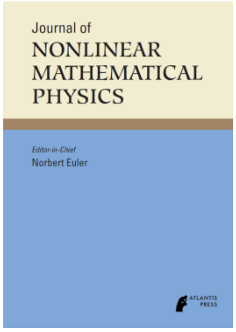

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1776-0852**
Journal Home Page:https://www.tandfonline.com/loi/tnmp20

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To cite this article: Mahouton Norbert Hounkonnou, Isiaka Aremua (2012) Landau Levels in a Two-Dimensional Noncommutative Space: Matrix and Quaternionic Vector Coherent States, Journal of Nonlinear Mathematical Physics 19:4, 551–579, DOI: https://doi.org/10.1142/S1402925112500337

To link to this article: https://doi.org/10.1142/S1402925112500337

Published online: 04 January 2021

World Scientific
www.worldscientific.com

ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics, Vol. 19, No. 4 (2012) 1250033 (29 pages)

DOI: 10.1142/S1402925112500337

LANDAU LEVELS IN A TWO-DIMENSIONAL NONCOMMUTATIVE SPACE: MATRIX AND QUATERNIONIC VECTOR COHERENT STATES

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Received 7 July 2012
Accepted 29 August 2012
Published 31 December 2012

The behavior of an electron in an external uniform electromagnetic background coupled to an harmonic potential, with noncommuting space coordinates, is considered in this work. The thermodynamics of the system is studied. Matrix vector coherent states (MVCS) as well as quaternionic vector coherent states (QVCS), satisfying required properties, are also constructed and discussed.

Keywords: Noncommutative space; quantum Hilbert space; vector coherent state.

1. Introduction

Noncommuting spatial coordinates and fields can (approximately) be realized in actual physical situations. Landau models and their quantum Hall limit have become the focus of intense research activity as a physical realization of the simplest example of noncommutative geometry [7, 8, 14–17, 19, 27, 29–31, 35–38, 43, 46–48, 53]. Similar structures also arise in specific approaches towards a theory of quantum gravity, such as M-theory in the presence of background fields [52] or tentative formulations of relativistic quantum theories of gravity through spacetime noncommutativity.[a]

As is well known [29], given a point particle of mass m, charge $\bar{q}$ and position $\vec{r} = (x, y)$ moving in a plane in the presence of a constant external magnetic field B perpendicular to that plane, the spectrum of the quantized theory is organized into infinitely degenerate Landau levels, with separation $\mathcal{O}(\bar{q}B/m)$. The limit $B \to \infty$ effectively projects onto the

[a]For recent material on Noncommutativity and Quantum Gravity, see for instance [39].

lowest Landau level and is equivalent to a negligibly small mass m, i.e. $m \to 0$. Consequently in each of the projected Landau levels, one obtains a noncommuting algebra for the space coordinates,

$$[x, y] = -\frac{i\hbar}{\bar{q}B}. \tag{1.1}$$

Historically, this is the Peierls substitution rule introduced seventy years ago [42]. As a matter of fact, this noncommuting character arises already at the classical level in terms of the Dirac brackets associated to the second-class constraints that follow upon taking the limit $m \to 0$ in the classical Hamiltonian formulation of the dynamics of the Landau problem and requiring finite energy configurations.

Given the simplicity and ubiquity of the above two-dimensional system in problems involving background fields, as is the case for instance in M-theory and its many D-brane constructions [52], the Landau problem and its noncommutative quantum Hall limit have played a central role in recent years as inspiration and guide towards tentative formulation of relativistic quantum theories of gravity through spacetime noncommutativity.

Besides, in most of the introductory references in the literature devoted to quantum mechanics and quantum field theory, it comes out that the natural appearance of noncommutativity in string theories has increasingly led to attempts to study physical problems in noncommutative spaces [11, 12]. Although noncommuting coordinates are operators even at the classical level, one can treat them as commuting by replacing operator products by $*$-products [9]. This approach allows one to generalize classical as well as quantum mechanics without altering their main physical interpretations and to recover the usual results when noncommutativity is switched off. In some recent works, the quantum Hall system has attracted considerable attention from the point of view of noncommutative quantum mechanics and quantum field theory (see e.g. [10, 16, 17, 27, 41]). Note that a noncommutative model valid for a constant magnetic field, with respect to the geometrical aspect of the problem, has been also investigated. For more details, see [40]. The description of such a system [24, 25] is adequately provided by the well known Landau model [33] as mentioned above. This latter describes the motion of an electron in a uniform magnetic field which can be assimilated to that of a two-dimensional harmonic oscillator. Since this discovery, the quantum states of a particle in a magnetic and electromagnetic fields on noncommutative plane have been attracting considerable attention, see for instance [3, 10, 13, 16, 17, 19, 21, 22, 27, 32] and more recently [56] (and references listed therein). In [51], based on a previous work [49], the thermodynamics of an ideal fermion gas confined in a two-dimensional noncommutative well has been investigated. The authors have shown that the thermodynamical properties of the fermion gas for the commutative and noncommutative cases agree at low densities, while at high densities they start diverging strongly due to the implied excluded area resulting from the noncommutativity. In [20] the possible occurrence of orbital magnetism for two-dimensional electrons confined by an harmonic potential [28] in various regimes of temperature and magnetic field has been studied. Coherent states (CS) have been applied to investigations on noncommutative geometries [26]. One of their primary properties is the minimization of the position and momentum uncertainty relation. They are also relevant for studying the Dirac operator in fuzzy quantum spaces, when the underlying spacetime or spatial cut can be treated as a phase space and quantized, as demonstrated in [1]. Besides, they have been formulated in terms of the diagonal coherent state matrix elements of operators and star products [4] in the analysis of models implying noncommutative Chern–Simons theory. Standard coherent states have been used to calculate symbols of various observables like the thermodynamical potential, the magnetic moment or the spatial distribution of the current in the system. In [32], an analogous treatment in a noncommutative framework has been achieved and the results of [20] in the commutative case have been recovered by switching off the θ-parameter.

In the noncommutative quantum mechanics formulation, a major role is played by the CS on the quantum Hilbert space denoted by $\mathcal{H}_q$ (the space of Hilbert Schmidt operators on the classical configuration space denoted by $\mathcal{H}_c$), which are expressed in terms of a projection operator on the usual Glauber–Klauder–Sudarshan CS in the classical configuration space. Based on the approach developed in [50], Gazeau–Klauder CS have been constructed in the noncommutative quantum mechanics [23]. These states share similar properties to those of ordinary canonical CS in the sense that they saturate the related position uncertainty relation, obey a Poisson distribution and possess a flat geometry. The thermodynamics of an ideal fermion gas has been investigated in [51].

This work deals with the study of the electron motion in an external uniform electromagnetic field, (the so-called Landau problem), coupled with an harmonic potential in a two-dimensional noncommutative space. The thermodynamics of this physical system is investigated following the method established in [20] by formulating at first CS on the quantum Hilbert space $\mathcal{H}_q$. Then, relevant inequalities are deduced and used to compute both the thermodynamical potential and magnetic moment. Besides, we complete this study with the construction of vector cs (VCS) and show that they fulfill a resolution of the identity on a suitable Hilbert space. This construction fits the general formulation of VCS by Ali *et al.* [2] instead of the general theory of vector-valued coherent state representations [45] (and references listed therein) and similar developments based on operators of unitary representations of groups, where the VCS are defined as orbits of vectors [5, 6, 44]. Furthermore, we extend the VCS construction used in [23] to a formal tensor product of quantum Hilbert spaces by using the primary formulation of [55]. This extension is implemented with complex matrices and quaternions as CS variables. The physical features of the quaternionic VCS (QVCS) are discussed. The expectation values and dispersions of the quadrature operators are provided. The constructed QVCS reveal some inequalities inherited from the parametrization and the noncommutativity of the space coordinates.

The paper is organized as follows. In Sec. 2, we describe the physical model and give the corresponding matrix formulation. The Hamiltonian spectrum and its spectral decomposition are provided. The definition of the passage operators from an orthonormal basis to another one is also supplied. Section 3 discusses the thermodynamical aspects of the studied model. In Sec. 4, relevant VCS and QVCS for Landau levels are constructed and analysed. Finally, there follow concluding remarks in Sec. 5.

2. The Electron in Noncommutative Plane

2.1. *Quantum model*

The physics of an electron in crossed constant uniform electric **E** and magnetic **B** fields coupled with a confining harmonic potential in a noncommutative space, is described, in