_{1,xx}, u_{2,xx}). \tag{5.9b}$$

The associated auxiliary system for (5.9a) and (5.9b) is

$$v_{1,x} = \Phi_1^t(x, u_1, u_2, u_{1,x}, u_{2,x}, \ldots), \quad v_{1,t} = -\Phi_1^x(x, u_1, u_2, u_{1,x}, u_{2,x}, \ldots) \tag{5.10a}$$

$$v_{2,x} = \Phi_2^t(x, u_1, u_2, u_{1,x}, u_{2,x}, \ldots), \quad v_{2,t} = -\Phi_2^x(x, u_1, u_2, u_{1,x}, u_{2,x}, \ldots). \tag{5.10b}$$