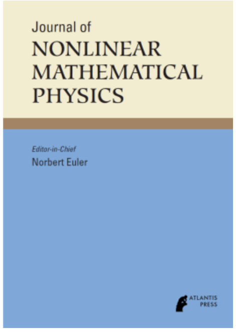

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A PROBLEM IN THE CLASSICAL THEORY OF WATER WAVES: WEAKLY NONLINEAR WAVES IN THE PRESENCE OF VORTICITY

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The classical water-wave problem is described, and two parameters (ε-amplitude; δ-long wave or shallow water) are introduced. We describe various nonlinear problems involving weak nonlinearity ($\varepsilon \to 0$) associated with equations of integrable type ("soliton" equations), but with vorticity. The familiar problem of propagation governed by the Korteweg–de Vries (KdV) equation is introduced, but allowing for an arbitrary distribution of vorticity. The effect of the constant vorticity on the solitary wave is described. The corresponding problem for the Nonlinear Schrödinger (NLS) equation is briefly mentioned but not explored here. The problem of two-way propagation (admitting head-on collisions), as described by the Boussinesq equation, is examined next. This leads to a new equation: the Boussinesq-type equation valid for constant vorticity. However, this cannot be transformed into an integrable Boussinesq equation (as is possible for the corresponding KdV and NLS equations). The solitary-wave solution for this new equation is presented. A description of the Camassa–Holm equation for water waves, with constant vorticity, with its solitary-wave solution, is described. Finally, we outline the problem of propagation of small-amplitude, large-radius ring waves over a flow with vorticity (representing a background flow in one direction). Some properties of this flow, for constant vorticity, are described.

Keywords: Water waves; soliton equations; asymptotic expansions; nonlinear waves; vorticity.

Mathematics Subject Classification: 76B15, 34E10

1. Introduction

There has been a considerable surge of interest, over the last decade or two, in the theory of water waves, an interest that has accelerated in recent years, with the emphasis on flows with vorticity. The main thrust has been directed towards the development of rigorous theories for steady, periodic waves in the presence of vorticity (and notably for the special case of constant vorticity). This work has provided general descriptions of the surface wave, e.g. the existence of symmetric profiles, the conditions that permit the appearance of stagnation points in the flow and the associated structure in the presence of critical layers. The main developments of this work can be found in [8–10, 13, 15, 19, 39, 40] (and the references therein). Additionally, there has been significant progress in the construction of numerical solutions for many of these problems (see [32, 33, 35]), which have built on earlier work that can be found, for example, in [16, 37]. Although these numerical solutions provide specific examples of possible solutions, it is likely that any analytical details will be accessed only by appropriate asymptotic approximations (because any general, exact solution is quite beyond our skills). Indeed, such results — some obtained well before the appearance of the recent work — can be used as a check on, and a clarification of, some of the properties of these flows.

The main aim in this review is to present some of the existing, and perhaps familiar, asymptotic results that describe the propagation of waves on water that is moving according to some prescribed vorticity distribution. We will, however, include some specific detailed comparisons of various flows, e.g. constant vorticity as compared with zero vorticity, and add some new results that relate to bidirectional propagation with vorticity (described by a new version of the Boussinesq equation). The comparisons will be made, in the main, by constructing various solitary-wave solutions (although we might mention soliton solutions as well); this choice is simply because, in our asymptotic description, this is far easier than working with the periodic solutions. The periodic scenario is, by and large, the preferred choice for the development of the rigorous approach to these flow problems. Nevertheless, we can expect that the predictions in these two formats will not differ greatly, because a large period in the periodic problem — and this is always allowed — should correspond quite closely to the problem on the infinite domain.

The various researches cited above (and the further references included therein) show how the surface wave and the underlying vorticity distribution interact; for arbitrary amplitude waves, up to the highest, this is a very involved process, although the structure is rather simpler for small-amplitude waves. For constant vorticity, the flow structure may be particularly accessible, indicating, for example, the important differences between zero, and non-zero, vorticity. It has been shown, in the simplest problem of this type — constant vorticity — that there exist regions of parameter space (defined by the constant vorticity, total energy, mass flux and maximum amplitude) which correspond to periodic-wave solutions with, for example, stagnation points and critical layers; see e.g. [14, 18, 41]. Although the details presented here cannot capture all of this complicated structure, we can rehearse some of the relevant background work in this area, and so provide results (and observations) that will exemplify some of the phenomena encountered, and can be used as a comparison or check on the more general properties deduced from the rigorous analyses.

The plan in this review is to introduce the general equations that describe the classical water wave problem, primarily for one-dimensional propagation, and then very briefly outline the procedure that leads to the familiar (and famous) Korteweg–de Vries (KdV) equation. This is extended to allow for a pre-existing background flow with some prescribed vorticity; the discussion of the possible appearance of a critical layer will not be pursued, although the special case of constant vorticity will be explored with some care (and no critical layer is possible in this case, for small-amplitude waves). The corresponding problem that leads to the nonlinear Schrödinger (NLS) equation will also be introduced, but the details in this case will not be developed. One reason is because of the complicated nature of the various coefficients involved in the construction of the equation but, more significantly, the type of solution represented by the NLS equation is not directly relevant in the context of this review. We also include the intriguing problem of wave propagation controlled by the Camassa–Holm (CH) equation, but in the presence of vorticity (with detailed results being given for constant vorticity). As a continuation of this theme, we present some new results for bidirectional propagation (governed by a suitable Boussinesq equation) with vorticity. We conclude with a brief discussion of a geometrically more complicated scenario: ring waves on a unidirectional flow with vorticity, with details presented for constant vorticity.

2. Governing Equations

The problem, typically referred to as the classical problem of water waves, takes as its model an incompressible, inviscid fluid with zero surface tension. Further, the water moves over an impermeable bed (which we take here to be stationary and horizontal), with a constant pressure — atmospheric pressure — at the free surface. The fundamental governing equations are then Euler's equation and the equation of mass conservation:

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \mathbf{F}; \quad \nabla \cdot \mathbf{u} = 0, \tag{2.1}$$

with

$$p = p_a = \text{constant} \quad \text{and} \quad w = \frac{\mathrm{D}h}{\mathrm{D}t} \quad \text{on } z = h(\mathbf{x}_\perp, t) \tag{2.2}$$

and

$$w = 0 \quad \text{on } z = 0, \tag{2.3}$$

where $\mathbf{F} = (0, 0, -g)$, for constant g, and ρ is the constant density of water; $\mathrm{D}/\mathrm{D}t$ is the familiar material derivative. The velocity in the fluid has been written as $\mathbf{u} = (\mathbf{u}_\perp, w)$, with $h = h(\mathbf{x}_\perp, t)$, where $\mathbf{x}_\perp$ is the 2-vectors perpendicular to the z-coordinate, with the associated velocity vector $\mathbf{u}_\perp$; w is the component of the velocity in the z-direction. It is useful, at this early stage, to retain the two-dimensionality of the surface (even though much of our analysis will be for one-dimensional plane waves) because we shall need this extra freedom in the final calculation that we present here.

The next stage in any problem of this type — or, indeed, in a systematic approach to any problem in applied mathematics — is to non-dimensionalize the set (2.1)–(2.3) by introducing suitable general scales that describe the class of problems under consideration. Let h_0 be the depth of the water in the absence of waves, and λ an average or typical wavelength of the wave; an associated speed scale is then $\sqrt{gh_0}$, with a corresponding time scale $\lambda/\sqrt{gh_0}$. This speed scale, however, is used only for the horizontal velocity components ($\mathbf{u}_\perp = (u, v)$); in order to be consistent with the equation of mass conservation (and so, equivalently, consistent with the existence of a stream function), the vertical component of the velocity (w) is non-dimensionalized by using $h_0\sqrt{gh_0}/\lambda$. Thus we non-dimensionalize according to the transformation

$$\begin{aligned} &\mathbf{x}_\perp \to \lambda\mathbf{x}_\perp, \quad z \to h_0 z, \quad t \to (\lambda/\sqrt{gh_0})t, \\ &\mathbf{u}_\perp \to \sqrt{gh_0}\,\mathbf{u}_\perp, \quad w \to (h_0\sqrt{gh_0}/\lambda)w, \end{aligned} \tag{2.4}$$

where "→" is to be read as "replace by" (so that, for convenience, the current notation is retained, but now all the variables are non-dimensional versions of those introduced earlier). Further, we also introduce

$$h = h_0 + a\eta \quad \text{and} \quad p \to p_a + \rho g(h_0 - z) + \rho g h_0 p, \tag{2.5}$$

where a is a measure of the wave amplitude and the new p is a non-dimensional pressure that measures the deviation away from the hydrostatic pressure distribution. (Note that, in the transformation for p, the original, dimensional z is used.) We now use (2.4) and (2.5) in equations (2.1)–(2.3) to give

$$\frac{\mathrm{D}\mathbf{u}_\perp}{\mathrm{D}t} = -\nabla_\perp p, \quad \delta^2 \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z}, \quad \nabla \cdot \mathbf{u} = 0, \tag{2.6}$$

with

$$p = \varepsilon\eta \quad \text{and} \quad w = \varepsilon \left\{ \frac{\partial \eta}{\partial t} + (\mathbf{u}_\perp \cdot \nabla_\perp)\eta \right\} \quad \text{on } z = 1 + \varepsilon\eta(\xi_\perp t) \tag{2.7}$$

and

$$w = 0 \quad \text{on } z = 0. \tag{2.8}$$

Here, we have introduced the two familiar, and fundamental, parameters that characterize the classical water wave problem: $\varepsilon = a/h_0$, the amplitude parameter, and $\delta = h_0/\lambda$, the long wavelength (or shallowness) parameter. The final stage is to scale these equations with respect to ε.

The case $\varepsilon = 0$ recovers undisturbed conditions in the absence of any waves; indeed, as $\varepsilon \to 0$, so the disturbance associated with the wave propagation vanishes. Although there are many problems for which we may not wish to take this limit — we could elect to examine the "fully nonlinear" problem — the equations must be consistent with this choice. Thus we further redefine our variables according to the additional transformation

$$(\mathbf{u}_\perp, w, p) \to \varepsilon(\mathbf{u}_\perp, w, p) \tag{2.9}$$

when the underlying flow configuration is that of stationary water; we will allow for an existing background vorticity shortly. The final form of our governing equations, at this stage, therefore becomes

$$\frac{D\mathbf{u}_\perp}{\mathrm{D}t} = -\nabla_\perp p, \quad \delta^2 \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z}, \quad \nabla \cdot \mathbf{u} = 0, \tag{2.10}$$

with

$$p = \eta \quad \text{and} \quad w = \frac{\partial \eta}{\partial t} + \varepsilon(\mathbf{u}_\perp \cdot \nabla_\perp)\eta \quad \text{on } z = 1 + \varepsilon\eta \tag{2.11}$$

and

$$w = 0 \quad \text{on } z = 0, \tag{2.12}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \varepsilon(\mathbf{u} \cdot \nabla).$$

This version of the problem allows for the identification of various problems of practical interest, accessed by making suitable choices of the two parameters: neglecting either one or the other, for example, or choosing some special relation between them.

Our main interest here is one-dimensional (plane wave) propagation, so we make a further simplification of the equations above. In addition, we take the opportunity, initially, to remove the explicit dependence on δ; this is accomplished by performing one more transformation:

$$\mathbf{x}_\perp = \frac{\delta}{\sqrt{\varepsilon}}\mathbf{X}_\perp, \quad t = \frac{\delta}{\sqrt{\varepsilon}}T, \quad w = \frac{\sqrt{\varepsilon}}{\delta}W \tag{2.13}$$

which produces the set (2.10)–(2.12) with δ^2 replaced by ε, for arbitrary δ. This version of the equations is used in a brief description of the classical derivation of the KdV equation.

3. The KdV Equation

We start with equations (2.10)–(2.12), with (2.13) incorporated, restricted to plane waves propagating in the X-direction, where $\mathbf{X}_\perp = (X, Y)$ and $\mathbf{u}_\perp = (u, 0)$; thus we introduce

$$\xi = X - T, \quad \tau = \varepsilon T,$$

which describes a far field for right-running waves. With all these choices, the governing equations become

$$\begin{gathered} -u_\xi + \varepsilon(u_\tau + uu_\xi + Wu_z) = -p_\xi; \quad \varepsilon\{-W_\xi + \varepsilon(W_\tau + uW_\xi + WW_z)\} = -p_z; \\ u_\xi + W_z = 0, \end{gathered} \tag{3.1}$$

with

$$p = \eta \quad \text{and} \quad W = -\eta_\xi + \varepsilon(\eta_\tau + u\eta_\xi) \quad \text{on } z = 1 + \varepsilon\eta \tag{3.2}$$

and

$$W = 0 \quad \text{on } z = 0. \tag{3.3}$$

In these equations, we have introduced subscripts to denote partial derivatives. We seek a solution of this set for $z \in [0, 1+\varepsilon\eta]$, $-\infty < \xi < \infty$ and $\tau \geq 0$, and for suitable initial data.

The procedure now is to seek a formal asymptotic solution of this system, expressed as

$$\eta(\xi, \tau; \varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n \eta_n(\xi, \tau) \quad \text{and} \quad q(\xi, \tau, z; \varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n q_n(\xi, \tau, z), \tag{3.4}$$

where q (and correspondingly q_n) represents each of u, w and p. These asymptotic expansions are, typically, uniformly valid as $\varepsilon \to 0$, for $0 \leq \tau < \tau_0$, for some fixed τ_0, provided that the initial data decays sufficiently rapidly as $|\xi| \to \infty$. Indeed, matching to the near field (defined by $T = \mathrm{O}(1)$, and containing $T = 0$) shows that the problem in the far field satisfies the initial data prescribed at $T = 0$. The procedure is altogether routine (and possibly familiar), resulting in, for example,

$$p \sim \eta_0 + \varepsilon\left\{\eta_1 + \frac{1}{2}(1 - z^2)\eta_{0\xi\xi}\right\}$$

and

$$W \sim -z\eta_{0\xi} + \varepsilon\left\{\left(\eta_{1\xi} + \eta_{0\tau} + \eta_0\eta_{0\xi} + \frac{1}{2}\eta_{0\xi\xi\xi}\right)z + \frac{1}{6}z^3\eta_{0\xi\xi\xi}\right\},$$

both defined for $0 \leq z \leq 1$. This approach has required the two boundary conditions at the surface to be expressed as Taylor expansions about $z = 1$, which is easily seen to be a valid manoeuvre here because the solution, at all orders, turns out to be polynomial in z. In the above, the function $\eta_0(\xi, \tau)$ satisfies the KdV equation

$$2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = 0. \tag{3.5}$$

This equation provides the basis for a discussion of this class of water waves, which includes the solitary wave and soliton interactions; for more background to water waves generally, and to soliton dynamics, see [17, 28].

This short reminder of the standard approach, and resulting equation, has been provided as a natural introduction to what follows. We now turn to the development of a theory for weakly nonlinear waves (i.e. associated with $\varepsilon \to 0$) in the presence of vorticity, with a detailed discussion of the case of constant vorticity.

4. The KdV Equation with Vorticity

Our primary interest in this paper is the propagation of weakly nonlinear waves, of various types, in a flow that is moving according to some prescribed vorticity. (Because such a flow can be regarded as a model for a real flow, which can exhibit viscous and turbulent properties, this is often called a "shear" flow.) We start by reformulating our governing equations, (2.10)–(2.12) with (2.13), by introducing the background vorticity — which we take to be O(1) relative to the scales previously defined — in the form (for plane waves)

$$\varepsilon\mathbf{u}_\perp = \varepsilon(u, 0) \quad \text{is replaced by } (U(z) + \varepsilon u, 0),$$

where $U(z)$ is given. The equations therefore become

$$u_T + Uu_X + U'W + \varepsilon(uu_X + Wu_z) = -p_X; \tag{4.1}$$

$$\varepsilon[W_T + UW_X + \varepsilon(uW_X + WW_z)] = -p_z; \quad u_X + W_z = 0, \tag{4.2}$$

with

$$p = \eta \quad \text{and} \quad W = \eta_T + U\eta_X + \varepsilon u\eta_X \quad \text{on } z = 1 + \varepsilon\eta \tag{4.3}$$

and

$$W = 0 \quad \text{on } z = 0, \tag{4.4}$$

where $U' = \mathrm{d}U/\mathrm{d}z$. The problem is now recast — as we did for the classical KdV equation — in suitable far-field variables, although in this case we do not know the speed of propagation, c, of (linear) waves at the outset; thus we set

$$\xi = X - cT, \quad \tau = \varepsilon T.$$

Equations (4.1)–(4.4) thus become

$$(U-c)u_\xi + U'W + \varepsilon(u_\tau + uu_\xi + Wu_z) = -p_\xi; \tag{4.5}$$

$$\varepsilon\{(U-c)W_\xi + \varepsilon(W_\tau + uW_\xi + WW_z)\} = -p_z; \quad u_\xi + W_z = 0, \tag{4.6}$$

with

$$p = \eta \quad \text{and} \quad W = (U-c)\eta_\xi + \varepsilon(\eta_\tau + u\eta_\xi) \quad \text{on } z = 1 + \varepsilon\eta \tag{4.7}$$

and

$$W = 0 \quad \text{on } z = 0. \tag{4.8}$$

We now seek an asymptotic solution exactly as before; see (3.4).

At leading order, Eqs. (4.5)–(4.8) yield the problem

$$(U-c)u_{0\xi} + U'W_0 = -p_{0\xi}; \quad p_{0z} = 0; \quad u_{0\xi} + W_{0z} = 0,$$

with

$$p_0 = \eta_0 \quad \text{and} \quad W_0 = (U-c)\eta_{0\xi} \quad \text{on } z = 1,$$

and

$$W_0 = 0 \quad \text{on } z = 0.$$

The appropriate solution of this set, defined for $z \in [0,1]$, is

$$p_0 = \eta_0, \quad W_0 = (U-c)I_2\eta_{0\xi}, \quad u_0 = -\eta_0\frac{\mathrm{d}}{\mathrm{d}z}[(U-c)I_2], \tag{4.9}$$

where

$$I_2(z) = \int_0^z \frac{\mathrm{d}z'}{[U(z')-c]^2} \tag{4.10}$$

and then c is determined from

$$I_2(1) = 1 \quad (\text{written } I_{21} = 1) \text{ or } \int_0^1 \frac{dz}{[U(z)-c]^2} = 1. \tag{4.11}$$

This last result is the famous *Burns condition*, [4, 38], which, for some choices of $U(z)$, can lead to the presence of critical layers (i.e. where $U(z) = c$ for some $z \in (0,1)$), even for infinitesimally small waves; for more on these ideas see [2, 25, 27, 34]. We will consider, here, only situations where critical layers do not appear in the linearized problem. The procedure at the next order follows, precisely, the pattern laid down for the classical KdV equation (§3); this produces

$$-2I_{31}\eta_{0\tau} + 3I_{41}\eta_0\eta_{0\xi} + J_1\eta_{0\xi\xi\xi} = 0, \tag{4.12}$$

where

$$I_{n1} = I_n(1) = \int_0^1 \frac{\mathrm{d}z}{[U(z)-c]^n} \quad \text{and} \quad J_1 = \int_0^1\int_z^1\int_0^\zeta \frac{[U(\zeta)-c]^2}{[U(z)-c]^2[U(Z)-c]^2}\mathrm{d}Z\mathrm{d}\zeta\mathrm{d}z.$$

(More details can be found in [21], which describes a development, and generalization, of the seminal work presented in [1].)

If we set aside the complications that arise when critical layers are present (see [25]), the results represented by (4.12) are most encouraging. This is a KdV equation, with appropriate constant coefficients, which has, for example, solitary-wave and soliton solutions — and this holds for any background vorticity. Thus we can expect that these solutions will be relevant to wave propagation in (almost) any realistic one-dimensional flow of water. In particular, let us briefly consider the special case of constant vorticity.

We suppose that the original equations (expressed in X, T, u variables) have been written in the Galilean frame in which $U(0) = 0$; indeed, this will be the physical frame if $U(z)$ models a "shear" flow with a no-slip condition on the bed. Now we write $U(z) = \gamma z$, for $\gamma =$ constant, and then the speeds of small-amplitude waves are given (from (4.11)) by

$$c = \frac{1}{2}(\gamma \pm \sqrt{4+\gamma^2}) \tag{4.13}$$

and no critical layers exist for this choice; then, correspondingly, the KdV equation (4.12) becomes

$$\pm\sqrt{4+\gamma^2}\eta_{0\tau} + (3+\gamma^2)\eta_0\eta_{0\xi} + \frac{1}{3}\left(1 - \frac{\gamma}{c}\right)\eta_{0\xi\xi\xi} = 0. \tag{4.14}$$

(We see immediately that the problem of zero vorticity, $\gamma = 0$, recovers our classical KdV equation, (3.5), for propagation to the right i.e. the upper sign.) It is of some interest, and possible relevance to more general theories of waves with vorticity, to investigate how the surface-wave profile is affected by the underlying shear flow, according to this model. To this end, we might consider two cases: $\gamma = \mp\omega$ (= constant), where the upper sign corresponds to *positive* vorticity in much of the rigorous work cited earlier. (This mismatch has arisen because here we have elected to use the (x, z) coordinate system, whereas the other work uses the (x, y) system, both being interpreted in the conventional (x, y, z) right-handed, rectangular Cartesian system, and then the two vorticities differ by a sign.) We should note, however, that here the choice of sign for the constant vorticity is unimportant: the underlying flow is then either to the left or the right — both generate the same mathematical problem — and then the crucial identification, in either case, is whether the surface wave is propagating upstream or downstream. To exemplify the essential character of the effect of constant vorticity, Fig. 1 shows three solitary waves, all of the same amplitude, and all exact solutions of Eq. (4.14). The middle profile is for zero vorticity — the irrotational case $\gamma = 0$; the outer ("wider") solitary wave is an example ($\gamma = 1$) of the profile obtained for

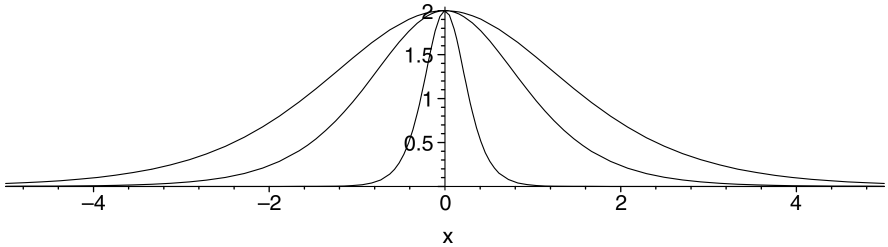

Fig. 1. The profile of the sech^2 solitary wave of the KdV equation, (4.14), for various γ: middle profile $\gamma = 0$; outer (broader) profile is for $\gamma = 1$, upstream propagation; inner (narrower) profile is for $\gamma = 1$, downstream propagation.

upstream propagation (the sign of γ is immaterial), and the narrower profile is that obtained for downstream propagation. This observation — equivalently a longer wave upstream and shorter wave downstream — is precisely that made in [1]. Similar observations apply to other solutions of this KdV equation, such as soliton interactions and cnoidal waves.

5. The NLS Equation with Vorticity

The analysis that leads to the NLS equation is, in one sense, in keeping with the theme of this review but, on two other counts, it is rather out on a limb. Firstly, the type of solution — a modulated harmonic wave — is very different from all the other types of solution that we discuss here (although the connection with inverse scattering theory still exists, of course). Secondly, the technical details are far more involved, even for the case of constant vorticity, so that the production of useful results is very lengthy. With both these points in mind, we give only a bare outline of the procedure and resulting solution-structure; the interested reader may explore the ideas further through [24].

We seek a solution for which the initial wave-profile takes the form

$$A(\varepsilon x)\exp(ikx) + \text{c.c.},$$

where "c.c." denotes the complex conjugate and k is a general wave number; the governing equations are those that retain the parameter δ, and which include a background flow, $U(z)$. Thus from (2.10) to (2.12), combined with the formulation used in (4.1)–(4.4), we have

$$u_t + Uu_x + U'w + \varepsilon(uu_x + wu_z) = -p_x; \tag{5.1}$$

$$\delta^2[w_t + Uw_x + \varepsilon(uw_x + ww_z)] = -p_z; \quad u_x + w_z = 0, \tag{5.2}$$

with

$$p = \eta \quad \text{and} \quad w = \eta_t + U\eta_x + \varepsilon u\eta_x \quad \text{on } z = 1 + \varepsilon\eta \tag{5.3}$$

and

$$w = 0 \quad \text{on } z = 0. \tag{5.4}$$

The appropriate "slow" evolution variables that we need here are

$$\xi = x - c_p t, \quad \zeta = \varepsilon(x - c_g t), \quad \tau = \varepsilon^2 t,$$

where $c_p(k)$ and $c_g(k)$ are the phase and group speeds, respectively. The asymptotic expansion, at fixed δ for $\varepsilon \to 0$, for the surface wave, takes the form

$$\eta(\xi, \zeta, \tau; \varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n \sum_{m=0}^{n+1} A_{nm}(\zeta, \tau)E^m + \text{c.c.},$$

where $E = \exp(ik\xi)$ and $A_{00} = 0$; corresponding expansions (with z as an additional argument) are used for each of u, w and p. This form of expansion admits the appropriate nonlinear interactions that generate higher harmonics. The procedure is altogether routine, but very involved by virtue of the z-dependence. In summary, we find that

$$p_{01} = P(z)A_{01}(\zeta, \tau) \quad \text{where } \frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{\mathrm{d}P/\mathrm{d}z}{[U(z) - c_p]^2}\right\} - \left(\frac{\delta k}{U(z) - c_p}\right)^2 P = 0, \tag{5.5}$$

a Rayleigh equation, with

$$P(1) = 1, \quad P'(0) = 0 \quad \text{and} \quad P'(1) = (\delta k)^2 [U(1) - c_p]^2, \tag{5.6}$$

this last condition being equivalent to

$$\int_0^1 \frac{P(z)}{[U(z) - c_p]^2} \mathrm{d}z = 1, \tag{5.7}$$

a generalization of the Burns condition, cf. (4.11), which determines the phase speed. We then obtain, for example,

$$u_{01} = -A_{01} \frac{\mathrm{d}}{\mathrm{d}z} \{[U(z) - c_p] I\} \quad \text{and} \quad w_{01} = ikA_{01}[U(z) - c_p] I,$$

where

$$I(z) = \int_0^z \frac{P(z')}{[U(z') - c_p]^2} \mathrm{d}z';$$

we also find that

$$c_g = c_p - \frac{\left(\int_0^1 \frac{P(z)}{U(z) - c_p} \mathrm{d}z \right) - 1}{[U(1) - c_p]^{-1} + \int_0^1 \frac{I(z)P(z)U'(z)}{[U(z) - c_p]^2} \mathrm{d}z},$$

which does satisfy the familiar identity $c_g = \frac{\mathrm{d}}{\mathrm{d}k}(kc_p)$. Finally, we obtain the classical NLS equation for the leading term that describes the amplitude modulation of the surface wave:

$$-2ik\alpha A_{01\tau} + \beta A_{01\zeta\zeta} + \kappa A_{01} |A_{01}|^2 = 0, \tag{5.8}$$

but here the constant coefficients α, β and κ are complicated expressions involving numerous integrals in z (which mirrors the type of result that was obtained for the KdV equation, (4.12)).

Although it is most gratifying that we have obtained the familiar NLS equation — and so the basic structure of an evolving wave packet holds for arbitrary vorticity — the details are very difficult to access. Certainly, the recovery of the standard variant of the NLS equation, for irrotational water waves (see [22, 24]), is quite straightforward, but even the case of constant vorticity is quite daunting. We will briefly indicate how this process can be started. We set $U(z) = \gamma z$ (γ = constant) and then the equation for $P(z)$, (5.5), becomes

$$(\gamma z - c_p) P'' - 2\gamma P' - \delta^2 k^2 (\gamma z - c_p) P = 0$$

which has the general solution

$$P = A(Z \sinh Z - \cosh Z) + B(Z \cosh Z - \sinh Z) \quad [Z = \delta k(\gamma z - c_p)/\gamma],$$

where A and B are arbitrary constants. This, in turn, gives the phase speeds as

$$c_p = \gamma \left(1 - \frac{1}{2} \frac{\tanh(\delta k)}{\delta k} \right) \pm \frac{1}{2\delta k} \sqrt{4\delta k \tanh(\delta k) + \gamma^2 \tanh^2(\delta k)}$$

which agrees with the speeds, for long waves ($\delta k \to 0$), as obtained for the KdV equation; see (4.13). The determination of all the other expressions, as developed in [24], eventually leads (after a lengthy sequence of calculations) to an explicit form of the NLS equation appropriate to this problem. This is not pursued here because this particular form of wave development — a modulated harmonic wave — is not directly relevant as a background to, or check of, the recent rigorous results (and providing this information is the main aim of this work).

6. The Boussinesq Equation with Vorticity

The asymptotic expansions that underpin the derivation of both the KdV and NLS equations have been developed and applied in their most natural and reliable form: the leading order, based on the asymptotic sequence $\{\varepsilon^n\}$, gives rise to the appropriate governing equation. A less satisfactory application of this approach arises when terms of different order are retained, and truncation, e.g. beyond a particular power of ε, is imposed. The first of two examples of this type — the second will be the CH equation (§7) — occurs in the problem of two-way propagation: the Boussinesq equation, [3, 23]. This equation admits, up to quadratic nonlinearity, exact solutions that describe waves that propagate to the left and to the right. Thus this equation can be used as a model for waves all moving in the same direction (just as for the KdV equation), or waves that suffer a head-on collision. The appropriate governing equations are written in terms of our original $(X,T) = \mathrm{O}(1)$ variables (i.e. not scaled into a far field); thus we have (cf. (4.1)–(4.4) and (5.1)–(5.4)), for waves on stationary water,

$$u_T + \varepsilon(uu_X + Wu_z) = -p_X;$$
$$\varepsilon[W_T + \varepsilon(uW_X + WW_z)] = -p_z; \quad u_X + W_z = 0,$$

with

$$p = \eta \quad \text{and} \quad W = \eta_t + \varepsilon u\eta_X \quad \text{on } z = 1 + \varepsilon\eta$$

and

$$W = 0 \quad \text{on } z = 0.$$

Because the asymptotic formulation used here does not follow the most satisfactory route, we briefly outline the standard derivation that leads to the Boussinesq equation (before we turn to a new derivation that includes constant vorticity). We seek a solution in the familiar form (cf. (3.4))

$$\eta(X,T;\varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n \eta_n(X,T) \quad \text{and} \quad q(X,T,z;\varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n q_n(X,T,z),$$

where q (and correspondingly q_n) represents each of u, W and p, which gives

$$p_0 = \eta_0, \quad u_{0T} = -\eta_{0X}, \quad W_0 = -zu_{0X}, \quad u_{0X} = -\eta_{0T} \quad (0 \le z \le 1) \tag{6.1}$$

(so that $\eta_{0TT} - \eta_{0XX} = 0$). Then, for example, at the next order, we find that

$$p_1 = -\frac{1}{2}(1 - z^2)u_{0XT} + \eta_1 \tag{6.2}$$

and

$$W_{1T} = \left[(u_0 u_{0X})_X + \eta_{1XX} - \frac{1}{2}u_{0XXXT}\right] z + \frac{1}{6}z^3 u_{0XXXT}, \tag{6.3}$$

which lead to

$$\eta_{1TT} - \eta_{1XX} - \left(\frac{1}{2}\eta_0^2 + u_0^2\right)_{XX} - \frac{1}{3}\eta_{0XXXX} = 0. \tag{6.4}$$

We now form the equation for $\eta = \eta_0 + \varepsilon\eta_1 + \mathrm{O}(\varepsilon^2)$:

$$\eta_{TT} - \eta_{XX} - \varepsilon\left\{\frac{1}{2}\eta^2 + \left(\int_X^\infty \eta_T \mathrm{d}X'\right)^2\right\}_{XX} - \frac{1}{3}\varepsilon\eta_{XXXX} = \mathrm{O}(\varepsilon^2), \tag{6.5}$$

where we have used

$$u \sim \int_X^\infty \eta_T(X', T)\mathrm{d}X' \tag{6.6}$$

(on the assumption that $u \to 0$ as $X \to \infty$). For a discussion of the relevance of this model to water waves, see [28].

However, Eq. (6.5) is not written in a "completely integrable" form; this is accomplished by expressing the equation in a Lagrangian (rather than Eulerian) frame i.e.

$$\bar{X} = X - \varepsilon\int_X^\infty \eta(X', T; \varepsilon)\mathrm{d}X' \quad \text{with } \eta - \varepsilon\eta^2 = H(\bar{X}, T; \varepsilon).$$

The resulting equation for H is then

$$H_{TT} - H_{\bar{X}\bar{X}} - \frac{3}{2}\varepsilon(H^2)_{\bar{X}\bar{X}} - \frac{1}{3}\varepsilon H_{\bar{X}\bar{X}\bar{X}\bar{X}} = \mathrm{O}(\varepsilon^2) \tag{6.7}$$

which is a version of the Boussinesq equation, completely integrable for $\forall\varepsilon > 0$ (see [23]) (after setting the error term to zero, of course).

The investigation of interest here is the development of theories for (weakly nonlinear) waves over a "shear" flow. In the light of the detailed results described earlier, and because of the complexities of constructing a Boussinesq equation for arbitrary vorticity, we limit the discussion to the case of constant vorticity ($U(z) = \gamma z$). The governing equations are therefore

$$u_T + \gamma z u_X + \gamma W + \varepsilon(u u_X + W u_z) = -p_X; \tag{6.8}$$

$$\varepsilon[W_T + \gamma z W_X + \varepsilon(u W_X + W W_z)] = -p_z; \quad u_X + W_z = 0, \tag{6.9}$$

with

$$p = \eta \quad \text{and} \quad W = \eta_T + \gamma(1 + \varepsilon\eta)\eta_X + \varepsilon u\eta_X \quad \text{on } z = 1 + \varepsilon\eta \tag{6.10}$$

and

$$W = 0 \quad \text{on } z = 0; \tag{6.11}$$

cf. Eqs. (4.1)–(4.4).

The form of the asymptotic expansion is exactly as used in the case for $\gamma = 0$; at leading order, we find that, for $z \in [0, 1]$,

$$p_0 = \eta_0, \quad u_{0X} = -(\eta_{0T} + \gamma\eta_{0X}), \quad u_{0T} = -\eta_{0X}, \quad W_0 = (\eta_{0T} + \gamma\eta_{0X})z, \tag{6.12}$$

which gives

$$\eta_{0TT} + \gamma\eta_{0XT} - \eta_{0XX} = 0 \tag{6.13}$$

and $u_0(X, T)$ satisfies this same equation (but related to $\eta_0(X, T)$ as described in (6.12)). At the next order, we find (for example) that

$$p_1 = \frac{1}{2}Q_T(1 - z^2) + \frac{1}{3}Q_X(1 - z^3) + \eta_1,$$

where $Q(X, T) = \eta_{0T} + \gamma\eta_{0X}$, and

$$W_{1T} = \left\{\frac{1}{2}Q_{XXT} + \frac{1}{3}\gamma Q_{XXX} + \eta_{1XX} + (u_0u_{0X})_X\right\}(z - z^3)$$
$$+ \{\eta_{1TT} + \gamma\eta_{1XT} + \gamma(\eta_0\eta_{0X})_T + (u_0\eta_{0X})_T - (Q\eta_0)_T\}z^3.$$

The equation for $\eta_1(X, T)$ is then given by

$$\eta_{1TT} + \gamma\eta_{1XT} - \eta_{1XX} = \frac{1}{3}(\eta_0 - \gamma u_0)_{XXXX} + \left(u_0^2 + \frac{1}{2}\eta_0^2 + \gamma u_0\eta_0 + \frac{1}{2}\gamma^2\eta_0^2\right)_{XX} \tag{6.14}$$

with

$$u_0 = -\gamma\eta_0 + \int_X^\infty \eta_{0T}(X', T)\mathrm{d}X', \tag{6.15}$$

assuming decay conditions ahead ($X \to \infty$) of the wave. The equation for $\eta \sim \eta_0 + \varepsilon\eta_1$, constructed from (6.13) to (6.15), recovers Eq. (6.5) for $\gamma = 0$. When $\gamma \neq 0$, the equation for η becomes

$$\eta_{TT} + \gamma\eta_{XT} - \eta_{XX} = \frac{1}{3}\varepsilon(\eta - \gamma u_0)_{XXXX} + \varepsilon\left[\frac{1}{2}(1 + \gamma^2)\eta^2 + u_0^2 + \gamma u_0\eta\right]_{XX} + \mathrm{O}(\varepsilon^2), \tag{6.16}$$

where u_0 satisfies (6.13) with $u_{0T} \sim -\eta_X$; this is therefore a generalization of Eq. (6.4) which, in that case, can be transformed into the Boussinesq equation.

The new equation, (6.16), cannot, however, be transformed into the — or a form of — classical Boussinesq equation, (6.7), by any transformation that corresponds to the Lagrangian form used above i.e. $X \to \bar{X}$ using a transform linear in integrals of η and u_0, together with $\eta \to H$ using all terms of degree two in η and u_0. (The difficulties are readily seen by writing (6.16), first, in a frame moving at speed $\frac{1}{2}\gamma$ (which removes the term $\gamma\eta_{XT}$), introducing $h = \eta - \gamma u_0$ and noting that

$$\left(1 + \frac{1}{2}\gamma^2\right)u_0 - \frac{1}{2}\gamma\eta \sim \int_\xi^\infty (\eta - \gamma u_0)\mathrm{d}T, \quad \text{with } \xi = X - \frac{1}{2}\gamma T;$$

all this should be compared with (6.4)–(6.6).) Nevertheless, the equation does admit a sech^2 solitary-wave solution:

$$\eta \sim \varepsilon\, a \operatorname{sech}^2 \left[\frac{1}{2}\sqrt{\frac{a}{2}}\sqrt{(3+\gamma^2)\{(2+\gamma^2) \pm \gamma\sqrt{4+\gamma^2}\}}(X - cT)\right],$$

where c is given in (4.13). This solution exhibits the same properties as observed for the solitary-wave solution of the KdV equation, (4.14), for upstream/downstream propagation when $\gamma \neq 0$. This is to be expected because the far-field approximation of (6.16), following either left- or right-going waves, recovers precisely the KdV equation (4.14), to leading order as $\varepsilon \to 0$. Now the fact that we obtain — obviously — the two variants of the KdV equation from the one Boussinesq equation, and each of these admits scaling transformations that convert them into any desired, standard version of the KdV equation for any γ, suggests a possible explanation for the difficulties mentioned above. Even though the KdV and NLS problems, for a flow with vorticity, generate equations that are equivalent to the corresponding propagation in still water, the failure of our new Boussinesq equation is to be expected. The scaling transformation depends on c, in particular the direction of travel, and we cannot define a single transformation that will accommodate both directions of travel at the same time: either one or the other (as in the KdV equation), but not both. Thus we submit that we have a new equation, (6.16), ripe for further study; this may describe new phenomena, but this is outside the aims and remit of this review, at this time.

7. The CH Equation with Vorticity

The next example, and the final one directly related to simple problems involving integrable equations with vorticity, is the CH equation (see [6, 7, 11, 20]) as it appears in the water wave context. We start with Eqs. (2.10)–(2.12), with both ε and δ retained as separate and independent parameters (and $\varepsilon \to 0$, $\delta \to 0$), and consider one-dimensional wave propagation (in the x-direction). First, we very briefly outline the conventional problem that produces the CH equation, relevant to still water; the details can be found in [29].

In order to obtain the CH equation appropriate to a suitable far-field, we introduce

$$\zeta = \sqrt{\varepsilon}(x - t), \quad \tau = \varepsilon\sqrt{\varepsilon}t, \quad w = \sqrt{\varepsilon}\bar{w},$$

for right-running waves (and the geometry is restricted to two dimensions, (x, z)); we form the equation for $\eta(\zeta, \tau; \varepsilon, \delta) \sim \eta_{00} + \varepsilon\eta_{10} + \delta^2\eta_{01} + \varepsilon\delta^2\eta_{11}$, which gives

$$2\eta_\tau + 3\eta\eta_\zeta + \frac{1}{3}\delta^2\eta_{\zeta\zeta\zeta} - \frac{3}{4}\varepsilon\eta^2\eta_\zeta = -\frac{1}{12}\varepsilon\delta^2(23\eta_\zeta\eta_{\zeta\zeta} + 10\eta\eta_{\zeta\zeta\zeta}) + \mathrm{O}(\varepsilon^2, \delta^4), \tag{7.1}$$

but this is not a CH equation. To proceed, we note that, to the same order of approximation, we have

$$u \sim \eta - \frac{1}{4}\varepsilon\eta^2 + \varepsilon\delta^2\left(\frac{1}{3} - \frac{1}{2}z^2\right)\eta_{\zeta\zeta} \quad (\text{for } 0 \leq z \leq 1) \tag{7.2}$$

and now we select a specific depth, denoted by $z = z_0$ $(0 \leq z_0 \leq 1)$ and introduce $\lambda = \frac{1}{3} - \frac{1}{2}z_0^2$; writing $\hat{u} = u(\zeta, \tau, z_0; \varepsilon, \delta)$, we invert (7.2), evaluated at $z = z_0$, to give

$$\eta \sim \hat{u} + \frac{1}{4}\varepsilon\hat{u}^2 - \varepsilon\delta^2\lambda\hat{u}_{\zeta\zeta}. \tag{7.3}$$

This is used in (7.1) to obtain the corresponding equation in $\hat{u}$:

$$2\hat{u}_\tau + 3\hat{u}\hat{u}_\zeta + \frac{1}{3}\delta^2\hat{u}_{\zeta\zeta\zeta} = -\varepsilon\delta^2\left\{\left(6\lambda + \frac{29}{12}\right)\hat{u}_\zeta\hat{u}_{\zeta\zeta} + \frac{5}{6}\hat{u}\hat{u}_{\zeta\zeta\zeta}\right\} + \mathrm{O}(\varepsilon^2, \delta^4)$$

or

$$2(\hat{u}_{\hat{t}} + \hat{u}_{\hat{x}}) + 3\varepsilon\hat{u}\hat{u}_{\hat{x}} + \frac{1}{3}\varepsilon\delta^2\hat{u}_{\hat{x}\hat{x}\hat{x}} = -\varepsilon^2\delta^2\left\{\left(6\beta + \frac{29}{12}\right)\hat{u}_{\hat{x}}\hat{u}_{\hat{x}\hat{x}} + \frac{5}{6}\hat{u}\hat{u}_{\hat{x}\hat{x}\hat{x}}\right\} + \mathrm{O}(\varepsilon^3, \varepsilon\delta^4), \qquad (7.4)$$

when expressed in (essentially) original variables: $\hat{x} = \sqrt{\varepsilon}x, \hat{t} = \sqrt{\varepsilon}t$. Finally, we add the term $\varepsilon\delta^2\mu(\hat{u}_{\hat{x}\hat{x}\hat{t}} - \hat{u}_{\hat{x}\hat{x}\hat{t}})$ to the left-hand side of (7.4), and use

$$\hat{u}_{\hat{t}} \sim -\left(\hat{u}_{\hat{x}} + \frac{3}{2}\varepsilon\hat{u}\hat{u}_{\hat{x}}\right)$$

in the first of these; now choose $\mu = \frac{1}{2} + 4\lambda$ and then $\mu = \frac{5}{6}$ (so that $\lambda = \frac{1}{12}$), to give

$$2(\hat{u}_{\hat{t}} + \hat{u}_{\hat{x}}) + 3\varepsilon\hat{u}\hat{u}_{\hat{x}} - \frac{1}{2}\varepsilon\delta^2\hat{u}_{\hat{x}\hat{x}\hat{x}} - \frac{5}{6}\varepsilon\delta^2\hat{u}_{\hat{x}\hat{x}\hat{t}} = \frac{5}{12}\varepsilon^2\delta^2\{2\hat{u}_{\hat{x}}\hat{u}_{\hat{x}\hat{x}} + \hat{u}\hat{u}_{\hat{x}\hat{x}\hat{x}}\} + \mathrm{O}(\varepsilon^3, \varepsilon\delta^4). \qquad (7.5)$$

This is a CH equation (because a simple frame shift allows the term $\hat{u}_{\hat{x}\hat{x}\hat{x}}$ to be subsumed into $\hat{u}_{\hat{x}\hat{x}\hat{t}}$), and the reversion to the standard form then requires no more than a simple scaling transformation; so $\hat{\zeta} = 2\sqrt{\frac{5}{3}}(\hat{x} - \frac{3}{5}\hat{t})$, $\hat{u} \to \sqrt{\frac{5}{3}}\hat{u}$ (and $\hat{t}$ unchanged) gives

$$\hat{u}_{\hat{t}} + 2\kappa\hat{u}_{\hat{\zeta}} + 3\varepsilon\hat{u}\hat{u}_{\hat{\zeta}} - \varepsilon\delta^2\hat{u}_{\hat{\zeta}\hat{\zeta}\hat{t}} = \varepsilon^2\delta^2\{2\hat{u}_{\hat{\zeta}}\hat{u}_{\hat{\zeta}\hat{\zeta}} + \hat{u}\hat{u}_{\hat{\zeta}\hat{\zeta}\hat{\zeta}}\}, \qquad (7.6)$$

at this order, where $\kappa = (2/5)\sqrt{3/5}$. Thus the CH equation describes a class of water waves, but this description applies only to the horizontal velocity component in the flow, at a specific depth ($z_0 = (1/\sqrt{2})$), and then, most importantly, if we retain (when expressed in (ζ, τ) variables) only terms $\mathrm{O}(1)$, $\mathrm{O}(\varepsilon)$, $\mathrm{O}(\delta^2)$ and $\mathrm{O}(\varepsilon\delta^2)$. The equation for the surface, (7.1), as we have previously observed, is not a CH equation; the behavior of the surface (at this order) is obtained from (7.3) with $\hat{u}$ determined by (7.5). Most gratifyingly, this analysis and derivation has been put on a rigorous basis in [12]. There, it is shown that the CH equation for the horizontal velocity component, at the depth we have found, is a proper approximation of the water wave problem provided that $\varepsilon \leq M\delta$, for some $M > 0$ (independent of ε and δ), with an overall error $\mathrm{O}(\varepsilon^3)$ in Eq. (7.6).

We now turn to the problem of primary interest here: the possibility of a CH equation describing a class of water waves with some background vorticity. Although the problem for general vorticity can be formulated — this procedure is outlined in [31] — we will consider the results for the case of constant vorticity. The method is developed as for the case of zero vorticity, except that we must use

$$\zeta = \sqrt{\varepsilon}(x - ct), \quad \tau = \varepsilon\sqrt{\varepsilon}t, \quad w = \sqrt{\varepsilon}\bar{w},$$

where c satisfies the Burns condition

$$\int_0^1 \frac{\mathrm{d}z}{(\gamma z - c)^2} = 1 \quad \text{i.e. } c = \frac{1}{2}(\gamma \pm \sqrt{\gamma^2 + 4}),$$

exactly as introduced in our earlier calculations. It turns out that, in the presence of an underlying vorticity, the equation for the horizontal component of the velocity in the flow at a specific depth ($\hat{u}$) must be replaced by

$$v = \hat{u} + \varepsilon\beta(c)\hat{u}^2 + \mathrm{O}(\varepsilon^2, \delta^4), \tag{7.7}$$

for some constant β (which is zero for $\gamma = 0$ i.e. $c = \pm 1$). (It is more convenient, and simpler, to express the various constants in terms of c rather than γ.)

The procedure follows precisely the approach outlined above, but now resulting in a CH equation for v, defined in (7.7). With the choice

$$\beta = \frac{c^5(c^4 + c^2 - 2)}{2(c^4 + c^2 + 1)(1 + c^2)^2}$$

and selecting, this time,

$$\mu = \frac{c^4 + 6c^2 + 3}{2(1 + c^2)(c^4 + c^2 + 1)},$$

we obtain

$$\begin{aligned}(1 + c^2)(v_{\hat{t}} + v_{\hat{x}}) + \varepsilon(c^4 + c^2 + 1)vv_{\hat{x}} + \varepsilon\delta^2\{((1/3c) - \mu)v_{\hat{x}\hat{x}\hat{x}} - \mu v_{\hat{x}\hat{x}\hat{t}}\} \\ = \varepsilon^2\delta^2\frac{c^4 + 6c^2 + 3}{6(1 + c^2)^2}(2v_{\hat{x}}v_{\hat{x}\hat{x}} + vv_{\hat{x}\hat{x}\hat{x}}) + \mathrm{O}(\varepsilon^3, \varepsilon\delta^4).\end{aligned} \tag{7.8}$$

Then, with the scaling transformation

$$\hat{\zeta} = \sqrt{\frac{1 + c^2}{\mu}}(\hat{x} - (1/3c\mu)\hat{t}), \quad v \to 3\frac{\sqrt{\mu(1 + c^2)}}{c^4 + c^2 + 1}v,$$

we obtain

$$v_{\hat{t}} + 2\kappa v_{\hat{\zeta}} + 3\varepsilon vv_{\hat{\zeta}} - \varepsilon\delta^2 v_{\hat{\zeta}\hat{\zeta}\hat{t}} = \varepsilon^2\delta^2\{2v_{\hat{\zeta}}v_{\hat{\zeta}\hat{\zeta}} + vv_{\hat{\zeta}\hat{\zeta}\hat{\zeta}}\}$$

(to this order) exactly as in (7.6), where

$$\kappa = \frac{\sqrt{2}}{3c}(1 + c^2)^2\left(\frac{c^4 + c^2 + 1}{c^4 + 6c^2 + 3}\right)^{3/2}.$$

The depth at which the horizontal component of the velocity is defined for (7.7) (to produce v) is now

$$z_0 = \frac{\sqrt{2c^{12} + 16c^{10} + 33c^8 + 31c^6 + 32c^4 - 3c^2 - 3}}{\sqrt{6}(1 + c^2)(c^4 + c^2 + 1)}.$$

In conclusion, therefore, we have demonstrated that, just as for the corresponding KdV and NLS problems — but not the Boussinesq case — the inclusion of vorticity (albeit constant here) does not fundamentally affect the underlying propagation structure. We still obtain a standard (constant coefficient) variant of the CH equation. The details of this result, and how it impinges on the properties of this water wave problem, are discussed in [31]; here, we will simply note how the constant vorticity distorts the solitary wave, based on

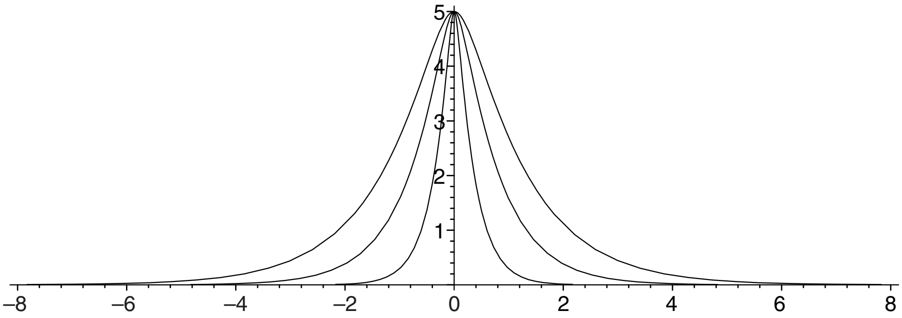

Fig. 2. The profile of the sech^2-like solitary wave of the CH equation, (7.8), for various γ: middle profile $\gamma = 0$; outer (broader) profile is for $\gamma = 1$, upstream propagation; inner (narrower) profile is for $\gamma = 1$, downstream propagation.

Eq. (7.8), for various γ. (The wave profile at the surface is recovered by using (7.7) in (7.3), but the dominant behavior is given by $\eta \sim v$.) In Fig. 2 we show three examples — just as in Fig. 1 — of exact solutions of (7.8); the previous observation (longer waves upstream, shorter waves downstream) is repeated here. (The relevant solutions of the CH equation, together with soliton solutions, are described in [30, 36].)

8. Ring Waves with Vorticity

For our final example, we consider a problem that takes us well beyond the simple propagation scenario that we have discussed so far in this review; these have been chosen, by and large, to relate directly to the rigorous theories and advances that are currently being developed. Here, we consider how a ring wave propagates on the surface of water that is moving (in one direction) with some prescribed vorticity. In order to formulate this problem, we first take equations (6.8)–(6.11), where the parameter δ has been scaled out in favor of ε, add the dependence on a second horizontal dimension (Y), and allow for a general $U(z)$; thus we get

$$u_T + Uu_X + U'W + \varepsilon(uu_X + vu_Y + Wu_z) = -p_X; \tag{8.1}$$

$$v_T + Uv_X + \varepsilon(uv_X + vv_Y + Wv_z) = -p_Y; \tag{8.2}$$

$$\varepsilon[W_T + UW_X + \varepsilon(uW_X + vW_Y + WW_z)] = -p_z; \quad u_X + v_Y + W_z = 0, \tag{8.3}$$

with

$$p = \eta \quad \text{and} \quad W = \eta_T + U\eta_X + \varepsilon(u\eta_X + v\eta_Y) \quad \text{on } z = 1 + \varepsilon\eta \tag{8.4}$$

and

$$W = 0 \quad \text{on } z = 0. \tag{8.5}$$

The intention is to provide a brief outline of this problem, indicating the particular complexities that are encountered, and producing some details for the case of constant vorticity. (This work is based on [26], although the specific choice of constant vorticity is new.) At the outset, we observe that the geometry of this problem poses a particular difficulty (requiring a mix of rectangular Cartesian and polar representations), and so only the general approach and main results will presented here. The full details, specifying the various coordinate transformations and much other information, can be found in [26], and the interested reader is directed there.

The first stage is to transform to a plane-polar coordinate system that is moving (in the x-direction) at a constant speed c (which will be chosen later). Thus we transform according to

$$X = cT + r\cos\theta, \quad Y = r\sin\theta \tag{8.6}$$

with

$$u \to u\cos\theta - v\sin\theta, \quad v \to u\sin\theta + v\cos\theta, \tag{8.7}$$

so that (u, v) now represents the horizontal velocity vector written in the polar system that is moving in the x-direction. Further, it is convenient to consider a large radius; it will then be possible to extend the analysis to a far field that accommodates (weak) nonlinearity and dispersion. The leading order, with this choice, will still correspond to the linear problem (which is appropriate for weak nonlinearity in the near field). Thus we also transform according to

$$\xi = rk(\theta) - T, \quad R = \varepsilon rk(\theta), \tag{8.8}$$

where $k(\theta)$ is to be determined; the wave front represented by $\xi =$ constant, for given T, is a circle if $k(\theta) =$ constant. When we seek an asymptotic solution in the usual form (see (3.4)), we find, at leading order, the system

$$-u_{0\xi} + (U - c)(k\cos\theta - k'\sin\theta)u_{0\xi} + U'W_0\cos\theta = -kp_{0\xi}; \tag{8.9}$$

$$-v_{0\xi} + (U - c)(k\cos\theta - k'\sin\theta)v_{0\xi} - U'W_0\sin\theta = -k'p_{0\xi}; \tag{8.10}$$

$$p_{0z} = 0; \quad ku_{0\xi} + k'v_{0\xi} + W_{0z} = 0, \tag{8.11}$$

with

$$p_0 = \eta_0 \quad \text{and} \quad W_0 = -\eta_{0\xi} + (U - c)(k\cos\theta - k'\sin\theta)\eta_{0\xi} \quad \text{on } z = 1 \tag{8.12}$$

and

$$W_0 = 0 \quad \text{on } z = 0, \tag{8.13}$$

where $k' = \mathrm{d}k/\mathrm{d}\theta$ (and $U' = \mathrm{d}U/\mathrm{d}z$, as used earlier).

It is altogether routine to find that a solution exist of the set (8.9)–(8.13) (giving η_0 arbitrary at this order, which is then determined at the next order; see below) provided that

$$(k^2 + k'^2)\int_0^1 \frac{\mathrm{d}z}{[1 - \{U(z) - c\}(k\cos\theta - k'\sin\theta)]^2} = 1, \tag{8.14}$$

which is another generalization of the Burns condition; cf. (5.7). We now choose to continue this calculation for a constant vorticity flow: $U(z) = \gamma z$ (and another choice, which models more realistic flows, is discussed in [26]); thus (8.14) becomes

$$\frac{k^2 + k'^2}{1 + \gamma(k\cos\theta - k'\sin\theta)} = 1, \tag{8.15}$$

where we have made the obvious choice $c = \gamma = U(1)$, and so the chosen frame moves at the surface speed of the water.

The first order, nonlinear ordinary differential equation, (8.15), has the general solution

$$k(\theta) = a\cos\theta \pm \sqrt{1 + a\gamma - a^2}\sin\theta, \tag{8.16}$$

which is real if the (real) parameter a satisfies $a^2 \leq 1 + a\gamma$, for a given γ. However, even when this is the case, the solution (8.16) has the property that $k(\theta) = 0$ where

$$\tan\theta = \mp\frac{a}{\sqrt{1 + a\gamma - a^2}}$$

and then $r \to \infty$ on the wave front: a ring wave does not exist. Further, (8.16) does not recover the circular ring-wave solution, corresponding to $k(\theta) =$ constant, on a stationary flow i.e. when $\gamma = 0$; we conclude that (8.16) is not the solution that we seek. However, the equation also possesses a singular solution

$$k(\theta) = \frac{1}{2}\gamma\cos\theta \pm \sqrt{1 + \frac{1}{4}\gamma^2} \tag{8.17}$$

and here we elect to use the upper sign, so that $k > 0$ (and then $k = 1$ for $\gamma = 0$). This solution accommodates all the properties that we seek and expect; the wave front, $\xi =$ constant, for a given T, is then expressed as

$$r = \frac{\frac{1}{2}\gamma + \sqrt{1 + \frac{1}{4}\gamma^2}}{\frac{1}{2}\gamma\cos\theta + \sqrt{1 + \frac{1}{4}\gamma^2}}, \tag{8.18}$$

which has been normalized so that $r = 1$ at $\theta = 0$. Three examples of the wave front, based on Eq. (8.18), including the circular case ($\gamma = 0$), are shown in Fig. 3. The distorting effect of the underlying flow, which is moving from left to right, is quite evident — the wave front is an ellipse — involving downstream propagation on the front edge of the wave front ($\theta = 0$), and upstream at the back ($\theta = \pm\pi$). There is no critical level at any point below this ring wave (although more general vorticity distributions can exhibit this complication; more information about this aspect, and many others, can be found in [26]).

The $\mathrm{O}(\varepsilon)$ problem generates the equation for η_0 (with η_1 now arbitrary at this order), because we are in an appropriate far field; cf. the KdV problem discussed in §§3, 4. This produces, after extensive manipulation, an equation of the form

$$A_1\eta_{0R} + \frac{A_2}{R}\eta_0 + \frac{A_3}{R}\eta_{0\theta} + A_4\eta_0\eta_{0\xi} + A_5\eta_{0\xi\xi\xi} = 0, \tag{8.19}$$

where each $A_i = A_i(\theta)$ is an involved function containing many integrals over z; for details see [26]. In the case of a ring wave over stationary water (so the wave front is a circle),