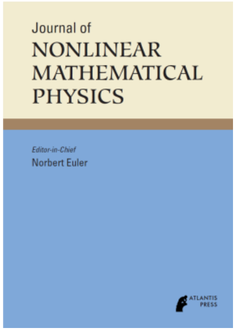

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page:https://www.tandfonline.com/loi/tnmp20

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To cite this article: Adán R. Rodríguez-Domínguez, Alejandro Martínez-González (2012) Group of Transformations with Respect to the Counterpart of Rapidity and Related Field Equations, Journal of Nonlinear Mathematical Physics 19:4, 580–594, DOI: https://doi.org/10.1142/S1402925112500349

To link to this article: https://doi.org/10.1142/S1402925112500349

Published online: 04 January 2021

World Scientific
www.worldscientific.com

ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics, Vol. 19, No. 4 (2012) 1250034 (15 pages)
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DOI: 10.1142/S1402925112500349

GROUP OF TRANSFORMATIONS WITH RESPECT TO THE COUNTERPART OF RAPIDITY AND RELATED FIELD EQUATIONS

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Received 6 July 2012
Accepted 6 September 2012
Published 31 December 2012

The Lorentz-group of transformations usually consists of linear transformations of the coordinates, keeping as invariant the norm of the four-vector in (Minkowski) space-time. Besides those linear transformations, one may construct different forms of nonlinear transformations of the coordinates keeping unchanged that respective invariant. In this paper we explore nonlinear transformations of second-order which have a natural interpretation within the framework of Yamaleev's concept of the *counterpart of rapidity* (co-rapidity). The purpose of developed concept is to show that the formulae for energy and momentum of the relativistic particle become regular near the zero-mass and speed of light states. Furthermore, in a covariant formulation, the co-rapidity is presented as a four-vector which admits an extension of the Lorentz-group of transformations. In this paper we additionally show, that in the same way as the rapidity is related to the electromagnetic field, the co-rapidity is related to the field of strengths, which are given by a four-vector. The corresponding equations of such a field are also constructed.

Keywords: Relativistic dynamics; inertial mass; rapidity; co-rapidity; energy-momentum; Lorentz-group; electromagnetic field; notoph field.

Mathematics Subject Classification 2010: 83A05, 78A35, 70G10, 22E43

1. Introduction

The group of linear transformations, the Lorentz-transformations,

$$\delta x'^{\mu} = M^{\mu}_{\nu}\delta x^{\nu}, \tag{1.1}$$

keep unchanged the pseudo-norm

$$\rho^2 = g_{\mu\nu}\, x^\mu x^\nu, \tag{1.2}$$

due to the antisymmetric form of the tensor $M_{\mu\nu} = -M_{\nu\mu}$. Besides of these transformations we will consider transformations of the coordinates of second-order given by

$$\delta x'^\mu = \rho^2\, \delta x^\mu - x^\mu (x_\nu\, \delta x^\nu). \tag{1.3}$$

It is easily seen that these transformations also keep invariant the pseudo-norm ρ^2.

The Lorentz-group of transformations is well known to require six independent parameters, which form some kind of antisymmetric tensor $\psi_{\mu\nu}$, including three parameters of Lorentz-boosts and other three parameters for space rotations in three-dimensional space [2, 8]. The transformations defined by (1.3) are fulfilled with respect to four parameters which form a set of four-vectors. Geometrical interpretation of these transformations is clear: they generate a motion along hyperplane $\rho^2 = \text{const}$. Noteworthy, this is however not the case when the curvature of the hyperplane is given by a universal constant, not as when the pseudo-norm ρ^2 is a constant only with respect to every transformation; that is, it is just constant of motion, not a universal constant of the theory. In order to realize the geometrical and physical essence of the second-order transformations (1.3), we will start by simplicity with a two-dimensional space-time. In these case, the transformations (1.3) are reduced into a transformation where the counterpart of rapidity (co-rapidity) appears as a parameter of the transformation.

Recently, Yamaleev has introduced the concept of co-rapidity [18–21], seeking a regular representation for energy-momentum of the relativistic particle near zero-mass and light-speed states. In the representation of energy-momentum as functions of the velocity and the proper-mass the formulae for energy and momentum have a singularity at the point where the proper mass is equal to zero ($m = 0$) and the velocity equals the speed of light ($v = c$). The speed of light is unattainable for a particle with non-vanishing proper mass; on the contrary, the particle moving with speed of light has to possess its proper mass equal to zero. This means, that in the formulation of the relativistic dynamics one is not allowed just to put conditions like $m = 0$ and $v = c$ separately; in the same way, one is not allowed to pass from formulae for energy-momentum of the massive particle onto those of energy-momentum of the massless particle, simply by taking $m = 0$ and $v = c$, because in this way, one is meeting an indeterminacy of the type $0/0$. Thus, the formalism of the relativistic dynamics in its present formulation does not provide us with some rule for solving this indeterminacy. The solution of this indeterminacy as in [18, 19, 21], has been found as follows. The relativistic physics requires two kinds of reference systems. The inertial reference systems linked to massive particles may stay at rest state, but they, however, can never reach the state of light speed. Besides, a different kind of reference systems do have their "rest state" at the state of light-speed. These systems, contrary to the earlier, never can reach the rest state. It seems, there is some symmetry between the class of systems of reference originating from rest state, and the class of systems of reference originating from light-speed state. In fact, a deeper investigation shows that this symmetry comes from symmetry between the mass and the momentum in mass-shell equation with respect to the energy. This symmetry also implies some reciprocity between rapidity and its counterpart. However this symmetry is valid only for a static picture, however in dynamics, the proper-mass is a constant of motion, whereas the momentum is a variable.

This is an issue emerging from the concept of co-rapidity: the hyperbolic angle corresponding to co-rapidity is proportional to the proper-mass of the massive particle. The rapidity in a covariant formulation is presented by a two-form, i.e. via an antisymmetric tensor with two indices, in that sense its tensorial structure coincides with tensorial structure of the electromagnetic strengths. Furthermore, one may interpret the concept of rapidity by introducing the rapidity as an auxiliary variable in the Lorentz-force equations. The co-rapidity in a covariant formulation is given by a one-form, i.e. via a four-vector. As the rapidity is related to electromagnetic field, the co-rapidity may be related to the field of electromagnetic strengths given by a four-vector. This field in the literature is known as the "notoph", or, as the Kalb–Ramond's fields in the string theoretical lore [1, 3, 10, 22].

The paper is presented as follows. Section 2 presents elements of the theory of co-rapidity and its consequences. In Sec. 3, the notion of co-rapidity will be extended to a four-vector. In Sec. 4, the Lorentz-group of transformations is extended by including a covariant form of the co-rapidity into the set of parameters of transformations. In Sec. 5, the Lorentz-force equations induced by the counterpart of electromagnetic fields are formulated. Finally, in Sec. 6 we include some concluding remarks.

2. Hyperbolic Angle and the Counterpart of Rapidity

Let us start with a two-dimensional Minkowski plane, for simplicity. Let a vector there be given by coordinates (x_0, x), with $x_0 = ct$, and introduce the indefinite length of the vector or pseudo-norm, by

$$x_0^2 - x^2 = \rho^2. \tag{2.1}$$

In polar coordinates this equality is satisfied by the parametrization via hyperbolic trigonometry

$$\begin{aligned} x_0 &= \rho \cosh(\psi), \\ x &= \rho \sinh(\psi). \end{aligned} \tag{2.2}$$

Note, however, that the parametrization used in (2.2) is not unique. In fact, instead of this parametrization we can take, for instance,

$$\begin{aligned} x_0 &= \rho \coth \chi, \\ x &= \frac{\rho}{\sinh \chi}. \end{aligned} \tag{2.3}$$

The velocity v defined with respect to time coordinate, in parametrization (2.2), is given by well-known formula

$$v = c \tanh(\psi), \tag{2.4}$$

whereas within parametrization (2.3) one gets

$$v = \frac{c}{\cosh(\chi)}. \tag{2.5}$$

Now, let us introduce a velocity complementary to v, namely $\bar{v}$, defined through the following relationship

$$v^2 + \bar{v}^2 = c^2. \tag{2.6}$$

Note that $\bar{v}$ is related with hyperbolic angle (χ) in a manner quite similar as v is expressed via rapidity (ψ). We will denote from now on this complementary velocity, as the co-rapidity. In fact,

$$\bar{v}^2 = c^2 - v^2 = c^2\left(1 - \frac{p^2}{p_0^2}\right) = c^2 \tanh^2(\chi). \tag{2.7}$$

Interrelation between χ and ψ is expressed by the following reciprocity formulae [20],

$$\begin{aligned} \exp(\psi) &= \coth\left(\frac{\chi}{2}\right), \\ \exp(\chi) &= \coth\left(\frac{\psi}{2}\right). \end{aligned} \tag{2.8}$$

From these formulae it follows, that the state with $\psi = 0$ or $\chi = \infty$ means a rest state with $v = 0$, and that the state with $\psi = \infty$ or $\chi = 0$, means light-speed state with $v = c$. In this sense the pair $(\bar{v}, \chi)$ is the counterpart of the pair (v, ψ).

The evident symmetry between two hyperbolic angles ψ and χ is displayed only when we restrict ourselves to the Minkowski plane. In covariant formulation they have to be considered as complementary parameters of some group of transformations generalizing the Lorentz-group [18, 19, 21].

Now let us underline one remarkable property of the co-rapidity, namely, we shall prove that χ is proportional to the proper mass of the particle, i.e. $\chi = mc\phi$. In order to prove this statement let us beforehand explore one interesting and useful mathematical relationship between fraction and exponential function.

Let q_2, q_1 be positive real numbers. Consider three following combinations of these values: $Q_1 = q_1 + q_2,\ Q_2^2 = q_1 q_2,\ Q_0 = q_2 - q_1$. Then, the quantities q_2, q_1 can be considered as solutions of the following quadratic equation

$$X^2 - Q_1 X + Q_2^2 = 0. \tag{2.9}$$

Extensions of the relativistic dynamics to higher characteristic polynomials may be found in [15].

Because q_1, q_2 satisfy the quadratic equation the following Euler formulae hold true [17]:

$$\begin{aligned} \exp(q_1\phi) &= g_0(\phi) + q_1\ g_1(\phi), \\ \exp(q_2\phi) &= g_0(\phi) + q_2\ g_1(\phi), \end{aligned} \tag{2.10}$$

inversely, the modified cosine–sine functions can be expressed via exponential functions:

$$g_1(\phi) = \frac{\exp(q_2\phi) - \exp(q_1\phi)}{Q_0}, \quad g_0(\phi) = \frac{q_2\exp(q_1\phi) - q_1\exp(q_2\phi)}{Q_0}. \tag{2.11}$$

Now consider fraction

$$\frac{\exp(q_2\phi)}{\exp(q_1\phi)} = \exp(Q_0\phi) = \frac{g_1(\phi)q_2 + g_0(\phi)}{g_1(\phi)q_1 + g_0(\phi)} = \frac{q_2 + U}{q_1 + U}, \tag{2.12}$$

where we denote by U the fraction

$$U := \frac{g_0(\phi)}{g_1(\phi)}. \tag{2.13}$$

Let at the point $\phi = \phi_0$ the function $g_0 = 0$. Then from (2.11), as well as from (2.12) we get

$$\exp(Q_0\phi_0) = \frac{q_2}{q_1}. \tag{2.14}$$

Note, that simultaneous translation of q_1, q_2 by Δ, $q_i = q_i + \Delta$, $i = 1, 2$, this shift keeps unchanged the difference between them $Q_0 = q_2 - q_1$. Hence, this translation induces a corresponding translation of the hyperbolic argument ϕ. We have,

$$\exp(Q_0(\phi_0 + \delta)) = \frac{q_2 + \Delta}{q_1 + \Delta}. \tag{2.15}$$

Inversely, translation of ϕ_0 by δ, namely $\phi = \phi_0 + \delta$, must be unchanged Q_0, whereas q_1, q_2 will undergo some simultaneous translations by $q_2 = q_2 + U,\ q_1 = q_1 + U$. Thus we come to the following formula

$$\frac{q_2}{q_1} = e^{(q_2 - q_1)\phi}, \tag{2.16}$$

which can be interpreted from different points of view. For instance, this formula can be considered as some interrelation between the fraction and the difference of two values via the hyperbolic exponential function. Another point of view is that this formula establishes some interrelation between simultaneous translations of denominator and numerator of the fraction and the hyperbolic rotation.

Once we have stated this formula, let us now make an application of it. Let us take as values for q_1, q_2 the following quantities of the relativistic mechanics $q_1 = p_0 - mc$, $q_2 = p_0 + mc$. Then according to formula (2.16), we write at once

$$\frac{p_0 + mc}{p_0 - mc} = \exp(2mc\phi), \tag{2.17}$$

where the mass m is a constant of the evolution with respect to parameter ϕ whereas p_0, p depend of ϕ, and this dependence is given by formulae of hyperbolic trigonometry

$$\begin{aligned} p_0 &= mc \coth(mc\phi), \\ p &= \frac{mc}{\sinh(mc\phi)}. \end{aligned} \tag{2.18}$$

Comparing these formulae with parametrization (2.2) we note that an advantage of the latter is that the hyperbolic angle χ is presented here as a quantity proportional to the mass, namely, $\chi = mc\phi$, where mc and ϕ are totally independent variables. It has to be pointed out also that formula (2.16) is applicable if and only if the difference between denominator and numerator of the fraction is a constant with respect to evolution parameter ϕ.

Exploring the behavior of the energy momentum near speed of light state when $m = 0$ we find

$$p(m = 0) = p_0(m = 0) = \frac{1}{\phi} = \pi_0. \tag{2.19}$$

Here we have introduced the new quantity π_0 which is equal to energy-momentum of the relativistic system at the state of speed of light with $m = 0$, $v = c$.

It is important to underline some duality between π_0 and the corresponding rest mass mc [20]. At the rest state, the energy equals mc^2, $\psi = 0$; and vice versa, at the state of speed of light energy is equal to $c\pi_0$, $\chi = 0$. The particle with $m > 0$ is never able to attain the state of speed of light; conversely, the particle possessing $\pi_0 > 0$ cannot fall into the rest state. The latter statement has its simple interpretation within classical mechanics: *the particle that possesses $\pi_0 > 0$, possessing kinetic energy, cannot consequently be in the rest state.* In the rest state the energy is equal to the proper inertial mass (in energy units) and, in the same manner, in the light-speed state, the energy is equal to $c\pi_0$. Thus, the relativistic dynamics of the relativistic particle is governed, besides the inertial mass m, by some special kind of energy, which we have denoted by $c\pi_0$. The parameter π_0 is, in some sense, dual to the inertial mass which determines the value of the kinetic energy of the motion. This quantity corresponds to the energy of the particle in its massless state.

Next, let us examine separately the two limits as we approach to the rest and speed of light states, respectively and compare them with each other.

(i) **Low-speed, or, non-relativistic limit**

In order to obtain the non-relativistic limit we start with formulae

$$\begin{aligned} p_0 &= mc \cosh(\psi), \\ p &= mc \sinh(\psi), \end{aligned} \tag{2.20}$$

where the point $\psi = 0$ corresponds to the rest state. Then for small values of $\psi \ll 1$ the following expansion holds true

$$\begin{aligned} p_0 &= mc\left(1 + \frac{1}{2}\psi^2\right), \\ p &= mc\psi. \end{aligned} \tag{2.21}$$

Hence,

$$p_0 = mc + \frac{p^2}{2mc}, \tag{2.22}$$

$$\mathcal{E}_{(\text{nonrel})} = c(p_0 - mc) = \frac{p^2}{2m}. \tag{2.23}$$

(ii) **High speed limit**

Near the speed of light we have to use the representation

$$\begin{aligned} p_0 &= mc \coth(\chi), \\ p &= \frac{mc}{\sinh(\chi)}. \end{aligned} \tag{2.24}$$

For small values of $\chi \ll 1$ we obtain

$$\begin{aligned} p_0 &= mc\left(\frac{1}{\chi} + \frac{1}{2}\chi\right), \\ p &= mc\frac{1}{\chi}. \end{aligned} \tag{2.25}$$

Removing from these equations χ we obtain

$$p_0 = p + \frac{m^2c^2}{2p}. \tag{2.26}$$

Thus, we note that at these limits the mass and momentum interchange their roles.

3. Covariant Form of the Co-Rapidity and Modified Poincaré-Group of Transformations

In this section we will follow [18, 19, 21]. The transformations of the energy and momentum $p_0,\ p$ under a translation of the hyperbolic angle ϕ, which we denominated co-rapidity, keep invariant the square of the proper-mass. From (2.18) these transformations for $p_0,\ p$, which under ϕ turn out to be bilinear of themselves, may be written as

$$\begin{aligned} dp &= -p\,p_0 d\phi, \\ dp_0 &= -p^2 d\phi. \end{aligned} \tag{3.1}$$

Our next goal is to extend these equations in such a way as to obtain Lorentz-covariant equations and, in that particular case, to cover Eq. (3.1). As it has been shown in references [18, 19, 21], in order to satisfy these conditions, it is only necessary to suppose that the parameter ϕ is a time-like component of some four-vector $\xi^\nu, \nu = 0, 1, 2, 3$, that is, we have $\xi^0 = \phi, \xi^k = 0,\ k = 1, 2, 3$.

The evolution of a four-vector x^μ, with respect to this set of parameters, is governed by the following equations

$$\frac{\partial x^\nu}{\partial \xi^\mu} = \rho^2\ \delta^\nu_\mu - x^\nu x_\mu, \quad \rho^2 = \eta_{\mu\nu}x^\mu x^\nu, \tag{3.2}$$

where $\eta_{\nu\mu} = \mathrm{diag}(1, -1, -1, -1)$ is the metric tensor of Minkowski space-time.

The generators of transformations with respect to the parameters $\xi^\nu, \nu = 0, 1, 2, 3$ are defined by the derivatives

$$G_\nu = \frac{\partial}{\partial \xi^\nu}, \quad \nu = 0, 1, 2, 3. \tag{3.3}$$

Obviously, they do not commute. Let us express these generators in terms of space-time coordinates x^μ as follows

$$G_\nu = \frac{\partial x^\mu}{\partial \xi^\nu}\frac{\partial}{\partial x^\mu} = \rho^2\frac{\partial}{\partial x^\nu} - x_\nu\left(x^\mu\frac{\partial}{\partial x^\mu}\right), \quad \nu, \mu = 0, 1, 2, 3. \tag{3.4}$$

Now we are able to construct commutation relations containing the generators $G_\mu, \mu = 0, 1, 2, 3$ as elements of the group, which we are seeking. Let us call this group Γ_g. First of all, calculate the commutators between the generators G_μ. They are

$$[G_\mu, G_\nu] = \rho^2 M_{\mu\nu}, \quad M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu. \tag{3.5}$$

Thus, in surplus, Γ_g contains elements of the Lorentz-group $M_{\mu\nu}$ and the factor ρ^2 which is also an element of the Γ_g. This length is invariant under the action of the generators G_μ and also under the action of the generators of the Lorentz-group,

$$[G_\mu, \rho^2] = 0, \quad [\rho^2, M_{\mu\nu}] = 0. \tag{3.6}$$

The G_μ generator is a sum of two operators, namely, the differential operator in Minkowski space, ∂_μ, and the generator of dilatation $D = x^\mu \partial_\mu$, that is, $G_\mu = \rho^2 \partial_\mu - x_\mu D$. By taking into account $[M_{\mu\nu}, D] = 0$ and $[\rho^2, M_{\mu\nu}] = 0$, we get

$$[M_{\mu\nu}, G_\lambda] = \eta_{\nu\lambda} G_\mu - \eta_{\mu\lambda} G_\nu. \tag{3.7}$$

The generators of the Lorentz-group obey ordinary commutation relations

$$[M_{\mu\nu}, M_{\lambda\eta}] = \eta_{\mu\lambda} M_{\nu\eta} - \eta_{\nu\lambda} M_{\mu\eta} + \eta_{\mu\eta} M_{\lambda\nu} - \eta_{\nu\eta} M_{\lambda\mu}. \tag{3.8}$$

The Γ_g is furnished with two Casimir operators, namely,

$$\begin{aligned} C_1 &= G^2, \\ C_2 &= M_{\mu\nu} M_{\mu\nu} G^2/2 - M_{\mu\lambda} M_{\mu\lambda} G_\mu G_\nu. \end{aligned} \tag{3.9}$$

The Γ_g can be extended by introducing the dilation operator $D = x^\mu \partial_\mu$. In this case, there appears two additional commutators

$$[D, G_\mu] = G_\mu, \quad [D, \rho^2] = 2\rho^2. \tag{3.10}$$

This extension is similar to the extension of Poincaré group into the Weyl group, by adding the dilation operator [7].

It is worth noticing the analogy between elements of special conformal group K_μ, ($\mu = 0, 1, 2, 3$) and the generators of Γ_g. A representation of the special conformal elements via differential operators in Minkowski space is given by

$$K_\mu = \rho^2 \partial_\mu - 2x_\mu D. \tag{3.11}$$

However, this operator differs from the operator G_μ just by a factor of 2 at the second term.

In same way, let us underline the similarity of the generator G_μ with the generator of translation on the surface with constant curvature [6, 16]. In fact, if the factor $\rho^2 = \text{const.}$, the operator G is transformed into the operator of translation along a hyperbolic surface imbedded into four-dimensional Minkowski space.

It is important to note that a realization of the commutation relations for the elements of Γ_g may be accomplished by a set of finite-dimensional matrices built out of the Dirac *gamma*-matrices. These matrices are well known to satisfy the anti-commutation relations

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\,\eta_{\mu\nu}\,1, \tag{3.12}$$

where $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ [14].