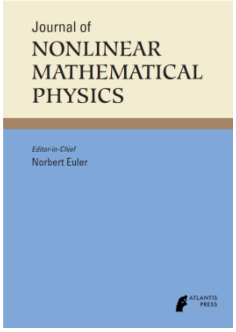

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BINARY DARBOUX TRANSFORMATION FOR THE SUPERSYMMETRIC PRINCIPAL CHIRAL FIELD MODEL

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The standard binary Darboux transformation is investigated and is used to obtain quasideterminant multisoliton solutions of the supersymmetric chiral field model in two dimensions.

Keywords: Supersymmetry; integrable systems; principal chiral model; Darboux transformation.

1. Introduction

The supersymmetric chiral field model is an important example of two dimensional integrable classical and quantum field theory [4, 7–9, 14, 15, 20, 23, 30, 36, 39]. The model belongs to a general class of supersymmetric sigma models with symmetric spaces as their target manifolds. The supersymmetric chiral field model exhibits various important features of four dimensional gauge theories and they are also useful in the sense that their exact analysis leads to the various applications of D-brane dynamics [1–3, 12, 13, 25]. We studied the elementary Darboux transformation of supersymmetric principal chiral field model in two dimensions in a recent work [23] and obtained the quasideterminant multisoliton solutions of the model. We also related these solutions with the solutions obtained by Mikhailov using the dressing method [30]. We also constructed the standard binary Darboux transformation of principal chiral model in two dimensions and expressed the grammian type multisoliton solutions in terms of quasideterminants in another work [24].

In this paper we use the method employed in [24] and construct the standard binary Darboux transformation to find the exact multisolitons of supersymmetric chiral field model in two dimensions. The binary Darboux transformation is one of the well known techniques used to construct grammian type multisolitons of integrable systems [31–34]. We study standard binary Darboux transformation by introducing superfield Darboux matrix for the supersymmetric chiral model for the direct and the adjoint spaces and then obtain superfield binary Darboux matrix by composing the two Darboux transformations. We obtain the superfield multisoliton solutions using the iterated binary Darboux transformations. We also show the relation between standard binary Darboux transformation and elementary binary Darboux transformation. The solutions are also obtained for the component fields. The main results of the paper are as follows: the superfield generalization of the standard binary Darboux transformation, the construction of superfield multisolitons of grammian type and their expression in terms of quasideterminants.

The superspace Lagrangian of the supersymmetric chiral field with target space $\mathcal{G}$ is given by

$$\mathcal{L}_{\mathrm{SCF}} = \frac{1}{2}\mathrm{Tr}(D_+G^{-1}D_-G), \tag{1.1}$$

and $G(x,\theta)G^{-1}(x,\theta) = 1 = G^{-1}(x,\theta)G(x,\theta)$. The superspace Lagrangian $\mathcal{L}_{\mathrm{SCF}}$ is invariant under the global transformation $\mathcal{G}_L \times \mathcal{G}_R : G(x^\pm,\theta^\pm) = \mathcal{U}_L G(x^\pm,\theta^\pm)\mathcal{U}_R^{-1}$, where $\mathcal{U}_L$ and $\mathcal{U}_R$ are $\mathcal{G}_L$ and $\mathcal{G}_R$ valued Grassmann even matrix superfields, respectively. The superfield equation of motion of the supersymmetric chiral field is the superfield conservation equation and the superfield zero-curvature condition[a]

$$D_+J_- - D_-J_+ = 0, \tag{1.2}$$

$$D_-J_+ + D_+J_- + \mathrm{i}\,\{J_+,J_-\} = 0. \tag{1.3}$$

The Noether conserved superfield currents associated with $\mathcal{G}_R$ are $J_\pm = -\mathrm{i}G^{-1}D_\pm G$ which are Grassmann odd superfields and are Lie algebra $\mathfrak{g}$ valued, i.e., $J_\pm = J_\pm^a T^a$, where anti-hermitian generators $\{T^a,\, a = 1,2,\ldots,\dim\mathfrak{g}\}$ of the Lie algebra $\mathfrak{g}$ obey $[T^a,T^b] = f^{abc}T^c$ and $\mathrm{Tr}(T^aT^b) = \delta^{ab}$. For any $X \in \mathfrak{g}$, $X = X^aT^a$. The conserved superfield current corresponding to the $\mathcal{G}_L$ transformation is $-GJ_\pm G^{-1}$ and $G(x^\pm,\theta^\pm)$ is the Grassmann even superfield with values in a Lie group $\mathcal{G}$ and $G(x,\theta)G^{-1}(x,\theta) = 1 = G^{-1}(x,\theta)G(x,\theta)$. We can expand the matrix superfield $G(x^\pm,\theta^\pm)$ in terms of bosonic and fermionic component fields as

$$G(x,\theta) = g(x)(1 + \mathrm{i}\theta^+\psi_+(x) + \mathrm{i}\theta^-\psi_-(x) + \mathrm{i}\theta^+\theta^-F(x)), \tag{1.4}$$

where $\psi_\pm$ are the Majorana spinors such that $\psi_\pm^R = g^{-1}\psi_\pm^L g$, and $F(x)$ is the auxiliary field, with an algebraic equation of motion.[b] The Majorana spinors $\psi_\pm(x)$ take values in the Lie

[a]In our notation the superspace is defined with coordinates $(x^\pm,\theta^\pm)$, where $x^\pm$ are the commuting light-cone coordinates of the two dimensional Minkowski space $\mathbb{R}^{1,1}$ with metric $\eta_{\mu\nu} = \mathrm{diag}(1,-1)$ and $\theta^\pm$ are anticommuting fermionic coordinates and are real Grassmann numbers. The light-cone coordinates $x^\pm$ are related to the orthonormal coordinates by $x^\pm = \frac{1}{2}(t \pm x)$ with the derivatives $\partial_\pm = \partial_t \pm \partial_x$. The $N = 1$ supercharges are $Q_\pm = \frac{\partial}{\partial\theta^\pm} + \mathrm{i}\theta^\pm\partial_\pm$, and the superspace covariant derivatives are $D_\pm = \frac{\partial}{\partial\theta^\pm} - \mathrm{i}\theta^\pm\partial_\pm$, which obey $\{Q_\pm,Q_\pm\} = \mathrm{i}2\partial_\pm$, $\{D_\pm,D_\pm\} = -\mathrm{i}2\partial_\pm$, with all other anti-commutators vanishing. Under the Lorentz transformation the coordinates of superspace and their derivatives transform as $x^\pm \mapsto e^{\pm\eta}x^\pm$, $\partial_\pm \mapsto e^{\mp\eta}\partial_\pm$, $\theta_\pm \mapsto e^{\pm\frac{1}{2}\eta}\theta_\pm$, and $D_\pm \mapsto e^{\mp\frac{1}{2}\eta}D_\pm$, where η is the rapidity of the Lorentz boost.

[b]Note that our conventions are such that the γ-matrices $\gamma_0 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}$, $\gamma_1 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$ satisfy $\{\gamma_\mu,\gamma_\nu\} = 2\eta_{\mu\nu}$. The Dirac spinor is $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$, where $\psi_\pm$ are chiral spinors and our assumption is that $\psi_\pm$ are real (Majorana) and the Lorentz behavior of these Majorana spinors is $\psi_\pm \mapsto e^{\mp\frac{1}{2}\eta}\psi_\pm$.

algebra $\mathfrak{g}$ of $\mathcal{G}$. The action of the symmetry $\mathcal{G}_L \times \mathcal{G}_R$ on component fields is given by

$$g \mapsto UgV^{-1}, \quad \psi^R_\pm \mapsto V\psi^R_\pm V^{-1}, \quad \psi^L_\pm \mapsto U\psi^L_\pm U^{-1}, \tag{1.5}$$

where U_L and U_R are the leading bosonic components of the matrix superfields $\mathcal{U}_L$ and $\mathcal{U}_R$, respectively, i.e. the fermions transform under $\mathcal{G}_R$ or $\mathcal{G}_L$. The component expansion of the superfield current $J_\pm$ is

$$J_\pm = \psi_\pm + \theta^\pm j_\pm - \frac{\mathrm{i}}{2}\theta^\mp\{\psi_+, \psi_-\} - \mathrm{i}\theta^+\theta^-\left(\partial_\pm\psi_\mp - [j_\pm, \psi_\mp] - \frac{\mathrm{i}}{2}[\psi^2_\pm, \psi_\mp]\right),$$

where the component bosonic conserved current is $j_\pm = -(g^{-1}\partial_\pm g + \mathrm{i}\psi^2_\pm)$. Substituting these into the superspace equations of motion, collecting terms and writing $h_\pm = \psi^2_\pm \Leftrightarrow h^a_\pm = \frac{1}{2}f^{abc}\psi^b_\pm\psi^c_\pm$, we get the equations of motion for fermionic fields $\psi_\pm$ and bosonic fields $j_\pm$ of the supersymmetric chiral field model,

$$\partial_\pm\psi_\mp - \frac{1}{2}[j_\pm, \psi_\mp] - \frac{\mathrm{i}}{4}[h_\pm, \psi_\mp] = 0, \tag{1.6}$$

$$\partial_- j_+ + \partial_+ j_- = 0. \tag{1.7}$$

We use the fermion equations of motion to get the following equations

$$\partial_- j_+ - \partial_+ j_- + [j_+, j_-] = \mathrm{i}(\partial_- h_+ - \partial_+ h_-), \tag{1.8}$$

$$\partial_\mp h_\pm + \frac{1}{2}\left[h_\pm, j_\mp + \frac{\mathrm{i}}{2}h_\mp\right] = 0, \tag{1.9}$$

$$\partial_- h_+ + \partial_+ h_- = 0. \tag{1.10}$$

The Eqs. (1.7) and (1.10) show the conservation of bosonic currents $j_\pm$ and $h_\pm$, respectively.

The supersymmetric chiral field Eqs. (1.2) and (1.3) can be written as the compatibility condition of the following superfield Lax pair

$$D_\pm\mathcal{U}(\lambda) = -\frac{\mathrm{i}}{1 \mp \lambda}J_\pm\mathcal{U}(\lambda), \tag{1.11}$$

where λ is a real (or complex) parameter and $\mathcal{U}$ is an $n \times n$ matrix superfield. The component expansion of the Lax pair (1.11) gives

$$\partial_\pm U(x^\pm; \lambda) = \frac{1}{1 \mp \lambda}j_\pm U(x^\pm; \lambda) + \mathrm{i}\left(\frac{1}{1 \mp \lambda}\right)^2 h_\pm U(x^\pm; \lambda), \tag{1.12}$$

where $U(x^\pm, \lambda)$ is the leading purely bosonic component of the superfield $\mathcal{U}(x^\pm, \theta^\pm, \lambda)$. In this case, the compatibility condition of (1.12) gives Eqs. (1.6) and (1.7). Therefore, the partial differential Eqs. (1.2) and (1.3) have the Lax pair (1.11), while Eqs. (1.6) and (1.10) have the Lax pair (1.12)).

2. Darboux Transformation on the Direct and Adjoint Lax Pairs

The one-fold Darboux transformation (see [5, 6, 10, 11, 26, 28, 29, 35, 38] for details) on the matrix solution of the Lax pair (1.11) is defined by

$$\widetilde{\mathcal{U}}(\lambda) = \mathcal{D}(x^+, x^-, \lambda)\mathcal{U}(\lambda), \tag{2.1}$$

where $\mathcal{D}(x,t,\lambda)$ is the superfield Darboux matrix given as

$$\mathcal{D}(x^+,x^-,\lambda) = \lambda I - \mathcal{S}(x^+,x^-), \tag{2.2}$$

and $\mathcal{S}(x^+,x^-)$ is the $n\times n$ matrix superfield and I is an $n\times n$ identity matrix. The superfield Darboux matrix transforms the matrix superfield solution $\mathcal{U}$ in space V to a new matrix superfield solution $\widetilde{\mathcal{U}}$ in $\widetilde{V}$, i.e.

$$\mathcal{D}(\lambda) : V \to \widetilde{V}. \tag{2.3}$$

The new superfield solution $\widetilde{\mathcal{U}}(x^+,x^-,\lambda)$ satisfies the following Lax pair,

$$D_\pm \widetilde{\mathcal{U}}(\lambda) = -\frac{\mathrm{i}}{1\mp\lambda}\tilde{J}_\pm \widetilde{\mathcal{U}}(\lambda), \tag{2.4}$$

where $\tilde{J}_\pm$ are the Lie algebra valued transformed superfield conserved currents that satisfy the equation of motion (1.2) and zero-curvature condition (1.3). Note that the Lax operators for supersymmetric chiral model are self-adjoint and under elementary Darboux transformation the transformed Lax operators are not self-adjoint. To solve this problem we use the binary Darboux transformation which preserves the self-adjointness of the Lax operators. We will define binary Darboux transformation in the next section. By substituting Eq. (2.1) in Eq. (2.4), we get

$$\tilde{J}_\pm = J_\pm \pm \mathrm{i}\, D_\pm \mathcal{S},$$

and the matrix superfield $\mathcal{S}$ is subjected to satisfy the following equations:

$$D_\pm \mathcal{S}(I\mp\mathcal{S}) = -\mathrm{i}[J_\pm,\mathcal{S}].$$

The bosonic superfield conserved currents $D_\pm J_\pm$ and $J_\pm^2$ transform as

$$D_\pm \tilde{J}_\pm = D_\pm J_\pm \pm \partial_\pm \mathcal{S}, \quad \tilde{J}_\pm^2 = (I\mp\mathcal{S})J_\pm^2(I\mp\mathcal{S})^{-1},$$

and the conditions on $\mathcal{S}$ are now written as

$$\partial_\pm \mathcal{S}(I\mp\mathcal{S}) = \pm D_\pm J_\pm \mathcal{S} \pm \mathcal{S} D_\pm J_\pm - \mathrm{i}J_\pm^2 + \mathrm{i}(I\mp\mathcal{S})J_\pm^2(I\mp\mathcal{S})^{-1}.$$

The matrix superfield $\mathcal{S}$ can be written in terms of the solutions of the linear system [23]

$$\mathcal{S} = \mathcal{M}\Lambda\mathcal{M}^{-1}, \tag{2.5}$$

where $\mathcal{M}$ is the matrix superfield defined by

$$\mathcal{M} = (\mathcal{U}(\lambda_1)|1\rangle,\ldots,\mathcal{U}(\lambda_n)|n\rangle) = (|\Theta_1\rangle,\ldots,|\Theta_n\rangle), \tag{2.6}$$

each column $|\Theta_i\rangle = \mathcal{U}(\lambda_i)|i\rangle$ in $\mathcal{M}$ is a column superfield solution of the superfield Lax pair (1.11) when $\lambda = \lambda_i$. i.e., it satisfies

$$D_\pm|\Theta_i\rangle = -\frac{\mathrm{i}}{1 \mp \lambda_i} J_\pm|\Theta_i\rangle, \tag{2.7}$$

and $i = 1, 2, \dots, n$. Assuming $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, the Eq. (2.7) can be written in matrix form as

$$D_\pm\mathcal{M} = -\mathrm{i}J_\pm\mathcal{M}(I \mp \Lambda)^{-1}.$$

The Darboux transformation of the chiral model in terms of particular matrix solution $\mathcal{M}$ with the particular eigenvalue matrix Λ is given as

$$\begin{aligned} \widetilde{\mathcal{U}} &= (\lambda I - \mathcal{M}\Lambda\mathcal{M}^{-1})\mathcal{U}, \\ \tilde{J}_\pm &= \mathcal{M}(I \mp \Lambda)\mathcal{M}^{-1}J_\pm\mathcal{M}(I \mp \Lambda)^{-1}\mathcal{M}^{-1}. \end{aligned} \tag{2.8}$$

The Darboux transformation on the chiral field $G(x)$ is now defined by

$$\tilde{G} = \widetilde{\mathcal{U}}(0) = -\mathcal{S}G = -(\mathcal{M}\Lambda\mathcal{M}^{-1}). \tag{2.9}$$

The result can be generalized to obtain K-fold Darboux transformation on matrix solution $\mathcal{U}$. Note that $\mathcal{M}$ and $\mathcal{U}$ are $n \times n$ matrices (i.e. matrix superfields) and since matrix multiplication is noncommutative in general, therefore the transformed solution $\mathcal{U}[K+1]$ can be written in terms of quasideterminant as[c] (for more details see [23])

$$\mathcal{U}[K+1] = \begin{vmatrix} \mathcal{M}_1 & \cdots & \mathcal{M}_K & \mathcal{U} \\ \mathcal{M}_1\Lambda_1 & \cdots & \mathcal{M}_K\Lambda_K & \lambda\mathcal{U} \\ \vdots & \ddots & \vdots & \vdots \\ \mathcal{M}_1\Lambda_1^K & \cdots & \mathcal{M}_K\Lambda_K^K & \boxed{\lambda^K\mathcal{U}} \end{vmatrix}.$$

The multisoliton superfield solution $G[K+1]$ of the supersymmetric chiral model can be readily obtained by taking $\lambda = 0$ in the expression of $\mathcal{U}[K+1]$ i.e.,

$$G\,[K+1] = \begin{vmatrix} \mathcal{M}_1 & \cdots & \mathcal{M}_K & I \\ \mathcal{M}_1\Lambda_1 & \cdots & \mathcal{M}_K\Lambda_K & O \\ \vdots & \ddots & \vdots & \vdots \\ \mathcal{M}_1\Lambda_1^K & \cdots & \mathcal{M}_K\Lambda_K^K & \boxed{O} \end{vmatrix} G.$$

[c]The quasideterminant for an $N \times N$ matrix over a ring R is defined as

$$|X|_{ij} = \begin{vmatrix} X^{ij} & c_j^i \\ r_i^j & \boxed{x_{ij}} \end{vmatrix} = x_{ij} - r_i^j(X^{ij})^{-1}c_j^i. \tag{2.10}$$

where for $1 \le i, j \le N$, r_i^j is the row matrix obatined by removing jth entry of X from the ith row. Similarly, c_j^i is the column matrix containing jth column of X without ith entry. There exist N^2 quasideterminants denoted by $|X|_{ij}$ for $i, j = 1, \dots, N$. For various properties and applications of quasideterminants in the theory of integrable systems, see e.g. [16–19, 27].

Similarly the expression for the superfield currents $J_{\pm}[K+1]$ is

$$J_{\pm}[K+1] = \begin{vmatrix} \mathcal{M}_1 & \mathcal{M}_2 & \cdots & \mathcal{M}_N & I \\ \mathcal{M}_1\,(I\mp\Lambda_1) & \mathcal{M}_2\,(I\mp\Lambda_2) & \cdots & \mathcal{M}_K\,(I\mp\Lambda_K) & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{M}_1\,(I\mp\Lambda_1)^K & \mathcal{M}_2\,(I\mp\Lambda_2)^K & \cdots & \mathcal{M}_K\,(I\mp\Lambda_K)^K & \boxed{O} \end{vmatrix}$$

$$\times J_{\pm} \times \begin{vmatrix} \mathcal{M}_1 & \cdots & \mathcal{M}_N & I \\ \mathcal{M}_1\,(I\mp\Lambda_1) & \cdots & \mathcal{M}_K\,(I\mp\Lambda_K) & O \\ \vdots & \ddots & \vdots & \vdots \\ \mathcal{M}_1\,(I\mp\Lambda_1)^K & \cdots & \mathcal{M}_K\,(I\mp\Lambda_K)^K & \boxed{O} \end{vmatrix}^{-1}. \tag{2.11}$$

The K-fold Darboux transformation on the matrix superfields $\mathcal{U}, G$ and $J_{\pm}$ can also be expressed in terms of superfield hermitian projectors $\mathcal{P}[K]$ i.e.,

$$\mathcal{U}[K+1] = \prod_{k=0}^{K}\left(I - \frac{\mu_{K-k+1}-\bar{\mu}_{K-k+1}}{\lambda-\bar{\mu}_{K-k+1}}\mathcal{P}[K-k+1]\right)\mathcal{U}, \tag{2.12}$$

$$J_{\pm}[K+1] = \prod_{k=0}^{K}\left(I \mp \frac{\mu_{K-k+1}-\bar{\mu}_{K-k+1}}{1-\bar{\mu}_{K-k+1}}\mathcal{P}[K-k+1]\right)J_{\pm}\prod_{l=1}^{K}\left(I \mp \frac{\bar{\mu}_l-\mu_l}{1-\bar{\mu}_l}\mathcal{P}[l]\right), \tag{2.13}$$

$$G\,[K+1] = \prod_{k=1}^{K}\left(I + \frac{\mu_{K-k+1}-\bar{\mu}_{K-k+1}}{\bar{\mu}_{K-k+1}}\mathcal{P}[K-k+1]\right)G, \tag{2.14}$$

where the superfield Hermitian projection in this case is

$$\mathcal{P}[k] = \sum_{i=1}^{n}\frac{|\Theta_i[k]\rangle\langle\Theta_i[k]|}{\langle\Theta_i[k]|\Theta_i[k]\rangle}, \qquad k = 1, 2, \ldots, K, \tag{2.15}$$

with $\mathcal{P}^{\dagger}[K] = \mathcal{P}[K]$ and $\mathcal{P}^2[K] = \mathcal{P}[K]$.

The equation of motion (1.2) and zero-curvature condition (1.3) can also be written as compatibility condition of the following linear system

$$D_{\pm}\mathcal{W}(\xi) = \frac{\mathrm{i}}{1\mp\xi}J_{\pm}^{\dagger}\mathcal{W}(\xi), \tag{2.16}$$

which is obtained by taking the formal adjoint of the system (1.11). Note that in Eq. (2.16) ξ is a real (or complex) parameter and $\mathcal{W}$ is an invertible $n\times n$ matrix superfield in the space $V^{\dagger} = \{\mathcal{W}\}$. The Darboux matrix $\mathcal{D}(\xi)$ transforms the matrix superfield solution $\mathcal{W}$ in space $V^{\dagger}$ to a new matrix solution $\widetilde{\mathcal{W}}$ in $\tilde{V}^{\dagger}$ i.e.,

$$\mathcal{D}(\xi) : V^{\dagger} \to \tilde{V}^{\dagger}. \tag{2.17}$$

The one-fold Darboux transformation on the matrix solution $\mathcal{W}$ is defined as i.e.,

$$\widetilde{\mathcal{W}} \equiv \mathcal{D}(\xi)\mathcal{W} = -(\xi I - \mathcal{H}\Xi\mathcal{H}^{-1})\mathcal{W}, \tag{2.18}$$

where $\Xi = \mathrm{diag}(\xi_1, \ldots, \xi_n)$ is the eigenvalue matrix. The matrix function $\mathcal{H}$ is an invertible non-degenerate $n \times n$ matrix and is given by

$$\mathcal{H} = (\mathcal{W}(\xi_1)|1\rangle, \ldots, \mathcal{W}(\xi_n)|n\rangle) = (|h_1\rangle, \ldots, |h_n\rangle).$$

The K-fold Darboux transformation on matrix superfield solutions $\mathcal{W}, G^\dagger$ and $J_\pm^\dagger$ can be expressed as

$$\mathcal{W}[K+1] = \begin{vmatrix} \mathcal{H}_1 & \cdots & \mathcal{H}_K & \mathcal{W} \\ \mathcal{H}_1\Xi_1 & \cdots & \mathcal{H}_N\Xi_K & \xi\mathcal{W} \\ \vdots & \ddots & \vdots & \vdots \\ \mathcal{H}_1\Xi_1^K & \cdots & \mathcal{H}_K\Xi_K^K & \boxed{\xi^K\mathcal{W}} \end{vmatrix}, \tag{2.19}$$

$$G^\dagger[K+1] = \begin{vmatrix} \mathcal{H}_1 & \cdots & \mathcal{H}_K & I \\ \mathcal{H}_1\Xi_1 & \cdots & \mathcal{H}_N\Xi_K & O \\ \vdots & \ddots & \vdots & \vdots \\ \mathcal{H}_1\Xi_1^K & \cdots & \mathcal{H}_K\Xi_K^K & \boxed{O} \end{vmatrix} G, \tag{2.20}$$

$$J_\pm^\dagger[K+1] = \begin{vmatrix} \mathcal{H}_1 & \cdots & \mathcal{H}_K & I \\ \mathcal{H}_1(I \mp \Xi_1) & \cdots & \mathcal{H}_K(I \mp \Xi_K) & O \\ \vdots & \ddots & \vdots & \vdots \\ \mathcal{H}_1(I \mp \Xi_1)^K & \cdots & \mathcal{H}_K(I \mp \Xi_K)^K & \boxed{O} \end{vmatrix} \times J_\pm^\dagger \times$$

$$\begin{vmatrix} \mathcal{H}_1 & \cdots & \mathcal{H}_K & I \\ \mathcal{H}_1(I \mp \Xi_1) & \cdots & \mathcal{H}_K(I \mp \Xi_K) & O \\ \vdots & \ddots & \vdots & \vdots \\ \mathcal{H}_1(I \mp \Xi_1)^K & \cdots & \mathcal{H}_K(I \mp \Xi_K)^K & \boxed{O} \end{vmatrix}^{-1}. \tag{2.21}$$

In terms of the Hermitian projector we write the above expressions as

$$\mathcal{W}[K+1] = \prod_{k=0}^{K} \left(I - \frac{\nu_{K-k+1} - \bar{\nu}_{K-k+1}}{\xi - \bar{\nu}_{K-k+1}} \mathcal{P}[K-k+1] \right) \mathcal{W}, \tag{2.22}$$

$$J_\pm^\dagger[K+1] = \prod_{k=0}^{K} \left(I \mp \frac{\nu_{K-k+1} - \bar{\nu}_{K-k+1}}{1 - \bar{\nu}_{K-k+1}} \mathcal{P}[K-k+1] \right) J_\pm^\dagger \prod_{l=1}^{K} \left(I \mp \frac{\bar{\nu}_l - \nu_l}{1 - \bar{\nu}_l} \mathcal{P}[l] \right), \tag{2.23}$$

$$G^\dagger\,[K+1] = \prod_{k=1}^{K} \left(I + \frac{\nu_{K-k+1} - \bar{\nu}_{K-k+1}}{\bar{\nu}_{K-k+1}} \mathcal{P}\,[K-k+1] \right) G^\dagger, \tag{2.24}$$

and the superfield hermitian projector in this case is defined as

$$\mathcal{P}[k] = \sum_{i=1}^{n} \frac{|h_i[k]\rangle\langle h_i[k]|}{\langle h_i[k]|h_i[k]\rangle}, \quad k = 1, 2, \ldots, K. \tag{2.25}$$

By making use of Eqs. (1.11) and (2.16) for the column solutions $|\Theta_i\rangle$ and the row solutions $\langle h_i|$ of the direct and adjoint Lax pair, respectively, it can be easily shown that the expressions (2.15) and (2.25) are equivalent.

3. Standard Binary Darboux Transformation

To define the binary transformation we follow the approach of [31–34] and consider a space $\hat{V}$, which is a copy of the direct space V and the corresponding solutions are $\hat{\mathcal{U}} \in \hat{V}$. Since it is a copy of the direct space, therefore the linear system, equation of motion and the zero curvature condition will have the similar form as given for the direct space. The equation of motion (1.2) and zero-curvature condition (1.3) can also be written as the compatibility condition of the following linear system for the matrix solution $\hat{\mathcal{U}}$

$$D_{\pm}\hat{\mathcal{U}}(\lambda) = -\frac{\mathrm{i}}{1 \mp \lambda}\hat{J}_{\pm}\hat{\mathcal{U}}(\lambda). \tag{3.1}$$

We have taken the specific solutions $\mathcal{M}$, $\mathcal{H}$ for the direct and adjoint spaces V and $V^{\dagger}$, respectively. The corresponding solutions for $\hat{V}$ are $\hat{\mathcal{M}} \in \hat{V}$ and $\hat{\mathcal{W}} \in \hat{V}^{\dagger}$. Also assuming that $i(\hat{\mathcal{M}}) \in \tilde{V}^{\dagger}$, then from Eqs. (2.3) and (2.17), we write the transformation as

$$\mathcal{D}^{(-1)\dagger}(\lambda) : V^{\dagger} \to \tilde{V}^{\dagger}.$$

Since $\mathcal{W} \in V^{\dagger}$, we have

$$i(\hat{\mathcal{M}}) = \mathcal{D}^{(-1)\dagger}(\lambda)\mathcal{W}. \tag{3.2}$$

Also from $\mathcal{D}^{\dagger}(\lambda)(i(\mathcal{M})) = 0$, we obtain $i(\mathcal{M}) = \mathcal{M}^{(-1)\dagger}$ and similarly $i(\hat{\mathcal{M}}) = \hat{\mathcal{M}}^{(-1)\dagger}$. Therefore we get from above equation

$$\begin{aligned} \hat{\mathcal{M}}^{(-1)\dagger} &= \mathcal{D}^{(-1)\dagger}(\lambda)\mathcal{W}, \\ \hat{\mathcal{M}} &= (\mathcal{D}^{(-1)\dagger}(\lambda)\mathcal{W})^{(-1)\dagger}. \end{aligned} \tag{3.3}$$

By using (2.2) and (2.5) in above equation

$$\begin{aligned} \hat{\mathcal{M}} &= ((\lambda I - \mathcal{M}\Lambda\mathcal{M}^{-1})^{(-1)\dagger}\mathcal{W})^{(-1)\dagger}, \\ &= (\lambda I - \mathcal{M}\Lambda\mathcal{M}^{-1})\mathcal{W}^{(-1)\dagger}, \\ &= \mathcal{M}(\lambda I - \Lambda)\mathcal{M}^{-1}\mathcal{W}^{(-1)\dagger}, \\ &= \mathcal{M}(\lambda I - \Lambda)(\mathcal{W}^{\dagger}\mathcal{M})^{-1}, \\ &= \mathcal{M}\Delta^{-1}, \end{aligned} \tag{3.4}$$

where the algebraic potential Δ is defined as

$$\Delta(\mathcal{M}, \mathcal{W}) = (\mathcal{W}^{\dagger}\mathcal{M})(\lambda I - \Lambda)^{-1}. \tag{3.5}$$

Note that the binary Darboux transformation is a rule to transform the solutions of the direct and adjoint Lax equations to the new direct and adjoint solutions of the transformed Lax equations. It has been shown in [31–34] that the binary Darboux matrix involves the formal integration ∂^{-1} which is replaced by the potential Δ that plays an important role in the construction of binary transformation. In the present case we represent the Darboux and binary Darboux transformation in terms of matrices and by using the Lax pair, we can replace the derivative ∂ by the eigenvalue matrix Λ which makes the expression for potential Δ explicit and is given by Eq. (3.5). Similarly for adjoint space,

$$\overset{\wedge}{\mathcal{H}} = \mathcal{H}\Delta^{(-1)\dagger},$$

we obtain

$$\Delta(\mathcal{U}, \mathcal{H}) = -(\lambda I - \Xi^{\dagger})^{-1}(\mathcal{H}^{\dagger}\mathcal{U}). \tag{3.6}$$

By writing Eqs. (3.5) and (3.6) in matrix form for the solutions $\mathcal{M}$ and $\mathcal{H}$, we get the following condition on Δ,

$$\Xi^{\dagger}\Delta(\mathcal{M}, \mathcal{H}) - \Delta(\mathcal{M}, \mathcal{H})\Lambda = \mathcal{H}^{\dagger}\mathcal{M}, \tag{3.7}$$

where Δ is a matrix. An entry Δ_{ij} from Eqs. (3.5)–(3.7) is given as

$$\Delta(\mathcal{M}, \mathcal{H})_{ij} = \frac{(\mathcal{H}^{\dagger}\mathcal{M})_{ij}}{\bar{\xi}_i - \lambda_j}. \tag{3.8}$$

Now we define the Darboux matrix in hat space as

$$\hat{\mathcal{D}}(\lambda) \equiv (\lambda I - \overset{\wedge}{\mathcal{S}}) = (\lambda I - \overset{\wedge}{\mathcal{M}}\Xi^{\dagger}\overset{\wedge}{\mathcal{M}}^{-1}), \tag{3.9}$$

where

$$\hat{\mathcal{D}}(\lambda)\overset{\wedge}{\mathcal{U}} = \widetilde{\mathcal{U}}. \tag{3.10}$$

We may summarize the above formulation as

$$\mathcal{D}(\lambda) : V \to \tilde{V}, \tag{3.11}$$

$$\hat{\mathcal{D}}(\lambda) : \hat{V} \to \tilde{V}, \tag{3.12}$$

$$\mathcal{D}(\xi) : V^{\dagger} \to \tilde{V}^{\dagger}. \tag{3.13}$$

The effect of $\hat{\mathcal{D}}(\lambda)$ is such that it leaves the linear system (3.1) invariant. i.e.,

$$D_{\pm}\widetilde{\mathcal{U}}(\lambda) = -\frac{\mathrm{i}}{1 \mp \lambda}\widetilde{\hat{J}}_{\pm}\widetilde{\mathcal{U}}(\lambda), \tag{3.14}$$

which implies the following transformation on the fermionic superfield currents $\hat{J}_{\pm}$

$$\widetilde{\hat{J}}_{\pm} = \hat{J}_{\pm} \pm \mathrm{i}\ D_{\pm}\overset{\wedge}{\mathcal{S}} \tag{3.15}$$

and the matrix superfield $\overset{\wedge}{\mathcal{S}}$ is subjected to satisfy the following equations

$$D_{\pm}\overset{\wedge}{\mathcal{S}}(I \mp \overset{\wedge}{\mathcal{S}}) = -\mathrm{i}[\hat{J}_{\pm}, \overset{\wedge}{\mathcal{S}}]. \tag{3.16}$$

Also for the bosonic superfield conserved currents $D_\pm \hat{J}_\pm$ and $\hat{J}^2_\pm$, the transformation is as follows

$$D_\pm \widetilde{\hat{J}}_\pm = D_\pm \hat{J}_\pm \pm \partial_\pm \overset{\wedge}{\mathcal{S}}, \quad \widetilde{\hat{J}^2}_\pm = (I \mp \overset{\wedge}{\mathcal{S}})\hat{J}^2_\pm (I \mp \overset{\wedge}{\mathcal{S}})^{-1},$$

with $\overset{\wedge}{\mathcal{S}}$ satisfying

$$\partial_\pm \overset{\wedge}{\mathcal{S}}(I \mp \overset{\wedge}{\mathcal{S}}) = \pm D_\pm \hat{J}_\pm \overset{\wedge}{\mathcal{S}} \pm \overset{\wedge}{\mathcal{S}} D_\pm \hat{J}_\pm - \mathrm{i}\hat{J}^2_\pm + \mathrm{i}(I \mp \overset{\wedge}{\mathcal{S}})\hat{J}^2_\pm (I \mp \overset{\wedge}{\mathcal{S}})^{-1}.$$

Again we make the following ansatz for the matrix superfield $\overset{\wedge}{\mathcal{S}}$

$$\overset{\wedge}{\mathcal{S}} = \overset{\wedge}{\mathcal{M}} \Xi^\dagger \overset{\wedge}{\mathcal{M}}^{-1}, \tag{3.17}$$

and the Darboux transformation on the matrix superfields $\overset{\wedge}{\mathcal{U}}$ and $\hat{J}_\pm$ in hat space $\hat{V}$ is

$$\widetilde{\overset{\wedge}{\mathcal{U}}} = (\lambda I - \overset{\wedge}{\mathcal{M}} \Xi^\dagger \overset{\wedge}{\mathcal{M}}^{-1})\overset{\wedge}{\mathcal{U}}, \tag{3.18}$$

$$\widetilde{\hat{J}}_\pm = \overset{\wedge}{\mathcal{M}}(I \mp \Xi^\dagger)\overset{\wedge}{\mathcal{M}}^{-1} \hat{J}_\pm \overset{\wedge}{\mathcal{M}}(I \mp \Xi^\dagger)^{-1}\overset{\wedge}{\mathcal{M}}^{-1}. \tag{3.19}$$

From Eq. (3.12) we know that

$$\hat{\mathcal{D}}(\lambda)\overset{\wedge}{\mathcal{U}} = \mathcal{D}(\lambda)\mathcal{U},$$

which implies

$$\overset{\wedge}{\mathcal{U}} = \hat{\mathcal{D}}^{-1}(\lambda)\mathcal{D}(\lambda)\mathcal{U}. \tag{3.20}$$

The Eq. (3.20) relates the two solutions $\mathcal{U}$ and $\overset{\wedge}{\mathcal{U}}$. This transformation is known as the standard binary Darboux transformation and we write it as $\mathcal{B}(\lambda) = \hat{\mathcal{D}}^{-1}(\lambda)\mathcal{D}(\lambda)$ i.e.,

$$\overset{\wedge}{\mathcal{U}} = \hat{\mathcal{D}}^{-1}(\lambda)\mathcal{D}(\lambda)\mathcal{U} = \mathcal{B}(\lambda)\mathcal{U}. \tag{3.21}$$

By substituting (3.9), (2.2) in Eq. (3.21), we obtain the explicit transformation on $\mathcal{U}$ as

$$\begin{aligned} \overset{\wedge}{\mathcal{U}} &= (\lambda I - \overset{\wedge}{\mathcal{M}} \Xi^\dagger \overset{\wedge}{\mathcal{M}}^{-1})^{-1}(\lambda I - \mathcal{M}\Lambda\mathcal{M}^{-1})\mathcal{U}, \\ &= \overset{\wedge}{\mathcal{M}}(\lambda I - \Xi^\dagger)^{-1}\overset{\wedge}{\mathcal{M}}^{-1}\mathcal{M}(\lambda I - \Lambda)\mathcal{M}^{-1}\mathcal{U}. \end{aligned} \tag{3.22}$$

By using (3.4) in Eq. (3.22), the expression of $\overset{\wedge}{\mathcal{U}}$ may be simplified as

$$\begin{aligned} \overset{\wedge}{\mathcal{U}} &= \mathcal{M}\Delta(\mathcal{M},\mathcal{H})^{-1}(\lambda I - \Xi^\dagger)^{-1}\Delta(\mathcal{M},\mathcal{H})\mathcal{M}^{-1}\mathcal{M}(\lambda I - \Lambda)\mathcal{M}^{-1}\mathcal{U} \\ &= \mathcal{M}\Delta(\mathcal{M},\mathcal{H})^{-1}(\lambda I - \Xi^\dagger)^{-1}\Delta(\mathcal{M},\mathcal{H})(\lambda I - \Lambda)\mathcal{M}^{-1}\mathcal{U} \\ &= \mathcal{M}\Delta(\mathcal{M},\mathcal{H})^{-1}(\lambda I - \Xi^\dagger)^{-1}(\lambda\Delta(\mathcal{M},\mathcal{H}) - \Delta(\mathcal{M},\mathcal{H})\Lambda)\mathcal{M}^{-1}\mathcal{U}. \end{aligned}$$

By substituting the value of $\Delta(\mathcal{M},\mathcal{H})\Lambda$ from (3.7), we get

$$\begin{aligned}\overset{\wedge}{\mathcal{U}} &= \mathcal{M}\Delta(\mathcal{M},\mathcal{H})^{-1}(\lambda I - \Xi^{\dagger})^{-1}(\lambda\Delta(\mathcal{M},\mathcal{H}) - \Xi^{\dagger}\Delta(\mathcal{M},\mathcal{H}) + \mathcal{H}^{\dagger}\mathcal{M})\mathcal{M}^{-1}\mathcal{U},\\ &= (I + \mathcal{M}\Delta(\mathcal{M},\mathcal{H})^{-1}(\lambda I - \Xi^{\dagger})^{-1\mathcal{H}^{\dagger}})\mathcal{U},\\ &= (I - \mathcal{M}\Delta(\mathcal{M},\mathcal{H})^{-1}\Delta(\cdot,\mathcal{H}))\mathcal{U},\\ &= \mathcal{U} - \mathcal{M}\Delta(\mathcal{M},\mathcal{H})^{-1}\Delta(\mathcal{U},\mathcal{H}),\end{aligned} \tag{3.23}$$

where we have used Eq. (3.6) in obtaining the last step. Equation (3.23) may be written in terms of quasideterminant as

$$\overset{\wedge}{\mathcal{U}} = \begin{vmatrix}\Delta(\mathcal{M},\mathcal{H}) & \Delta(\mathcal{U},\mathcal{H})\\ \mathcal{M} & \boxed{\mathcal{U}}\end{vmatrix}. \tag{3.24}$$

The adjoint binary transformation for $\overset{\wedge}{\mathcal{W}} \in \hat{V}^{\dagger}$ is obtained in a similar way and gives

$$\begin{aligned}\overset{\wedge}{\mathcal{W}} &= \mathcal{W} - \mathcal{H}\Delta(\mathcal{M},\mathcal{H})^{(-1)\dagger}\Delta^{\dagger}(\mathcal{M},\mathcal{W}),\\ &= \begin{vmatrix}\Delta^{\dagger}(\mathcal{M},\mathcal{H}) & \Delta^{\dagger}(\mathcal{M},\mathcal{W})\\ \mathcal{H} & \boxed{\mathcal{W}}\end{vmatrix}.\end{aligned}$$

Again from Eq. (3.22), we have

$$\begin{aligned}\hat{G} = \overset{\wedge}{\mathcal{U}}\Big|_{\lambda=0} &= (\overset{\wedge}{\mathcal{M}}\Xi^{\dagger}\overset{\wedge}{\mathcal{M}}^{-1})^{-1}(\mathcal{M}\Lambda\mathcal{M}^{-1})\mathcal{U}\Big|_{\lambda=0},\\ &= \mathcal{M}\Delta(\mathcal{M},\mathcal{H})^{-1}\Xi^{(-1)\dagger}\Delta(\mathcal{M},\mathcal{H})\Lambda\mathcal{M}^{-1}G.\end{aligned}$$

By using Eq. (3.7) for $\Delta(\mathcal{M},\mathcal{H})\Lambda$ in above equation

$$\begin{aligned}\hat{G} &= \mathcal{M}\Delta(\mathcal{M},\mathcal{H})^{-1}\Xi^{(-1)\dagger}(\Xi^{\dagger}\Delta(\mathcal{M},\mathcal{H}) - \mathcal{H}^{\dagger}\mathcal{M})G,\\ &= (I - \mathcal{M}\Delta(\mathcal{M},\mathcal{H})^{-1}\Xi^{(-1)\dagger}\mathcal{H}^{\dagger})G,\\ &= \begin{vmatrix}\Delta(\mathcal{M},\mathcal{H}) & \Xi^{(-1)\dagger}\mathcal{H}^{\dagger}\\ \mathcal{M} & \boxed{I}\end{vmatrix}G.\end{aligned} \tag{3.25}$$

To find out how the conserved superfield currents $\hat{J}_{\pm}$ transform under binary Darboux transformation, we start with the Eq. (3.21)

$$\overset{\wedge}{\mathcal{U}} = \mathcal{B}(\lambda)\mathcal{U}.$$

Taking derivative on both sides

$$\begin{aligned}D_{\pm}\overset{\wedge}{\mathcal{U}} &= D_{\pm}(\mathcal{B}(\lambda)\mathcal{U}) = D_{\pm}\mathcal{B}\mathcal{U} + \mathcal{B}\partial_{\pm}\mathcal{U},\\ -\frac{\mathrm{i}}{1 \mp \lambda}\hat{J}_{\pm}\overset{\wedge}{\mathcal{U}} &= D_{\pm}\mathcal{B}\mathcal{U} - \frac{\mathrm{i}}{1 \mp \lambda}\mathcal{B}J_{\pm}\mathcal{U}.\end{aligned} \tag{3.26}$$

Now we use $\mathcal{B}(\lambda) = \hat{\mathcal{D}}^{-1}(\lambda)\mathcal{D}(\lambda)$ to find $D_\pm\mathcal{B}$,

$$D_\pm\mathcal{B} = (\lambda I - \overset{\wedge}{\mathcal{S}})^{-1}(D_\pm\overset{\wedge}{\mathcal{S}}\mathcal{B} - D_\pm\mathcal{S}). \tag{3.27}$$

By substituting the expression of $D_\pm\mathcal{B}$ from (3.27) in (3.26) and simplifying, we get

$$(\lambda I - \overset{\wedge}{\mathcal{S}})\hat{J}_\pm\mathcal{B} = (1 \mp \lambda)(\mathrm{i}D_\pm\overset{\wedge}{\mathcal{S}}\mathcal{B} - \mathrm{i}D_\pm\mathcal{S}) + (\lambda I - \mathcal{S})J_\pm. \tag{3.28}$$

Again by using $\mathcal{B} = I + \mathcal{M}\Delta(\mathcal{M},\mathcal{H})^{-1}(\lambda I - \Xi^\dagger)^{-1}\mathcal{H}^\dagger$, and comparing the coefficients of λ^0 and λ in Eq. (3.28), we obtain the following equation:

$$\hat{J}_\pm \pm \mathrm{i}D_\pm\overset{\wedge}{\mathcal{S}} = J_\pm \pm \mathrm{i}D_\pm\mathcal{S}, \tag{3.29}$$

$$D_\pm\overset{\wedge}{\mathcal{S}}(I \mp \overset{\wedge}{\mathcal{S}}) + \mathrm{i}[\hat{J}_\pm, \overset{\wedge}{\mathcal{S}}] = D_\pm\mathcal{S}(I \mp \mathcal{S}) + \mathrm{i}[J_\pm, \mathcal{S}]. \tag{3.30}$$

By using (3.19) on left-hand side and (2.8) on the right-hand side of the above Eq. (3.29), we get

$$(I \mp \overset{\wedge}{\mathcal{S}})\hat{J}_\pm(I \mp \overset{\wedge}{\mathcal{S}})^{-1} = (I \mp \mathcal{S})J_\pm(I \mp \mathcal{S})^{-1}. \tag{3.31}$$

By writing $J_\pm$ in terms of a primitive field $\mathcal{F}_\pm$, as $J_\pm = \mathcal{F}_\pm\mathcal{F}_\pm^{-1}$, we may write (3.31) as

$$\hat{J}_\pm = (I \mp \overset{\wedge}{\mathcal{S}})^{-1}(I \mp \mathcal{S})J_\pm(I \mp \mathcal{S})^{-1}(I \mp \overset{\wedge}{\mathcal{S}}), \tag{3.32}$$

$$= \hat{\mathcal{F}}_\pm\hat{\mathcal{F}}_\pm^{-1}, \tag{3.33}$$

where

$$\hat{\mathcal{F}}_\pm = (I \mp \overset{\wedge}{\mathcal{S}})^{-1}(I \mp \mathcal{S})\mathcal{F}_\pm, \tag{3.34}$$

$$= (I \mp \overset{\wedge}{\mathcal{M}}\Xi^\dagger\overset{\wedge}{\mathcal{M}}^{-1})^{-1}(I \mp \mathcal{M}\Lambda\mathcal{M}^{-1})\mathcal{F}_\pm,$$

$$= (I \pm \mathcal{M}\Delta(\mathcal{M},\mathcal{H})^{-1}(I \mp \Xi^\dagger)^{-1}\mathcal{H}^\dagger)\mathcal{F}_\pm,$$

$$= \begin{vmatrix} \Delta(\mathcal{M},\mathcal{H}) & \mp(I \mp \Xi^\dagger)^{-1}\mathcal{H}^\dagger \\ \mathcal{M} & \boxed{I} \end{vmatrix}\mathcal{F}_\pm. \tag{3.35}$$

Note that from Eq. (3.31), we may write

$$\tilde{\hat{J}}_\pm = \tilde{J}_\pm,$$

which is a direct consequence of Eqs. (2.4) and (3.14). For the next iteration of binary Darboux transformation, we take $\mathcal{M}_1$, $\mathcal{M}_2$ to be two particular solutions of the Lax pair (1.11) at $\Lambda = \Lambda_1$ and $\Lambda = \Lambda_2$, respectively. Similarly $\mathcal{H}_1, \mathcal{H}_2$ are two particular solutions of the Lax pair (2.16) at $\Xi = \Xi_1$ and $\Xi = \Xi_2$. Using the notation $\mathcal{U}[1] = \mathcal{U}, G[1] = G, J_\pm[1] = J_\pm, \mathcal{F}_\pm[1] = \mathcal{F}_\pm$ and $\mathcal{U}[2] = \overset{\wedge}{\mathcal{U}}, G[2] = \hat{G}, J_\pm[2] = \hat{J}_\pm, \mathcal{F}_\pm[2] = \hat{\mathcal{F}}_\pm$, we write two-fold binary

Darboux transformation on $\mathcal{U}$ as

$$\mathcal{U}[3] = \mathcal{U}[2] - \mathcal{M}[2]\Delta(\mathcal{M}[2], \mathcal{H}[2])^{-1}\Delta(\mathcal{U}[2], \mathcal{H}[2]), \tag{3.36}$$

where $\mathcal{M}[1] = \mathcal{M}_1, \mathcal{H}[1] = \mathcal{H}_1, \mathcal{M}[2] = \mathcal{U}[2]|_{\mathcal{U}\to\mathcal{M}_2}, \mathcal{H}[2] = \mathcal{W}[2]|_{\mathcal{W}\to\mathcal{H}_2}$. Also note that by using the definition of the potential Δ and Eq. (3.8), we have

$$\begin{aligned}\Delta(\mathcal{U}[2], \mathcal{W}[2]) &= \Delta(\mathcal{U}_1, \mathcal{W}_1) - \Delta(\mathcal{M}_1, \mathcal{W}_1)\Delta(\mathcal{M}_1, \mathcal{H}_1)^{-1}\Delta(\mathcal{U}_1, \mathcal{H}_1),\\ &= \begin{vmatrix} \Delta(\mathcal{M}_1, \mathcal{H}_1) & \Delta(\mathcal{U}, \mathcal{H}_1) \\ \Delta(\mathcal{M}_1, \mathcal{W}) & \boxed{\Delta(\mathcal{U}, \mathcal{W})} \end{vmatrix}.\end{aligned} \tag{3.37}$$

The equation (3.37) implies that

$$\begin{aligned}\Delta(\mathcal{M}[2], \mathcal{H}[2]) &= \Delta(\mathcal{M}_2, \mathcal{H}_2) - \Delta(\mathcal{M}_1, \mathcal{H}_2)\Delta(\mathcal{M}_1, \mathcal{H}_1)^{-1}\Delta(\mathcal{M}_2, \mathcal{H}_1),\\ &= \begin{vmatrix} \Delta(\mathcal{M}_1, \mathcal{H}_1) & \Delta(\mathcal{M}_2, \mathcal{H}_1) \\ \Delta(\mathcal{M}_1, \mathcal{H}_2) & \boxed{\Delta(\mathcal{M}_2, \mathcal{H}_2)} \end{vmatrix}.\end{aligned} \tag{3.38}$$

By using Eqs. (3.37) and (3.38) and the notation defined above in Eq. (3.36), we get

$$\begin{aligned}\mathcal{U}[3] &= \begin{vmatrix} \Delta(\mathcal{M}_1, \mathcal{H}_1) & \Delta(\mathcal{U}, \mathcal{H}_1) \\ \mathcal{M}_1 & \boxed{\mathcal{U}} \end{vmatrix} - \begin{vmatrix} \Delta(\mathcal{M}_1, \mathcal{H}_1) & \Delta(\mathcal{M}_2, \mathcal{H}_1) \\ \mathcal{M}_1 & \boxed{\mathcal{M}_2} \end{vmatrix}\\ &\quad \times \begin{vmatrix} \Delta(\mathcal{M}_1, \mathcal{H}_1) & \Delta(\mathcal{M}_2, \mathcal{H}_1) \\ \Delta(\mathcal{M}_1, \mathcal{H}_2) & \boxed{\Delta(\mathcal{M}_2, \mathcal{H}_2)} \end{vmatrix}^{-1} \begin{vmatrix} \Delta(\mathcal{M}_1, \mathcal{H}_1) & \Delta(\mathcal{U}, \mathcal{H}_1) \\ \Delta(\mathcal{M}_1, \mathcal{W}) & \boxed{\Delta(\mathcal{U}, \mathcal{H}_2)} \end{vmatrix},\\ &= \begin{vmatrix} \Delta(\mathcal{M}_1, \mathcal{H}_1) & \Delta(\mathcal{M}_2, \mathcal{H}_1) & \Delta(\mathcal{U}, \mathcal{H}_1) \\ \Delta(\mathcal{M}_1, \mathcal{H}_2) & \Delta(\mathcal{M}_2, \mathcal{H}_2) & \Delta(\mathcal{U}, \mathcal{H}_2) \\ \mathcal{M}_1 & \mathcal{M}_2 & \boxed{\mathcal{U}} \end{vmatrix},\end{aligned} \tag{3.39}$$

where we have used the noncommutative Jacobi identity[d] in obtaining (3.39). The Kth iteration of binary Darboux transformation leads to

$$\begin{aligned}\mathcal{U}[K+1] &= \mathcal{U}[K] - \mathcal{M}[K]\Delta(\mathcal{M}[K], \mathcal{H}[K])^{-1}\Delta(\mathcal{U}[K], \mathcal{H}[K]),\\ &= \begin{vmatrix} \Delta(\mathcal{M}[K], \mathcal{H}[K]) & \Delta(\mathcal{U}[K], \mathcal{H}[K]) \\ \mathcal{M}[K] & \boxed{\mathcal{U}[K]} \end{vmatrix},\end{aligned}$$

[d]For quasideterminants, the noncommutative Jacobi identity is given as

$$\begin{vmatrix} E & F & G \\ H & A & B \\ J & C & \boxed{D} \end{vmatrix} = \begin{vmatrix} E & G \\ J & \boxed{D} \end{vmatrix} - \begin{vmatrix} E & F \\ J & \boxed{C} \end{vmatrix} \begin{vmatrix} E & F \\ H & \boxed{A} \end{vmatrix}^{-1} \begin{vmatrix} E & G \\ H & \boxed{B} \end{vmatrix}.$$

For the definition and more properties of quasideterminants see e.g. [16–19, 27].

$$= \begin{vmatrix} \Delta(\mathcal{M}_1,\mathcal{H}_1) & \cdots & \Delta(\mathcal{M}_K,\mathcal{H}_1) & \Delta(\mathcal{U},\mathcal{H}_1) \\ \vdots & \cdots & \vdots & \vdots \\ \Delta(\mathcal{M}_1,\mathcal{H}_K) & \cdots & \Delta(\mathcal{M}_K,\mathcal{H}_K) & \Delta(\mathcal{U},\mathcal{H}_K) \\ \mathcal{M}_1 & \cdots & \mathcal{M}_K & \boxed{\mathcal{U}} \end{vmatrix}. \tag{3.40}$$

Above result can be proved by induction by using the properties of quasideterminants. The procedure is same as we did for the bosonic case. Similarly the Kth iteration of adjoint binary Darboux transformation gives

$$\begin{aligned} \mathcal{W}[K+1] &= \mathcal{W}[K] - \mathcal{H}[K]\Delta(\mathcal{M}[K],\mathcal{H}[K])^{(-1)\dagger}\Delta(\mathcal{M}[K],\mathcal{W}[K])^{\dagger}, \\ &= \begin{vmatrix} \Delta(\mathcal{M}[K],\mathcal{H}[K])^{\dagger} & \Delta(\mathcal{M}[K],\mathcal{W}[K])^{\dagger} \\ \mathcal{H}[K] & \boxed{\mathcal{W}[K]} \end{vmatrix}, \\ &= \begin{vmatrix} \Delta(\mathcal{M}_1,\mathcal{H}_1)^{\dagger} & \cdots & \Delta(\mathcal{M}_K,\mathcal{H}_1)^{\dagger} & \Delta(\mathcal{M}_1,\mathcal{W})^{\dagger} \\ \vdots & \cdots & \vdots & \vdots \\ \Delta(\mathcal{M}_1,\mathcal{H}_K)^{\dagger} & \cdots & \Delta(\mathcal{M}_K,\mathcal{H}_K)^{\dagger} & \Delta(\mathcal{M}_K,\mathcal{W})^{\dagger} \\ \mathcal{H}_1 & \cdots & \mathcal{H}_K & \boxed{\mathcal{W}} \end{vmatrix}. \end{aligned} \tag{3.41}$$

The superfield multisoliton $G[K+1]$ can be obtained by putting $\lambda = 0$ in the expression for $\mathcal{U}[K+1]$ (3.40) and using $G = \mathcal{U}|_{\lambda=0}$, which on simplification gives

$$G[K+1] = \begin{vmatrix} \Delta(\mathcal{M}_1,\mathcal{H}_1) & \cdots & \Delta(\mathcal{M}_K,\mathcal{H}_1) & \Xi_1^{(-1)\dagger}\mathcal{H}_1^{\dagger} \\ \vdots & \cdots & \vdots & \vdots \\ \Delta(\mathcal{M}_1,\mathcal{H}_K) & \cdots & \Delta(\mathcal{M}_K,\mathcal{H}_K) & \Xi_K^{(-1)\dagger}\mathcal{H}_K^{\dagger} \\ \mathcal{M}_1 & \cdots & \mathcal{M}_K & \boxed{I} \end{vmatrix} G.$$

The Kth iteration of the conserved currents is

$$J_{\pm}[K+1] = \mathcal{F}_{\pm}[K+1]\mathcal{F}_{\pm}[K+1]^{-1},$$

where

$$\begin{aligned} \mathcal{F}_{\pm}[K+1] &= (I \pm \mathcal{M}[K]\Delta(\mathcal{M}[K],\mathcal{H}[K])^{-1}(I \mp \Xi_K^{\dagger})^{-1}\mathcal{H}[K]^{\dagger})\mathcal{F}_{\pm}[K], \\ &= \begin{vmatrix} \Delta(\mathcal{M}_1,\mathcal{H}_1) & \cdots & \Delta(\mathcal{M}_K,\mathcal{H}_1) & \mp(I \mp \Xi_1^{\dagger})^{-1}\mathcal{H}_1^{\dagger} \\ \vdots & \cdots & \vdots & \vdots \\ \Delta(\mathcal{M}_1,\mathcal{H}_K) & \cdots & \Delta(\mathcal{M}_K,\mathcal{H}_K) & \mp(I \mp \Xi_K^{\dagger})^{-1}\mathcal{H}_K^{\dagger} \\ \mathcal{M}_1 & \cdots & \mathcal{M}_K & \boxed{I} \end{vmatrix} F_{\pm}. \end{aligned} \tag{3.42}$$

Similar expressions can be obtained for the Kth iteration of $G^{\dagger}$ and $J_{\pm}^{\dagger}$.

4. Binary Darboux Transformation on Component Fields

To find the binary Darboux transformation on component fields, we expand all superfields and collect different coefficients of θ's. We first collect the results obtained in [23]. The Darboux transformation on the component fields of the supersymmetric chiral field model can be defined by a Darboux matrix D

$$D(\lambda) = \lambda I - S, \tag{4.1}$$

where D and S are the leading bosonic components of the matrix superfields $\mathcal{D}$ and $\mathcal{S}$, respectively. The matrix D is defined such that the solution U of the Lax pair (1.12) transforms as

$$\tilde{U} = D(\lambda)U, \tag{4.2}$$

Now following the arguments of the previous section, let

$$M = (U(\lambda_1)|1\rangle, \ldots, U(\lambda_n)|n\rangle) = (|m_1\rangle, \ldots, |m_n\rangle), \tag{4.3}$$

be an invertible $n \times n$ matrix, representing the leading bosonic component of the matrix superfield $\mathcal{M}$ with each column $|m_i\rangle = U(\lambda_i)|i\rangle$ in M representing a column solution of the Lax pair (1.12), when $\lambda = \lambda_i$. i.e.,

$$\partial_\pm M = j_\pm M(I \mp \Lambda)^{-1} + \mathrm{i}h_\pm M((I \mp \Lambda)^{-1})^2. \tag{4.4}$$

Note that the matrix S is now written in terms of Λ and M as

$$S = M\Lambda M^{-1}. \tag{4.5}$$

Thus, we have the Darboux transformation on the component fields $j_\pm \equiv j_\pm[1]$ and $\psi_\pm \equiv \psi_\pm[1]$ given by

$$(j_\pm, \psi_\pm, V) \mapsto (\tilde{j}_\pm, \tilde{\psi}_\pm, \tilde{V}).$$

Again substituting (4.2) into the Eq. (1.12), we have

$$\tilde{j}_\pm = j_\pm \pm \partial_\pm S, \tag{4.6}$$

$$\tilde{\psi}_\pm = (I \mp S)\psi_\pm(I \mp S)^{-1}, \tag{4.7}$$

$$\tilde{h}_\pm = (I \mp S)h_\pm(I \mp S)^{-1}\ , \tag{4.8}$$

with the conditions on S given by

$$\partial_+ S(I - S) = j_+ S - S j_+ - \mathrm{i}h_+ + \mathrm{i}(I - S)h_+(I - S)^{-1}, \tag{4.9}$$

$$\partial_- S(I + S) = -j_- S + S j_- + \mathrm{i}h_- - \mathrm{i}(I + S)h_-(I + S)^{-1}. \tag{4.10}$$

In fact $j_\pm$ and $h_\pm$ are the leading bosonic components of the matrix superfields $D_\pm J_\pm$ and $J_\pm^2$, respectively. Note that $\psi_\pm$ and $h_\pm$ transform exactly in the same manner [23]. Therefore once we write the expression for $\psi_\pm$, similar expression will hold for $h_\pm$. By using Eqs. (4.9) and (4.10) in Eq. (4.6), we get

$$\begin{aligned}\tilde{j}_\pm &= (I \mp S) j_\pm (I \mp S)^{-1} - (I \mp S)[(I \mp S)^{-1}, \mathrm{i}h_\pm](I \mp S)^{-1},\\ &= Q_\pm[1](j_\pm + [\mathrm{i}h_\pm, Q_\pm[1]^{-1}])Q_\pm[1]^{-1},\\ &= \begin{vmatrix} M & I \\ M(I \mp \Lambda) & \boxed{O} \end{vmatrix} \left\{ j_\pm - \mathrm{i}\left[h_\pm, \begin{vmatrix} M\,(I \mp \Lambda) & I \\ M & \boxed{O} \end{vmatrix} \right] \right\} \begin{vmatrix} M & I \\ M(I \mp \Lambda) & \boxed{O} \end{vmatrix}^{-1},\end{aligned}$$

where by analogy of superspace calculations, we write

$$Q_\pm[i] = \prod_{i=1}^{k} (I \mp S[k]).$$

Since M and Λ are matrices, that do not commute, therefore we can write Eq. (4.2) in terms of quasideterminants as

$$\tilde{U} = (\lambda I - M\Lambda M^{-1})U = \begin{vmatrix} M & I \\ M\Lambda & \boxed{\lambda I} \end{vmatrix} U, \tag{4.11}$$

and the chiral field $\widetilde{g}$ is expressed as

$$\widetilde{g} = \tilde{U}(0) = -Sg = -(M\Lambda M^{-1})g = \begin{vmatrix} M & I \\ M\Lambda & \boxed{O} \end{vmatrix} g. \tag{4.12}$$

and

$$\begin{aligned}\tilde{\psi}_\pm &= M(I \mp \Lambda)M^{-1}\psi_\pm M(I \mp \Lambda)^{-1}M^{-1},\\ &= \begin{vmatrix} M & I \\ M(I \mp \Lambda) & \boxed{O} \end{vmatrix} \psi_\pm \begin{vmatrix} M & I \\ M(I \mp \Lambda) & \boxed{O} \end{vmatrix}^{-1}.\end{aligned} \tag{4.13}$$

We can iterate the Darboux transformation K times and obtain the quasideterminant multisoliton solution of the supersymmetric model. For each $k = 1, 2, \ldots, K$, let M_k be an invertible $n \times n$ matrix solution of the Lax pair (1.12) at $\Lambda = \Lambda_k$, then the Kth solution $U[K+1]$ is expressed as

$$U[K+1] = \prod_{k=1}^{K} (\lambda I - S[K-k+1])U = \begin{vmatrix} M_1 & \cdots & M_K & I \\ M_1\Lambda_1 & \cdots & M_K\Lambda_K & \lambda I \\ \vdots & \cdots & \vdots & \vdots \\ M_1\Lambda_1^K & \cdots & M_K\Lambda_K^K & \boxed{\lambda^K I} \end{vmatrix} U. \tag{4.14}$$

The multisoliton solution $g[K+1]$ of the supersymmetric model can be readily obtained by taking $\lambda = 0$ in the expression of $V[K+1]$ i.e.,

$$g[K+1] = \begin{vmatrix} M_1 & \cdots & M_K & I \\ M_1\Lambda_1 & \cdots & M_K\Lambda_K & O \\ \vdots & \cdots & \vdots & \vdots \\ M_1\Lambda_1^K & \cdots & M_K\Lambda_K^K & \boxed{O} \end{vmatrix} g. \tag{4.15}$$

Similarly the K times iteration of Darboux transformation gives the following expression of the conserved currents

$$j_\pm[K+1] = \begin{vmatrix} M_1 & \cdots & M_K & I \\ M_1(I \mp \Lambda_1) & \cdots & M_K(I \mp \Lambda_K) & O \\ \vdots & \cdots & \vdots & \vdots \\ M_1(I \mp \Lambda_1)^K & \cdots & M_K(I \mp \Lambda_K)^K & \boxed{O} \end{vmatrix}$$

$$\left(j_\pm - \left[\mathrm{i}h_\pm, \begin{vmatrix} M_1(I \mp \Lambda_1)^K & \cdots & M_K(I \mp \Lambda_K)^K & O \\ \vdots & \cdots & \vdots & \vdots \\ M_1(I \mp \Lambda_1)^2 & \cdots & M_K(I \mp \Lambda_K)^2 & O \\ M_1(I \mp \Lambda_1) & \cdots & M_K(I \mp \Lambda_K) & I \\ M_1 & \cdots & M_K & \boxed{O} \end{vmatrix} \right] \right) \tag{4.16}$$

$$\times \begin{vmatrix} M_1 & \cdots & M_K & I \\ M_1(I \mp \Lambda_1) & \cdots & M_K(I \mp \Lambda_K) & O \\ \vdots & \cdots & \vdots & \vdots \\ M_1(I \mp \Lambda_1)^K & \cdots & M_K(I \mp \Lambda_K)^K & \boxed{O} \end{vmatrix}^{-1}. \tag{4.17}$$

For the spinor fields of the model the K-fold Darboux transformation gives

$$\psi_\pm[K+1] = \begin{vmatrix} M_1 & \cdots & M_K & I \\ M_1(I \mp \Lambda_1) & \cdots & M_K(I \mp \Lambda_K) & O \\ \vdots & \cdots & \vdots & \vdots \\ M_1(I \mp \Lambda_1)^K & \cdots & M_K(I \mp \Lambda_K)^K & \boxed{O} \end{vmatrix}$$

$$\times \psi_\pm \begin{vmatrix} M_1 & \cdots & M_K & I \\ M_1(I \mp \Lambda_1) & \cdots & M_K(I \mp \Lambda_K) & O \\ \vdots & \cdots & \vdots & \vdots \\ M_1(I \mp \Lambda_1)^K & \cdots & M_K(I \mp \Lambda_K)^K & \boxed{O} \end{vmatrix}^{-1}. \tag{4.18}$$

Similar quasideterminant expressions can be obtained for the adjoint space. Now we consider the linear system for the component fields in hat space

$$\partial_\pm \hat{U}(x^\pm;\lambda) = \frac{1}{1\mp\lambda}\hat{\jmath}_\pm \hat{U}(x^\pm;\lambda) + \mathrm{i}\left(\frac{1}{1\mp\lambda}\right)^2 \hat{h}_\pm \hat{U}(x^\pm;\lambda), \tag{4.19}$$

and define the Darboux matrix as

$$\hat{D}(\lambda) \equiv (\lambda I - \hat{S}) = (\lambda I - \hat{M}\Xi^\dagger \hat{M}^{-1}). \tag{4.20}$$

We see that the Darboux matrix defined in Eq. (4.20) leaves the system (4.19) invariant and we obtain

$$\widetilde{\hat{\jmath}}_\pm = \hat{\jmath}_\pm \pm \partial_\pm \hat{S}, \quad i\widetilde{\hat{h}^2}_\pm = (I\mp\hat{S})\hat{h}^2_\pm (I\mp\hat{S})^{-1},$$

with

$$\partial_\pm \hat{S}(I\mp\hat{S}) = \pm\hat{\jmath}_\pm \hat{S} \pm \hat{S}\hat{\jmath}_\pm - \mathrm{i}\hat{h}^2_\pm + \mathrm{i}(I\mp\hat{S})\hat{h}^2_\pm(I\mp\hat{S})^{-1}.$$

The standard binary Darboux transformation for component fields is now given as

$$\hat{U} = \hat{D}^{-1}(\lambda)D(\lambda)U = B(\lambda)U. \tag{4.21}$$

Now we find the expressions for the Kth iteration of the component fields. From Eq. (3.32) we obtain

$$\begin{aligned} j_\pm[2] &= \hat{Q}_\pm\{j_\pm - \mathrm{i}[h^2_\pm, Q^{-1}_\pm(I-\hat{Q}_\pm)]\}\hat{Q}^{-1}_\pm, \\ \psi_\pm[2] &= \hat{Q}_\pm \psi_\pm \hat{Q}^{-1}_\pm \\ h^2_\pm[2] &= \hat{Q}_\pm h^2_\pm \hat{Q}^{-1}_\pm, \end{aligned} \tag{4.22}$$

where $Q_\pm$ and $\hat{Q}_\pm$ are defined by

$$\begin{aligned} Q_\pm &= (I\mp S), \\ \hat{Q}_\pm &= (I\mp\hat{S})^{-1}(I\mp S). \end{aligned} \tag{4.23}$$

In terms of quasideterminants we write the expressions (4.23) as

$$\begin{aligned} j_\pm[2] = &\begin{vmatrix} \Delta(M,H) & \mp(I\mp\Xi^\dagger)^{-1}H^\dagger \\ M & \boxed{I} \end{vmatrix} \\ &\times \left\{ j_\pm - \mathrm{i}\left[h^2_\pm, \begin{vmatrix} \Delta(M,H) & (I\mp\Xi^\dagger)^{-1}H^\dagger \\ M(I\mp\Lambda)^{-1} & \boxed{O} \end{vmatrix}\right]\right\} \\ &\times \begin{vmatrix} \Delta(M,H) & \mp(I\mp\Xi^\dagger)^{-1}H^\dagger \\ M & \boxed{I} \end{vmatrix}^{-1}, \end{aligned} \tag{4.24}$$

$$\psi_\pm[2] = \begin{vmatrix} \Delta(M,H) & \mp(I\mp\Xi^\dagger)^{-1}H^\dagger \\ M & \boxed{I} \end{vmatrix} \psi_\pm \begin{vmatrix} \Delta(M,H) & \mp(I\mp\Xi^\dagger)^{-1}H^\dagger \\ M & \boxed{I} \end{vmatrix}^{-1}. \tag{4.25}$$

The K-times iteration of standard binary Darboux transformation on $j_\pm$ and $\psi_\pm$ gives

$$\begin{aligned} j_\pm[K+1] &= \hat{Q}_\pm[K](j_\pm - \mathrm{i}[h_\pm^2, A])\hat{Q}_\pm^{-1}[K], \\ \psi_\pm[K+1] &= \hat{Q}_\pm[K]\psi_\pm\hat{Q}_\pm^{-1}[K], \end{aligned} \tag{4.26}$$

where

$$\begin{aligned} \hat{Q}_\pm[K] &= (I \mp \hat{S}[K])^{-1}(I \mp S[K]) \cdots (I \mp \hat{S}[1])^{-1}(I \mp S[1]), \\ A &= \sum_{k=1}^{K}(\hat{Q}_\pm^{-1}[k-1](I \mp S[k])^{-1}(I - (I \mp \hat{S}[k])^{-1}(I \mp S[k]))\hat{Q}_\pm[k-1]). \end{aligned} \tag{4.27}$$

The quasideterminant expression of $\hat{Q}_\pm[K]$ is similar to that of $F_\pm[K]$ (see [23]) i.e.

$$\hat{Q}_\pm[K] = \begin{vmatrix} \Delta(M_1, H_1) & \cdots & \Delta(M_K, H_1) & \mp(I \mp \Xi_1^\dagger)^{-1}H_1^\dagger \\ \vdots & \cdots & \vdots & \vdots \\ \Delta(M_1, H_K) & \cdots & \Delta(M_K, H_K) & \mp(I \mp \Xi_K^\dagger)^{-1}H_K^\dagger \\ M_1 & \cdots & M_K & \boxed{I} \end{vmatrix}. \tag{4.28}$$

Now we consider the second equation of (4.27) viz

$$\begin{aligned} A &= \sum_{k=1}^{K}(\hat{Q}_\pm^{-1}[k-1](I \mp S[k])^{-1}(I - (I \mp \hat{S}[k])^{-1}(I \mp S[k]))\hat{Q}_\pm[k-1]), \\ &= \hat{Q}_\pm^{-1}[0](I \mp S[1])^{-1}(I - (I \mp \hat{S}[1])^{-1}(I \mp S[1]))\hat{Q}_\pm[0] + \\ &\quad \cdots \hat{Q}_\pm^{-1}[K-1](I \mp S[K])^{-1}(I - (I \mp \hat{S}[K])^{-1}(I \mp S[K]))\hat{Q}_\pm[K-1], \\ &= \begin{vmatrix} \Delta(M_1, H_1) & \cdots & \Delta(M_K, H_1) & (I \mp \Xi_1^\dagger)^{-1}H_1^\dagger \\ \vdots & \cdots & \vdots & \vdots \\ \Delta(M_1, H_K) & \cdots & \Delta(M_K, H_K) & (I \mp \Xi_K^\dagger)^{-1}H_K^\dagger \\ M_1(I \mp \Lambda_1)^{-1} & \cdots & M_K(I \mp \Lambda_K)^{-1} & \boxed{O} \end{vmatrix}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} j_\pm[K+1] &= \begin{vmatrix} \Delta(M_1, H_1) & \cdots & \Delta(M_K, H_1) & \mp(I \mp \Xi_1^\dagger)^{-1}H_1^\dagger \\ \vdots & \cdots & \vdots & \vdots \\ \Delta(M_1, H_K) & \cdots & \Delta(M_K, H_K) & \mp(I \mp \Xi_K^\dagger)^{-1}H_K^\dagger \\ M_1 & \cdots & M_K & \boxed{I} \end{vmatrix} \\ &\quad \times \left(j_\pm + \left[\mathrm{i}h_\pm^2, \begin{vmatrix} \Delta(M_1, H_1) & \cdots & \Delta(M_K, H_1) & (I \mp \Xi_1^\dagger)^{-1}H_1^\dagger \\ \vdots & \cdots & \vdots & \vdots \\ \Delta(M_1, H_K) & \cdots & \Delta(M_K, H_K) & (I \mp \Xi_K^\dagger)^{-1}H_K^\dagger \\ M_1(I \mp \Lambda_1)^{-1} & \cdots & M_K(I \mp \Lambda_K)^{-1} & \boxed{O} \end{vmatrix} \right] \right) \end{aligned}$$

$$\times\begin{vmatrix} \Delta(M_1,H_1) & \cdots & \Delta(M_K,H_1) & \mp(I\mp\Xi_1^{\dagger})^{-1}H_1^{\dagger} \\ \vdots & \cdots & \vdots & \vdots \\ \Delta(M_1,H_K) & \cdots & \Delta(M_K,H_K) & \mp(I\mp\Xi_K^{\dagger})^{-1}H_K^{\dagger} \\ M_1 & \cdots & M_K & \boxed{I} \end{vmatrix}^{-1}. \tag{4.29}$$

Equation (4.29) gives the quasideterminant expression for the Kth iteration of conserved component currents $j_{\pm}$ and $ih_{\pm}$. The proof of the expression for $ih_{\pm}$ is simple by induction. Now we consider the expression (4.29). We see from (4.24) that the expression is true for $K=1$. Now we consider the case of $K+2$,

$$j_{\pm}[K+2] = \hat{Q}_{\pm}[K+1]\times\Bigg(j_{\pm}+\Bigg[\mathrm{i}J^2_{\pm},\sum_{k=1}^{K}(\hat{Q}_{\pm}^{-1}[k](I\mp S[k+1])^{-1} \times(I-(I\mp\hat{S}[k+1])^{-1}(I\mp S[k+1])))\hat{Q}_{\pm}[k]\Bigg]\Bigg)\times\hat{Q}_{\pm}^{-1}[K+1],$$

where

$$\begin{aligned}
&\sum_{k=1}^{K}(\hat{Q}_{\pm}^{-1}[k](I\mp S[k+1])^{-1}(I-(I\mp\hat{S}[k+1])^{-1}(I\mp S[k+1]))\hat{Q}_{\pm}[k]) \\
&\quad=\sum_{k=1}^{K}(\hat{Q}_{\pm}^{-1}[k-1](I\mp S[k])^{-1}(I-(I\mp\hat{S}[k])^{-1}(I\mp S[k]))\hat{Q}_{\pm}[k-1]) \\
&\qquad+\hat{Q}_{\pm}^{-1}[k](I\mp S[k+1])^{-1}(I-(I\mp\hat{S}[k+1])^{-1}(I\mp S[k+1]))\hat{Q}_{\pm}[k], \\
&\quad=\begin{vmatrix} \Delta(M_1,H_1) & \cdots & \Delta(M_K,H_1) & (I\mp\Xi_1^{\dagger})^{-1}H_1^{\dagger} \\ \vdots & \cdots & \vdots & \vdots \\ \Delta(M_1,H_K) & \cdots & \Delta(M_K,H_K) & (I\mp\Xi_K^{\dagger})^{-1}H_K^{\dagger} \\ M_1(I\mp\Lambda_1)^{-1} & \cdots & M_K(I\mp\Lambda_K)^{-1} & \boxed{O} \end{vmatrix} \\
&\qquad-\begin{vmatrix} \Delta(M_1,H_1) & \cdots & \Delta(M_K,H_1) \\ \vdots & \cdots & \vdots \\ \Delta(M_1,H_{K+1}) & \cdots & \Delta(M_K,H_{K+1}) \\ M_1 & \cdots & \boxed{M_K(I\mp\Lambda_{K+1})^{-1}} \end{vmatrix} \\
&\qquad\times\begin{vmatrix} \Delta(M_1,H_1) & \cdots & \Delta(M_K,H_1) \\ \vdots & \cdots & \vdots \\ \Delta(M_1,H_K) & \cdots & \Delta(M_K,H_K) \\ M_1(I\mp\Lambda_1)^{-1} & \cdots & \boxed{M_K(I\mp\Lambda_K)^{-1}} \end{vmatrix}^{-1}
\end{aligned}$$

$$
\times \begin{vmatrix} \Delta(M_1, H_1) & \cdots & (I \mp \Xi_1^\dagger)^{-1} H_1^\dagger \\ \vdots & \cdots & \vdots \\ \Delta(M_1, H_K) & \cdots & (I \mp \Xi_K^\dagger)^{-1} H_K^\dagger \\ M_1 (I \mp \Lambda_1)^{-1} & \cdots & \boxed{(I \mp \Xi_{K+1}^\dagger)^{-1} H_{K+1}^\dagger} \end{vmatrix}
$$

$$
= \begin{vmatrix} \Delta(M_1, H_1) & \cdots & \Delta(M_{K+1}, H_1) & (I \mp \Xi_1^\dagger)^{-1} H_1^\dagger \\ \vdots & \cdots & \vdots & \vdots \\ \Delta(M_1, H_{K+1}) & \cdots & \Delta(M_{K+1}, H_{K+1}) & (I \mp \Xi_{K+1}^\dagger)^{-1} H_{K+1}^\dagger \\ M_1 (I \mp \Lambda_1)^{-1} & \cdots & M_{K+1}(I \mp \Lambda_{K+1})^{-1} & \boxed{O} \end{vmatrix}.
$$

The expression for $\hat{Q}_\pm[K+1]$ can be easily obtained by induction and the proof is similar as we did in [23]. Hence Eq. (4.29) is true for all values of K.

5. Elementary Binary Darboux Transformation

Now we relate the standard binary Darboux transformation obtained in the previous section to the elementary binary Darboux transformation. The elementary binary transformation is useful in the sense that we can present the solutions in the projector form which can further be related to the well known dressing method initially introduced by Mikhailov [30] to obtain the superfield multisoliton solutions of the supersymmetric chiral field model. Following discussion shows that the terms containing potential Δ in standard binary transformation reduce to the projector of elementary binary transformation when written in vector form. Let $|\theta\rangle$ be a column solution and $\langle\rho|$ be a row solution of the direct and adjoint Lax pairs (1.11), (2.16) with spectral parameters μ and ν, respectively ($\mu \neq \nu$). Through a projection operator $\mathcal{P}$, the one-fold binary Darboux transformation can be constructed to obtain new matrix solutions $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{W}}$ satisfying the direct and dual Lax pairs (1.11) and (2.16), respectively. The new solutions $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{W}}$ are related to the old solutions $\mathcal{U}$ and $\mathcal{W}$, respectively by the following transformation:

$$
\begin{aligned}
\hat{\mathcal{U}} &= \left(I - \frac{\mu - \nu}{\lambda - \nu} \mathcal{P} \right) \mathcal{U}, \\
\hat{\mathcal{W}} &= \left(I - \frac{\mu - \nu}{\mu - \xi} \mathcal{P} \right) \mathcal{W},
\end{aligned}
\tag{5.1}
$$

where the superfield projector $\mathcal{P}$ in this case is defined as ($\mathcal{P}^\dagger = \mathcal{P} = \mathcal{P}^2$)

$$
\mathcal{P} = \frac{|\theta\rangle\langle\rho|}{\langle\rho \mid \theta\rangle}, \tag{5.2}
$$

with

$$
\langle\rho \mid \theta\rangle = \sum_{i=1}^{n} \rho_i \theta_i. \tag{5.3}
$$

The projector $\mathcal{P}$ has been expressed in terms of the solutions of Lax pairs (1.11) and (2.16). Substituting (5.1) into the linear system (2.4), we obtain the following equations for the transformed conserved currents

$$\mathrm{i}\tilde{J}_{\pm} = \mathrm{i}J_{\pm} \pm (\mu - \nu)\partial_{\pm}\mathcal{P},$$

and the condition on the projector is

$$(\mu - \nu)D_{\pm}\mathcal{P}\mathcal{P} \mp (1 \mp \nu)D_{\pm}\mathcal{P} = \pm\mathrm{i}[\mathcal{P}, J_{\pm}].$$

One can easily prove the above condition by using (5.2), with

$$D_{\pm}|\theta\rangle = -\frac{\mathrm{i}}{1 \mp \mu}J_{\pm}|\theta\rangle,$$
$$\partial_{\pm}|\rho\rangle = \frac{\mathrm{i}}{1 \mp \bar{\nu}}J_{\pm}^{\dagger}|\rho\rangle,$$

Similar condition for the formal adjoint is obtained by using the second equation of (5.1) in the system (2.16). The successive iterations of binary Darboux transformation produces the transformed matrix solutions of direct and dual Lax pairs as

$$\begin{aligned}\mathcal{U}[K+1] &= \left(I - \frac{\mu_{K+1} - \nu_{K+1}}{\lambda - \nu_{K+1}}\mathcal{P}[K+1]\right)\cdots\left(I - \frac{\mu_1 - \nu_1}{\lambda - \nu_1}\mathcal{P}[1]\right)\mathcal{U} \\ &= \prod_{k=0}^{K}\left(I - \frac{\mu_{K-k+1} - \bar{\mu}_{K-k+1}}{\lambda - \bar{\mu}_{K-k+1}}\mathcal{P}[K-k+1]\right)\mathcal{U},\end{aligned} \tag{5.4}$$

$$\begin{aligned}\mathcal{W}[K+1] &= \left(I - \frac{\mu_1 - \nu_1}{\mu_1 - \lambda'}\mathcal{P}[1]\right)\cdots\left(I - \frac{\mu_{K+1} - \nu_{K+1}}{\mu_{K+1} - \lambda'}\mathcal{P}[K+1]\right)\mathcal{W} \\ &= \prod_{k=0}^{K}\left(I - \frac{\nu_{K-k+1} - \bar{\nu}_{K-k+1}}{\xi - \bar{\nu}_{K-k+1}}\mathcal{P}[K-k+1]\right)\mathcal{W},\end{aligned} \tag{5.5}$$

where

$$\mathcal{P}[i] = \frac{|\theta_i[i-1]\rangle\langle\rho_i[i-1]|}{\langle\rho_i[i-1] \mid \theta_i[i-1]\rangle}, \tag{5.6}$$

and $|\theta_i[i-1]\rangle$ and $\langle\rho_i[i-1]|, (i = 1, 2, 3, \ldots, K)$ defined as

$$\begin{aligned}|\theta_i[i-1]\rangle &= \left(I - \frac{\mu_{i-1} - \nu_{i-1}}{\mu_i - \nu_{i-1}}\mathcal{P}[i-1]\right)\cdots\left(I - \frac{\mu_i - \nu_i}{\mu_i - \nu_i}\mathcal{P}[i]\right)|\theta_i\rangle, \\ \langle\rho_i[i-1]| &= \langle\rho_i|\left(I - \frac{\mu_i - \nu_i}{\mu_i - \nu_i}\mathcal{P}[i]\right)\cdots\left(I - \frac{\mu_{i-1} - \nu_{i-1}}{\mu_{i-1} - \nu_i}\mathcal{P}[i-1]\right),\end{aligned} \tag{5.7}$$

are the matrix-column and matrix-row solutions of direct and dual Lax pairs, with spectral parameters μ_i and ν_i, respectively.

To find the relation between the standard binary Darboux transformation defined in the previous section and the elementary binary Darboux transformation, we start from the expression of standard binary Darboux transformation given in Eq. (3.23) i.e.,

$$\hat{\mathcal{U}} = \mathcal{U} - \mathcal{M}\Delta(\mathcal{M}, \mathcal{H})^{-1}\Delta(\mathcal{U}, \mathcal{H}),$$

where $\mathcal{M}, \mathcal{H}$ and Δ are the matrices. To obtain the elementary binary Darboux transformation from the above equation, we replace the matrices by vectors and get

$$\hat{\mathcal{U}} = \left(I - \frac{\mu - \nu}{\lambda - \nu} \frac{|\theta\rangle\langle\rho|}{\langle\rho \mid \theta\rangle} \right) \mathcal{U}, \tag{5.8}$$

where we have used Eqs. (3.6) and (3.7) for the potential Δ and the fact that $|\theta\rangle$ and $\langle\rho|$ are the column and row solutions of the direct and adjoint Lax pairs with spectral parameters μ and ν, respectively. Note that we have obtained Eq. (5.8) from the expression of standard binary Darboux transformation. The right-hand side of the above equation is same as that of Eq. (5.1) which is obtained by using elementary binary Darboux transformation. We therefore conclude that by replacing the matrix solutions with the vector solutions we can reduce standard binary Darboux transformation to elementary binary Darboux transformation.

6. Conclusions

In this paper, we have composed the elementary Darboux transformations of the two dimensional supersymmetric principal chiral field model and obtained the standard binary Darboux transformation of the model. By iterating the standard binary Darboux transformation we have generated the superfield multisolitons of the model. We have also obtained the quasideterminant expression for the potential Δ. From the expression of standard binary Darboux transformation, the elementary binary Darboux transformation has been obtained in terms of the particular vector solutions of the direct and adjoint linear systems.

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