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ON A PERIODIC 2-COMPONENT CAMASSA–HOLM EQUATION WITH VORTICITY

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We consider a periodic 2-component Camassa–Holm equation with vorticity in the paper. We first give the local well-posedness and the blow-up criterion for strong solutions to the equation in the Sobolev space H^s , $s > \frac{3}{2}$. We then present a global existence result for strong solutions to the equation. We finally obtain several blow-up results and the blow-up rate of strong solutions to the equation. The obtained results cover and improve the earlier results for a periodic 2-component Camassa–Holm equation without vorticity.

Keywords: A periodic 2-component Camassa–Holm equation; vorticity; global existence; blow-up; blow-up rate.

Mathematics Subject Classification 2000: 35G25, 35L05

1. Introduction

In the paper we consider the Cauchy problem of the following periodic 2-component Camassa–Holm equation with vorticity

$$\left\{ \begin{array}{ll} y_t + Ay_x + y_x u + 2yu_x - Au_x + \rho\rho_x = 0, & t > 0, x \in \mathbb{R}, \\ \rho_t + A\rho_x + (\rho u)_x = 0, & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t, x) = u(t, x + 1), & t \geq 0, x \in \mathbb{R}, \\ \rho(t, x) = \rho(t, x + 1), & t \geq 0, x \in \mathbb{R}, \end{array} \right. \quad (1.1)$$

where $y = u - u_{xx}$.

The 2-component Camassa–Holm Eq. (1.1) was recently derived by Ivanov [47], describing water waves in the shallow water regime with nonzero constant vorticity, where the nonzero vorticity case indicates the presence of an underlying current. For example, tidal

currents are realistically modeled by underlying shear flows of constant vorticity (see the discussions in [21, 28, 48]). $u(t, x)$ describes the horizontal velocity of the fluid and $\rho(t, x)$ is in connection with the horizontal deviation of the surface from equilibrium. All measured in dimensionless units. The parameter $A > 0$ characterizes a linear underlying shear flow so that (1.1) models wave-current interactions [48].

With $\rho = A = 0$ in Eq. (1.1), we find the classical Camassa–Holm equation, which models the unidirectional propagation of shallow water waves over a flat bottom, $u(t, x)$ representing the fluid velocity at time $t \geq 0$ in the spatial x direction [6, 24, 29, 46, 48]. The classical Camassa–Holm equation has many interesting features. It has a bi-Hamiltonian structure [11, 32] and is completely integrable [6, 14]. Its solitary waves are peaked, orbitally stable and interact like solitons [3, 6, 27]. These peaked waves are analogous to the exact traveling wave solutions of the governing equations for water waves representing waves of great height (see the discussions in [15, 20, 53]). And there is a geometric interpretation of Eq. (1.1) in terms of geodesic flow on the diffeomorphism group of the circle [23]. More remarkably, it has not only global strong solutions modeling permanent waves [17, 19, 25], but also blow-up solutions modeling wave breaking [13, 16–18, 25, 49, 52, 55]. The Cauchy problem of the Camassa–Holm equation has been studied extensively. It has been shown that this equation is locally well-posed [12, 17, 49, 52] for initial data $u_0 \in H^s(\mathbb{S})$ with $s > \frac{3}{2}$. On the other hand, it has global weak solutions with initial data $u_0 \in H^1$, cf. [4, 26, 54]. Moreover, the Camassa–Holm equation has not only global conservative solutions [38, 40] and dissipative solutions [5, 42] but also global conservative multipeakon solutions [39] and dissipative multipeakon solutions [41].

Equation (1.1) with $A = 0$ was rigorously justified by Constantin and Ivanov [22] in the context of shallow water waves without vorticity. The 2-component Camassa–Holm equation was also introduced by Falqui [31]. Chen, Liu and Zhang show that the 2-component Camassa–Holm equation can be identified with the first negative flow of the AKNS hierarchy and possesses the interesting peakon and multi-kink solutions [10]. Moreover Eq. (1.1) with $A = 0$ is connected with the time dependent Schrödinger spectral problem [1, 10]. Popowicz has been observed that Eq. (1.1) with $A = 0$ is related to the bosonic sector of an $N = 2$ supersymmetric extension of the classical Camassa–Holm equation [51].

For $A = 0$ in Eq. (1.1), there are many further works to study its mathematical properties. In [30], Lechtenfeld and Yin establish the local well-posedness and present the precise blow-up scenarios and several blow-up results of strong solutions to Eq. (1.1) on the line. In [22], Constantin and Ivanov investigate the global existence and blow-up phenomena of strong solutions of Eq. (1.1) on the line. Later, Guan and Yin obtain a new global existence result for strong solutions to Eq. (1.1) and get several blow-up results [33] which improve the recent results in [22]. Recently, Guan and Yin study the global existence of weak solutions Eq. (1.1) [34]. Henry studies the infinite propagation speed for Eq. (1.1) in [37]. Gui and Liu establish the local well-posedness for Eq. (1.1) in a range of the Besov spaces, they also derive a wave breaking mechanism for strong solutions [35]. Mustafa gives a simple proof of existence for the smoothing traveling waves for Eq. (1.1) [50]. Hu and Yin study the blow-up phenomena [44, 45] and the global existence of Eq. (1.1) on the circle [45].

For $A \neq 0$, the Cauchy problem of Eq. (1.1) on the line (nonperiodic case) has been discussed in [36]. Therein Gui and Liu determine a wave breaking criterion for strong solutions in the lowest Sobolev space H^s , $s > \frac{3}{2}$ by using the localization analysis in the

transport equation theory. Moreover, they establish an improved result of global solutions with only a nonzero initial profile of the free surface component of the system in H^s , $s > \frac{3}{2}$. In [8], Chen and Liu investigate the wave breaking and global existence for a generalized 2-component Camassa–Holm system. In [9], Chen, Liu and Qiao studied the existence and stability of solitary wave solutions of a generalized 2-component Camassa–Holm system. However, Eq. (1.1) with $A \neq 0$ on the circle (periodic case) has not been studied yet. The aim of this paper is to present a global existence result, and to investigate the blow-up phenomena for strong solutions to Eq. (1.1) with $A \neq 0$.

The paper is organized as follows. In Sec. 2, we give the local well-posedness of Eq. (1.1), the precise blow-up scenarios and some useful lemmas to pursue our goal. Considering a Lyapunov function introduced in [22] and using an important conservation law, we address a global existence of Eq. (1.1) in Sec. 3. Finally, we give several blow-up results and the precise blow-up rate for strong solutions of Eq. (1.1) in Sec. 4.

2. Preliminaries

In the section, we recall the local well-posedness and the blow-up criteria for strong solutions of Eq. (1.1). We first present the local well-posedness for the Cauchy problem of Eq. (1.1) in $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s > \frac{3}{2}$, with $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ (the circle of unit length).

Let $G(x) := \frac{\cosh(x-[x]-1/2)}{2\sinh(1/2)}$, $x \in \mathbb{R}$. Then $(1 - \partial_x^2)^{-1}f = G * f$ for all $f \in L^2(\mathbb{S})$ and $G * y = u$. Here, we denote by $*$ the convolution. With a Galilean transformation $x \rightarrow x - At, t \rightarrow t$, as used in [47], we can drop the terms Ay_x and $A\rho_x$ in (1.1) and hence obtain the 2-component Camassa–Holm system as follows:

$$\begin{cases} u_t + uu_x + \partial_x G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 - Au \right) = 0, & t > 0, x \in \mathbb{R}, \\ \rho_t + u\rho_x + u_x\rho = 0, & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t, x) = u(t, x + 1), & t \geq 0, x \in \mathbb{R}, \\ \rho(t, x) = \rho(t, x + 1), & t \geq 0, x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Or the equivalent form:

$$\begin{cases} u_t + uu_x = -\partial_x(1 - \partial_x^2)^{-1} \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 - Au \right), & t > 0, x \in \mathbb{R}, \\ \rho_t + u\rho_x + u_x\rho = 0, & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t, x) = u(t, x + 1), & t \geq 0, x \in \mathbb{R}, \\ \rho(t, x) = \rho(t, x + 1), & t \geq 0, x \in \mathbb{R}. \end{cases} \tag{2.2}$$

Applying transport equation theory combined with the method of Besov spaces, one may follow the similar argument as in [35, 36] to obtain the following local well-posedness result for the system (2.1).

Theorem 2.1. Given $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s > \frac{3}{2}$, there exists a maximal $T = T(A, z_0) > 0$, and a unique solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to Eq. (2.1) such that

$$z = z(\cdot, z_0) \in C([0, T]; H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})) \cap C^1([0, T]; H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S})).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping $z_0 \rightarrow z(\cdot, z_0) : H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}) \rightarrow C([0, T]; H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})) \cap C^1([0, T]; H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S}))$ is continuous. And T is independent of s .

Proof. The proof is similar to that of Theorem 1.1 in [35], so we omit it. \square

Using Littlewood–Paley analysis for the transport equation and Moser-type estimates [7] and performing the same argument as in [36], we can obtain the following blow-up criterion.

Theorem 2.2 [36]. Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s > \frac{3}{2}$, and let T be the maximal existence time of the solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to Eq. (2.1) with the initial data z_0 . Then $T < \infty \Rightarrow \int_0^T \|u_x(t, \cdot)\|_{L^\infty} dt = \infty$.

Proof. The proof is very similar to that of Theorem 4.1 in [36], and hence is omitted. \square

Next we state the following precise blow-up scenario.

Theorem 2.3 [36]. Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s > \frac{3}{2}$, and let T be the maximal existence time of the solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to Eq. (2.1) with the initial data z_0 . Then the corresponding solution blows up in finite time if and only if

$$\liminf_{t \rightarrow T} \inf_{x \in \mathbb{S}} \{u_x(t, x)\} = -\infty.$$

Proof. The proof of the theorem is similar to the proof of Theorem 4.2 in [36], so we omit it here. \square

Remark 2.1. If $\rho = A = 0$, then Theorems 2.2 and 2.3 cover the corresponding results for the Camassa–Holm equation in [49, 52]. If $A = 0$, Theorems 2.2 and 2.3 improve the corresponding results for the 2-component Camassa–Holm equation in [44, 45].

Given initial data $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s > \frac{3}{2}$, Theorem 2.1 ensures the existence and uniqueness of strong solutions to Eq. (2.1).

Consider the following initial value problem

$$\begin{cases} q_t = u(t, q), & t \in [0, T), x \in \mathbb{R}, \\ q(0, x) = x, & x \in \mathbb{R}, \end{cases} \quad (2.3)$$

where u denotes the first component of the solution z to Eq. (2.1) with the initial data z_0 . Since $u(t, \cdot) \in H^2(\mathbb{S}) \subset C^m(\mathbb{S})$ with $0 \leq m \leq \frac{3}{2}$, it follows that $u \in C^1([0, T) \times \mathbb{R}, \mathbb{R})$. Applying the classical results in the theory of ordinary differential equations, one can obtain the following results of q which is the key in the proof of global existence of solutions to Eq. (2.1) in Theorem 3.2.

Lemma 2.1 [22, 30, 43]. Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > \frac{3}{2}$, and let $T > 0$ be the maximal existence time of corresponding solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to Eq. (2.1) with the initial data z_0 . Then Eq. (2.3) has a unique solution $q \in C^1([0, T) \times \mathbb{R}, \mathbb{R})$. Moreover, the map $q(t, \cdot)$ is an increasing diffeomorphism of \mathbb{R} with

$$q_x(t, x) = \exp\left(\int_0^t u_x(s, q(s, x)) ds\right) > 0, \quad (t, x) \in [0, T) \times \mathbb{R}.$$

Lemma 2.2 [30, 45]. Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > \frac{3}{2}$, and let $T > 0$ be the maximal existence time of corresponding solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to Eq. (2.1) with the initial data z_0 . Then we have

$$\rho(t, q(t, x))q_x(t, x) = \rho_0(x), \quad (t, x) \in [0, T) \times \mathbb{R}. \tag{2.4}$$

Moreover, if there exists $x_0 \in \mathbb{S}$ such that $\rho_0(x_0) = 0$, then $\rho(t, q(t, x_0)) = 0$ for all $t \in [0, T)$.

We then give several useful conservation laws of strong solutions to Eq. (2.1).

Lemma 2.3. Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > \frac{3}{2}$, and let T be the maximal existence time of the solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to Eq. (2.1) with the initial data z_0 . Then for all $t \in [0, T)$, we have

$$\begin{aligned} \int_{\mathbb{S}} u(t, x) dx &= \int_{\mathbb{S}} u_0(x) dx, \\ \int_{\mathbb{S}} \rho(t, x) dx &= \int_{\mathbb{S}} \rho_0(x) dx. \end{aligned}$$

Proof. Integrating the first equation in (2.1) by parts, in view of the periodicity of u and G , we get

$$\frac{d}{dt} \int_{\mathbb{S}} u dx = - \int_{\mathbb{S}} u u_x dx - \int_{\mathbb{S}} \partial_x G * \left(u^2 + \frac{1}{2} u_x^2 + \frac{1}{2} \rho^2 - Au \right) dx = 0.$$

On the other hand, integrating the second equation in (2.1) by parts, in view of the periodicity of u and ρ , we get

$$\frac{d}{dt} \int_{\mathbb{S}} \rho dx = - \int_{\mathbb{S}} (u\rho)_x dx = 0.$$

This completes the proof of the lemma. □

Lemma 2.4. Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > \frac{3}{2}$, and let T be the maximal existence time of the solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to Eq. (2.1) with the initial data z_0 . Then for all $t \in [0, T)$, we have

$$\int_{\mathbb{S}} (u^2(t, x) + u_x^2(t, x) + \rho^2(t, x)) dx = \int_{\mathbb{S}} (u_0^2(x) + u_{0x}^2(t, x) + \rho_0^2(x)) dx.$$

Proof. Differentiating the first equation in (2.1) with respect to x , we get

$$u_{tx} = -\frac{1}{2}u_x^2 - uu_{xx} + u^2 + \frac{1}{2}\rho^2 - Au - G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 - Au\right). \tag{2.5}$$

Multiplying the first equation in (2.1) by u and multiplying (2.5) by u_x , then adding them together, finally integrating by parts, we have

$$\frac{d}{dt} \int_{\mathbb{S}} (u^2(t, x) + u_x^2(t, x)) dx = \int_{\mathbb{S}} u_x(t, x) \rho^2(t, x) dx.$$

Multiplying the second equation in (2.1) by ρ and integrating by parts, we get

$$\frac{d}{dt} \int_{\mathbb{S}} \rho^2(t, x) dx = - \int_{\mathbb{S}} u_x(t, x) \rho^2(t, x) dx.$$

Adding the above two equalities, we obtain

$$\frac{d}{dt} \int_{\mathbb{S}} (u^2(t, x) + u_x^2(t, x) + \rho^2(t, x)) dx = 0.$$

This completes the proof of the lemma. □

We state the following lemmas to investigate the blow-up phenomena in Sec. 4.

Lemma 2.5 [16]. *Let $T > 0$ and $v \in C^1([0, T]; H^2(\mathbb{R}))$. Then for every $t \in [0, T)$, there exists at least one point $\xi(t) \in \mathbb{R}$ with*

$$m(t) := \inf_{x \in \mathbb{R}} [v_x(x, t)] = v_x(t, \xi(t)).$$

The function $m(t)$ is almost everywhere differentiable on $(0, t)$ with

$$\frac{dm(t)}{dt} = v_{tx}(t, \xi(t)), \quad \text{a.e. on } (0, t).$$

Lemma 2.6 [55].

(i) *For every $f \in H^1(\mathbb{S})$, we have*

$$\max_{x \in [0, 1]} f^2(x) \leq \frac{e + 1}{2(e - 1)} \|f\|_{H^1(\mathbb{S})}^2,$$

where the constant $\frac{e+1}{2(e-1)}$ is sharp.

(ii) *For every $f \in H^3(\mathbb{S})$, we have*

$$\max_{x \in [0, 1]} f^2(x) \leq c \|f\|_{H^1(\mathbb{S})}^2,$$

with the best possible constant c lying within the range $(1, \frac{13}{12}]$. Moreover, the best constant c is $\frac{e+1}{2(e-1)}$.

By the conservation law stated in Lemmas 2.4 and 2.6(i), we have the following corollary.

Corollary 2.1. *Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s > \frac{3}{2}$ be given and assume that T is the maximal existence time of the corresponding solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to Eq. (2.1) with the initial data z_0 . Then for all $t \in [0, T)$, we have*

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{S})}^2 \leq \frac{e+1}{2(e-1)} \|u(t, \cdot)\|_{H^1(\mathbb{S})}^2 \leq \frac{e+1}{2(e-1)} (\|u_0\|_{H^1(\mathbb{S})}^2 + \|\rho_0\|_{L^2(\mathbb{S})}^2).$$

Lemma 2.7 [43]. *If $f \in H^3(\mathbb{S})$ is such that $\int_{\mathbb{S}} f(x)dx = \frac{a_0}{2}$, then for every $\epsilon > 0$, we have*

$$\max_{x \in [0,1]} f^2(x) \leq \frac{\epsilon+2}{24} \int_{\mathbb{S}} f_x^2 dx + \frac{\epsilon+2}{4\epsilon} a_0^2.$$

Moreover,

$$\max_{x \in [0,1]} f^2(x) \leq \frac{\epsilon+2}{24} \|f\|_{H^1(\mathbb{S})}^2 + \frac{\epsilon+2}{4\epsilon} a_0^2.$$

3. Global Existence

By Lemmas 2.1, 2.2 and 2.4, we obtain a global existence of strong solutions of Eq. (2.1).

Theorem 3.1. *Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s > \frac{3}{2}$ be given. If $\rho_0(x) \neq 0$ for all $x \in \mathbb{S}$, then the corresponding strong solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to Eq. (2.1) with the initial data z_0 exists globally in time.*

Proof. Assumed that T is the maximal existence time of the corresponding solution z to Eq. (2.1) with the initial data z_0 . In view of Theorem 2.3, it suffices to prove that there exists $M > 0$, such that $\inf_{x \in \mathbb{S}} u_x(t, x) \geq -M$ for all $t \in [0, T)$.

By Lemmas 2.1 and 2.2, we know that $\rho(0, x)$ has the same sign with $\rho(t, q(t, x))$. Since $\rho(0, x) \neq 0$ for all $x \in \mathbb{S}$, it follows that $\rho(t, q(t, x)) \neq 0$ for all $(t, x) \in [0, T) \times \mathbb{S}$.

By Lemma 2.1, we have that the map $q(t, \cdot)$ is an increasing diffeomorphism of \mathbb{R} . By the periodicity of u_x and the property of $q(t, \cdot)$, we have $\inf_{x \in \mathbb{R}} u_x(t, q(t, x)) = \inf_{x \in \mathbb{R}} u_x(t, x) = \inf_{x \in \mathbb{S}} u_x(t, x)$. Set $m(t, x) = u_x(t, q(t, x))$.

Next, we consider the function introduced in [22],

$$w(t, x) = \rho(0, x)\rho(t, q(t, x)) + \frac{\rho(0, x)}{\rho(t, q(t, x))}(1 + m^2(t, x)).$$

By Sobolev imbedding theorem, we have

$$0 < w(0, x) \leq \|\rho_0\|_{L^\infty(\mathbb{S})}^2 + \|u_0\|_{H^1(\mathbb{S})}^2 + 1 \leq \|z_0\|_{H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})} + 1.$$

By the definition of $m(t, x)$ and the first equation in (2.1), we have

$$\frac{\partial m}{\partial t} = (u_{tx} + uu_{xx})(t, q(t, x)). \tag{3.1}$$

By Eq. (2.3) and the second equation in (2.1), we obtain

$$\frac{\partial \rho(t, q(t, x))}{\partial t} = -\rho(t, q(t, x))m(t, x). \tag{3.2}$$

Substituting $(t, q(t, x))$ into (2.5), we obtain

$$\begin{aligned} \frac{\partial m}{\partial t} &= -\frac{1}{2}m^2(t) + \left(u^2 + \frac{1}{2}\rho^2 - Au\right)(t, q(t, x)) \\ &\quad - G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 - Au\right)(t, q(t, x)). \end{aligned} \tag{3.3}$$

Differentiating $w(t, x)$ with respect to t and using (3.2) and (3.3) combined with Young’s inequality and Corollary 2.1, we have

$$\begin{aligned} \frac{dw}{dt} &= \frac{2\rho(0, x)}{\rho(t, q(t, x))}m(t, x) \left[u^2 - Au - G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 - Au \right) + \frac{1}{2} \right], \\ &\leq \frac{\rho(0, x)}{\rho(t, q(t, x))}(1 + m^2(t, x)) \left| u^2 - Au - G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 - Au \right) + \frac{1}{2} \right|, \\ &\leq \left| u^2 - Au - G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 - Au \right) + \frac{1}{2} \right| w(t, x), \\ &\leq \left[\frac{e + 1}{2(e - 1)}E_0 + A\sqrt{\frac{e + 1}{2(e - 1)}E_0} + \frac{\cosh(1/2)}{2\sinh(1/2)} \left(E_0 + A\sqrt{\frac{e + 1}{2(e - 1)}E_0} \right) + \frac{1}{2} \right] w(t, x), \end{aligned}$$

where $E_0 = \|u_0\|_{H^1(\mathbb{S})}^2 + \|\rho_0\|_{L^2(\mathbb{S})}^2$. Here we use the facts that $\frac{1}{2\sinh(1/2)} \leq G(x) \leq \frac{\cosh(1/2)}{2\sinh(1/2)}$ and $\|u\|_{L^1(\mathbb{S})} \leq \|u\|_{L^\infty(\mathbb{S})}$.

By Gronwall’s inequality, we have

$$w(t, x) \leq w(0, x)e^{Kt} \leq (\|z_0\|_{H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})} + 1)e^{KT},$$

where

$$K = \left[\frac{e + 1}{2(e - 1)}E_0 + A\sqrt{\frac{e + 1}{2(e - 1)}E_0} + \frac{\cosh(1/2)}{2\sinh(1/2)} \left(E_0 + A\sqrt{\frac{e + 1}{2(e - 1)}E_0} \right) + \frac{1}{2} \right].$$

On the other hand, we get

$$w(t, x) \geq 2\sqrt{\rho^2(0, x)(1 + m^2)} \geq 2a|m(t, x)|,$$

where $a = \inf_{x \in \mathbb{S}} |\rho_0(x)| > 0$.

Thus, we deduce that

$$m(t, x) \geq -\frac{1}{2a}w(t, x) \geq -\frac{1}{2a}((\|z_0\|_{H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})} + 1)e^{KT}) := -M.$$

This completes the proof of the theorem. □

4. Blow-up Phenomena

In the section we investigate the blow-up phenomena of strong solutions to Eq. (2.1). We will show that the 2-component Camassa–Holm system has the similar wave breaking phenomena as the classical Camassa–Holm equation in a lower Sobolev space $H^s \times H^{s-1}$ for $s > \frac{3}{2}$. Now we give the first blow-up result.

Theorem 4.1. Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s > \frac{3}{2}$, and let T be the maximal existence time of solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to Eq. (2.1) with the initial data z_0 . If there is some $x_0 \in \mathbb{S}$ such that $\rho_0(x_0) = 0$ and

$$u'_0(x_0) < - \left[\frac{e+1}{2(e-1)} E_0 + A \sqrt{\frac{8(e+1)}{e-1}} E_0^{\frac{1}{2}} \right]^{\frac{1}{2}},$$

where $E_0 = \|u_0\|_{H^1(\mathbb{S})}^2 + \|\rho_0\|_{L^2(\mathbb{S})}^2$, then the corresponding solution to Eq. (2.1) blows up in finite time.

Proof. Let z be the solution to Eq. (2.1) with the initial data $z_0 \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s > \frac{3}{2}$, and let $T > 0$ be the maximal time of existence of the solution z with the initial data z_0 . Note that $\partial_x^2 G * f = G * f - f$.

Define $m(t) = u_x(t, q(t, x_0))$ and $h(t) = \rho(t, q(t, x_0))$. By Eqs. (2.1) and (2.3), we have

$$\frac{dm}{dt} = (u_{tx} + u_{xx}q_t)(t, q(t, x_0)) = (u_{tx} + uu_{xx})(t, q(t, x_0))$$

and

$$\frac{dh}{dt} = \rho_t + \rho_x q_t = -hm.$$

Substituting $(t, q(t, x_0))$ into Eq. (2.5), we obtain

$$\begin{aligned} m'(t) &= -\frac{1}{2}m^2(t) + u^2(t, q(t, x_0)) + \frac{1}{2}h^2 - Au(t, q(t, x_0)), \\ &\quad - G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 - Au \right) (t, q(t, x_0)), \\ &\leq -\frac{1}{2}m^2(t) + \left(\frac{1}{2}u^2 - Au + AG * u \right) (t, q(t, x_0)) + \frac{1}{2}h^2. \end{aligned} \tag{4.1}$$

Here we use the relation $G * (u^2 + \frac{1}{2}u_x^2) \geq \frac{1}{2}u^2$, and $G * \rho^2 \geq 0$.

Recalling Corollary 2.1, we have

$$\|u\|_{L^\infty(\mathbb{S})} \leq \sqrt{\frac{e+1}{2(e-1)}} (\|u_0\|_{H^1(\mathbb{S})}^2 + \|\rho_0\|_{L^2(\mathbb{S})}^2)^{\frac{1}{2}}.$$

Using Young’s inequality and the above estimate, we have

$$\|G * u\|_{L^\infty} \leq \|G\|_{L^1(\mathbb{S})} \|u\|_{L^\infty(\mathbb{S})} \leq \sqrt{\frac{e+1}{2(e-1)}} (\|u_0\|_{H^1(\mathbb{S})}^2 + \|\rho_0\|_{L^2(\mathbb{S})}^2)^{\frac{1}{2}},$$

where we use the fact that $\|G\|_{L^1(\mathbb{S})} = 1$.

Note that $h(0) = \rho(0, q(0, x_0)) = \rho_0(x_0) = 0$. By Lemma 2.2, we have $h(t) = 0$ for all $t \in [0, T)$. Setting $E_0 = \|u_0\|_{H^1(\mathbb{S})}^2 + \|\rho_0\|_{L^2(\mathbb{S})}^2$, we deduce that

$$m'(t) \leq -\frac{1}{2}m^2(t) + \frac{e+1}{4(e-1)} E_0 + 2A \sqrt{\frac{e+1}{2(e-1)}} E_0^{\frac{1}{2}}.$$

Set $K = (\frac{e+1}{4(e-1)}E_0 + 2A\sqrt{\frac{e+1}{2(e-1)}E_0^2})^{\frac{1}{2}}$. Note that if $m(0) < -\sqrt{2}K$, then $m(t) < -\sqrt{2}K$, for all $t \in [0, T)$. Therefore, we can solve the above inequality to obtain

$$\frac{m(0) + \sqrt{2}K}{m(0) - \sqrt{2}K}e^{\sqrt{2}Kt} - 1 \leq \frac{2\sqrt{2}K}{m(t) - \sqrt{2}K} \leq 0.$$

Due to $0 < \frac{m(0)+\sqrt{2}K}{m(0)-\sqrt{2}K} < 1$, there exists T_1 , and $0 < T_1 < \frac{1}{\sqrt{2}K} \ln(\frac{m(0)-\sqrt{2}K}{m(0)+\sqrt{2}K})$, such that $\lim_{t \uparrow T_1} m(t) = -\infty$. Applying Theorem 2.3, the solution z does not exist globally in time. \square

Remark 4.1. Theorem 4.1 is also true if $\int_{\mathbb{S}} \rho_0(x)dx = 0$ or ρ is odd. Moreover, Theorem 4.1 with $A = 0$ improves Theorem 4.2 in [45], where the Sobolev space $H^s \times H^{s-1}$ for $s \geq 2$ is considered.

We now give the second blow-up result.

Theorem 4.2. Let $z_0 = (\frac{u_0}{\rho_0}) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geq 3$, and let T be the maximal existence time of solution $z = (\frac{u}{\rho})$ to Eq. (2.1) with the initial data z_0 . Assume that $\int_{\mathbb{S}} u_0 dx = \frac{a_0}{2}$. If there is some $x_0 \in \mathbb{S}$ such that $\rho_0(x_0) = 0$ and for any $\epsilon > 0$

$$u'_0(x_0) < - \left(\frac{\epsilon + 2}{24}E_0 + \frac{\epsilon + 2}{4\epsilon}a_0^2 + A\sqrt{\frac{2(\epsilon + 2)}{3}E_0 + \frac{4(\epsilon + 2)}{\epsilon}a_0^2} \right)^{\frac{1}{2}},$$

where $E_0 = \|u_0\|_{H^1(\mathbb{S})}^2 + \|\rho_0\|_{L^2(\mathbb{S})}^2$, then the corresponding solution to Eq.(2.1) blows up in finite time.

Proof. By Lemma 2.3, we have $\int_{\mathbb{S}} u(t, x)dx = \int_{\mathbb{S}} u_0(x)dx = \frac{a_0}{2}$. Using Lemma 2.7, Corollary 2.1 and the above conservation law, we have

$$\|u\|_{L^\infty(\mathbb{S})} \leq \sqrt{\frac{\epsilon + 2}{24}E_0 + \frac{\epsilon + 2}{4\epsilon}a_0^2},$$

and

$$\|G * u\|_{L^\infty(\mathbb{S})} \leq \sqrt{\frac{\epsilon + 2}{24}E_0 + \frac{\epsilon + 2}{4\epsilon}a_0^2},$$

where we use Young’s inequality and the fact that $\|G\|_{L^1(\mathbb{S})} = 1$. Let $m(t)$ and $h(t)$ be the same as those defined in Theorem 4.1. Using (4.1) and the above inequality, we have

$$m'(t) \leq -\frac{1}{2}m^2(t) + \left(\frac{\epsilon + 2}{48}E_0 + \frac{\epsilon + 2}{8\epsilon}a_0^2 + 2A\sqrt{\frac{\epsilon + 2}{24}E_0 + \frac{\epsilon + 2}{4\epsilon}a_0^2} \right).$$

Set $K = \left(\frac{\epsilon+2}{48}E_0 + \frac{\epsilon+2}{8\epsilon}a_0^2 + 2A\sqrt{\frac{\epsilon+2}{24}E_0 + \frac{\epsilon+2}{4\epsilon}a_0^2} \right)^{\frac{1}{2}}$. Following the same argument as in Theorem 4.1, we deduce that the solution z blows up in finite time. \square

Remark 4.2. Theorem 4.2 with $A = 0$ covers the result of Theorem 4.3 in [45].

Letting $a_0 = 0$ and $\epsilon \rightarrow 0$ in Theorem 4.2, we have the following corollary immediately.

Corollary 4.1. *Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s \geq 3$, and let T be the maximal existence time of solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to Eq. (2.1) with the initial data z_0 . Assume that $\int_{\mathbb{S}} u_0 dx = 0$. If there is some $x_0 \in \mathbb{S}$ such that $\rho_0(x_0) = 0$ and*

$$u'_0(x_0) < - \left(\frac{E_0}{12} + 2A\sqrt{\frac{E_0}{3}} \right)^{\frac{1}{2}},$$

where $E_0 = \|u_0\|_{H^1(\mathbb{S})}^2 + \|\rho_0\|_{L^2(\mathbb{S})}^2$, then the corresponding solution to Eq. (2.1) blows up in finite time.

Remark 4.3. Corollary 4.1 is also true for u_0 and ρ_0 being odd. Furthermore, Corollary 4.1 with $A = 0$ covers the result of Corollary 4.2 in [45].

Next, we give a blow-up result if u_0 is odd and ρ_0 is even.

Theorem 4.3. *Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > \frac{3}{2}$, and let T be the maximal existence time of solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to Eq. (2.1) with the initial data z_0 . Assume that u_0 is odd, ρ_0 is even, $\rho_0(0) = 0$ and $u'_0(0) < 0$, then the corresponding solution to Eq. (2.1) blows up in finite time and $T < -2/u'_0(0)$.*

Proof. Let z be the solution to Eq. (2.1) with the initial data $z_0 \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > \frac{3}{2}$, and let $T > 0$ be the maximal time of existence of the solution z with the initial data z_0 .

Note that Eq. (2.1) is invariant under the transformation $(u, x) \rightarrow (-u, -x)$ and $(\rho, x) \rightarrow (\rho, -x)$. Thus, we deduce that if $u_0(x)$ is odd and $\rho_0(x)$ is even, then $u(t, x)$ is odd and $\rho(t, x)$ is even for any $t \in [0, T)$. By the oddness of $u(t, x)$, we have that $u(t, 0) = 0$. Define $m(t) = u_x(t, 0)$ and $h(t, x) = \rho(t, q(t, x))$. Note that $h(0, 0) = \rho(0, q(0, 0)) = \rho_0(0) = 0$. By Eq. (2.3) and the second equation in (2.1), we have

$$\frac{dh}{dt} = \rho_t + \rho_x q_t = -h(t, x)u_x(t, q(t, x)).$$

In view of Eq. (2.3), we deduce that if $u(t, x)$ is odd with respect to x , then $q(t, x)$ is also odd with respect to x . Then we have $q(t, 0) = 0$. By Lemma 2.2, we have $h(t, 0) = \rho(t, q(t, 0)) = \rho(t, 0) = 0$ for all $t \in [0, T)$.

Since u is odd and G is even, hence $(\partial_x^2 G * u)(t, 0) = 0$. Substituting $(t, 0)$ into Eq. (2.5), we obtain

$$\begin{aligned} m'(t) &= -\frac{1}{2}m^2(t) + u^2(t, 0) + \frac{1}{2}h^2(t, 0), \\ &\quad - G * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2)(t, 0) + A(\partial_x^2 G * u)(t, 0), \\ &\leq -\frac{1}{2}m^2(t). \end{aligned} \tag{4.2}$$

Here we use the relations $u(t, 0) = 0, h(t, 0) = 0$ and $G * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2) \geq 0$.

Note that $m(0) = u'_0(0) < 0$ and $m'(t) \leq 0$. We deduce that $m(t) < 0$ for all $t > 0$. Solving (4.2), we have

$$\frac{1}{m(0)} + \frac{1}{2}t \leq \frac{1}{m(t)} < 0.$$

The above inequality implies that $T < -\frac{2}{m(0)}$, and $u_x(t, 0)$ tends to negative infinity as t goes to T . This completes the proof of the theorem. □

Remark 4.4. If the condition $u'_0(0) < 0$ in Theorem 4.3 is replaced by the conditions $u'_0(0) \leq 0$ and $u_0 \neq 0$, then one can also deduce that the solution of Eq. (2.1) blows up in finite time. Moreover, Theorem 4.3 with $A = 0$ improves Theorem 4.4 in [45], where the Sobolev space $H^s \times H^{s-1}$ for $s \geq 2$ is considered.

Finally we give more insight into the blow-up rate for the wave breaking solutions to Eq. (2.1).

Theorem 4.4. *Let $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ be the solution to Eq. (2.1) with the initial data $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > \frac{3}{2}$ satisfying the assumption of Theorem 4.1, and let $T > 0$ be the maximal time of existence of the solution z . If $T < \infty$, we have*

$$\lim_{t \rightarrow T} \left(\inf_{x \in \mathbb{S}} u_x(t, x)(T - t) \right) = -2,$$

while the solution remains uniformly bounded.

Proof. By Corollary 2.1, we get the uniform bound of u . By (2.5) and the proof of Theorem 4.1, we find a constant $K > 0$ such that

$$\left| m'(t) + \frac{1}{2}m^2 \right| \leq K,$$

where K depends only on E_0 and A . It follows that

$$-K \leq m'(t) + \frac{1}{2}m^2 \leq K \quad \text{a.e. on } (0, T). \tag{4.3}$$

Choose $\epsilon \in (0, \frac{1}{2})$. Since $\liminf_{t \rightarrow T} \{ \inf_{x \in \mathbb{S}} u_x(t, x) \} = -\infty$ by Theorem 2.3, there is some $t_0 \in (0, T)$ with $m(t_0) < 0$ and $m^2(t_0) > \frac{K}{\epsilon}$. Let us first prove that

$$m^2(t) > \frac{K}{\epsilon} \quad t \in [t_0, T]. \tag{4.4}$$

Since m is locally Lipschitz (it belongs to $W^{1,\infty}_{loc}(\mathbb{R})$ by Lemma 2.5), there is some $\delta > 0$ such that

$$m^2(t) > \frac{K}{\epsilon} \quad t \in (t_0, t_0 + \delta).$$

Pick $\delta > 0$ maximal with this property. If $\delta < T - t_0$ we would have $m^2(t_0 + \delta) = \frac{K}{\epsilon}$ while

$$m'(t) \leq -\frac{1}{2}m^2 + K < -\frac{1}{2}m^2 + \epsilon m^2 \quad \text{a.e. on } (t_0, t_0 + \delta).$$

Note that m is locally Lipschitz and therefore absolutely continuous. Integrating the previous relation on $[t_0, t_0 + \delta]$ yields that

$$m(t_0 + \delta) \leq m(t_0) < 0.$$

It follows from the above inequality that

$$m^2(t_0 + \delta) \geq m^2(t_0) > \frac{K}{\epsilon}.$$

The obtained contradiction completes the proof of the relation (4.4).

By (4.3) and (4.4), we infer

$$\frac{1}{2} - \epsilon \leq -\frac{m'(t)}{m^2} \leq \frac{1}{2} + \epsilon \quad \text{a.e. on } (0, T). \tag{4.5}$$

Since m is locally Lipschitz on $[0, T]$ and (4.5) holds, it is easy to check $\frac{1}{m}$ is locally Lipschitz on (t_0, T) . Differentiating the relation $m(t) \cdot \frac{1}{m(t)} = 1, t \in (t_0, T)$, we get

$$\frac{d}{dt} \frac{1}{m(t)} = -\frac{m'(t)}{m^2(t)} \quad \text{a.e. on } (0, T).$$

For $t \in (t_0, T)$, integrating (4.5) on (t, T) to get

$$\left(\frac{1}{2} - \epsilon\right) (T - t) \leq -\frac{1}{m(t)} \leq \left(\frac{1}{2} + \epsilon\right) (T - t), \quad t \in (t_0, T).$$

Since $m(t) < 0$ on $[t_0, T)$, it follows that

$$\frac{1}{\frac{1}{2} + \epsilon} \leq -m(t)(T - t) \leq \frac{1}{\frac{1}{2} - \epsilon}, \quad t \in (t_0, T).$$

By the arbitrariness of $\epsilon \in (0, \frac{1}{2})$, the statement of the theorem follows. □

By the similar argument in Theorem 4.4, we have the following theorem.

Theorem 4.5. *Let $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ be the solution to Eq. (2.1) with the initial data $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > \frac{3}{2}$ satisfying the assumption of Theorem 4.3, and let $T > 0$ be the maximal time of existence of the solution z . If $T < \infty$, we have*

$$\lim_{t \rightarrow T} \left(\inf_{x \in \mathbb{S}} u_x(t, x)(T - t) \right) = -2,$$

while the solution remains uniformly bounded.

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