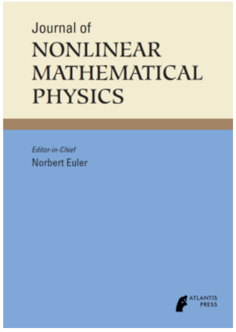

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1776-0852**
Journal Home Page:https://www.tandfonline.com/loi/tnmp20

Quasi-Periodic Solutions of the Relativistic Toda Hierarchy

Dong Gong, Xianguo Geng

Article

Journal of Nonlinear Mathematical Physics, Vol. 19, No. 4 (2012) 1250030 (35 pages)

DOI: 10.1142/S1402925112500301

QUASI-PERIODIC SOLUTIONS OF THE RELATIVISTIC TODA HIERARCHY

DONG GONG* and XIANGUO GENG†

Department of Mathematics, Zhengzhou University
100 Kexue Road, Zhengzhou
Henan 450001, P. R. China
**gongdong2004@126.com*
†xggeng@zzu.edu.cn

Received 7 May 2012
Accepted 18 August 2012
Published 31 December 2012

On the basis of the theory of algebraic curves, the continuous flow and discrete flow related to the relativistic Toda hierarchy are straightened out using the Abel–Jacobi coordinates. The meromorphic function and the Baker–Akhiezer function are introduced on the hyperelliptic curve. Quasi-periodic solutions of the relativistic Toda hierarchy are constructed with the help of the asymptotic properties and the algebro-geometric characters of the meromorphic function and the hyperelliptic curve.

Keywords: Relativistic Toda hierarchy; quasi-periodic solutions.

1. Introduction

The relativistic Toda lattice

$$\begin{aligned} a_t(n) &= \frac{1}{2}a(n)\left[a^2(n+1) - a^2(n-1) + b(n+1) - b(n)\right], \\ b_t(n) &= b(n)\left[a^2(n) - a^2(n-1)\right]. \end{aligned} \tag{1.1}$$

was first introduced by Ruijsenaars [16] and has been extensively studied by many authors. For instance, its integrability has been proved by the author himself, and in a list of papers by Bruschi and Ragnisco, in the case of a finite lattice (periodic case and free ends) [2–4]. Oevel, Fuchssteiner and Zhang investigated the integrability structure of the infinite relativistic Toda lattice from the Lie-algebraic point of view [14]. Alber found a nonlinear family of relativistic Toda lattices with corresponding stationary and dynamical systems, and reduced the finite-gap problem to the Jacobi inversion problem [1]. Ohta, Kajiwara, Matusukidara and Satsuma obtained its Casorati determinant solution [15]. Suris studied algebraic structure of discrete-time and relativistic Toda lattices [17]. Under a constraint between potentials and eigenfunctions, the discrete spectral problem associated with the relativistic Toda lattice is nonlinearized so as to be an integrable symplectic map of Neumann type, and the calculation of the finite-band solutions of the relativistic Toda lattice is reduced to the solution of a system of ordinary differential equations plus a simple iterative process of the symplectic map [18]. The relation between 2+1-dimensional modified Toda lattice and the relativistic Toda flows are revealed in [7]. A Darboux transformation for the relativistic Toda lattice is constructed, by which exact solutions of the relativistic Toda lattice equation are presented [19].

We recall here the basic definitions for the relativistic Toda lattice and its various different forms. In terms of the canonically conjugate variables $(q(n,t), p(n,t))$, the infinite relativistic Toda system is defined by the following Hamiltonian [2, 14, 16]

$$\begin{aligned} H(q,p) = \sum_{n\in\mathbb{Z}} \Big\{ & \exp(p(n))[1+\exp(q(n-1)-q(n))]^{\frac{1}{2}} \\ & \times [1+\exp(q(n)-q(n+1))]^{\frac{1}{2}} - 2 \Big\}, \end{aligned} \tag{1.2}$$

whose canonical equations of motion of the Hamiltonian system reads

$$\begin{aligned} q_t(n) &= c(n), \\ p_t(n) &= \frac{1}{2}d(n-1)(c(n)+c(n-1)) - \frac{1}{2}d(n)(c(n)+c(n+1)), \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} c(n) &= \exp(p(n))[1+\exp(q(n-1)-q(n-1))]^{\frac{1}{2}}[1+\exp(q(n)-q(n+1))]^{\frac{1}{2}}, \\ d(n) &= \frac{\exp(q(n)-q(n+1))}{1+\exp(q(n)-q(n+1))}. \end{aligned} \tag{1.4}$$

Substituting (1.4) into (1.3) and canceling variable $p(n)$, the equations of motion (1.3) are written in Newtonian form

$$q_{tt}(n) = q_t(n)\left(q_t(n-1)\frac{\exp(q(n-1)-q(n))}{1+\exp(q(n-1)-q(n))} - q_t(n+1)\frac{\exp(q(n)-q(n+1))}{1+\exp(q(n)-q(n+1))} \right). \tag{1.5}$$

Differentiating (1.4) with respect to t and substituting (1.3) yield the following lattice [3]

$$\begin{aligned} c_t(n) &= c(n-1)c(n)d(n-1) - c(n)c(n+1)d(n), \\ d_t(n) &= d(n)(1-d(n))(c(n)-c(n+1)). \end{aligned} \tag{1.6}$$

A further useful form of the relativistic Toda lattice can be obtained from (1.6) by setting

$$r(n) = c(n)d(n), \quad s(n) = c(n)(1-d(n)),$$

so that we have [4]

$$\begin{aligned} s_t(n) &= s(n)(r(n-1)-r(n)), \\ r_t(n) &= r(n)(s(n)-s(n+1)+r(n-1)-r(n+1)), \end{aligned} \tag{1.7}$$

which can be written as (1.1) in terms of new variables

$$r(n) = -a^2(n), \quad s(n) = -b(n). \tag{1.8}$$

The main aim of this paper is to construct quasi-periodic solutions of the relativistic Toda hierarchy based on the approaches in [5, 6, 8–10]. This is realized through three main steps. First, based on the resulting Lax representation, the relativistic Toda hierarchy is decomposed into the solvable ordinary differential equations. Second, the determinant of Lax matrix serves as a base to construct the associated algebraic curve and the Abel–Jacobi coordinates, through which the continuous flows and discrete ones determined by the time variable $t_{\underline{m}}$ and discrete variable n, respectively, are straightened out and the linear superposition yields the solutions of the relativistic Toda hierarchy, expressed in the Abel–Jacobi coordinates. Here straightening out of the various flows means that the Abel–Jacobi coordinate is a linear function of the associated flow variables $t_{\underline{m}}$ and n. Third, an inverse procedure is indispensable in transforming the explicit solution in the original coordinates. The main tool in our theory is the asymptotic properties and the algebro-geometric characters of the meromorphic function and the hyperelliptic curve.

The outline of this paper is as follows. In Sec. 2, we introduce the Lenard gradients and derive the relativistic Toda hierarchy with the aid of the stationary zero-curvature equation, in which the first nontrivial member is the relativistic Toda lattice. In Sec. 3, we introduce a Lax matrix and establish a direct relation between the elliptic variables and the potentials. The relativistic Toda hierarchy is separated into solvable ordinary differential equations. In Sec. 4, the hyperelliptic Riemann surface of arithmetic genus N and the Abel–Jacobi coordinates are introduced from which the corresponding various flows are straightened out, including the continuous and discrete ones. In Sec. 5, quasi-periodic solutions of the relativistic Toda hierarchy are constructed in terms of the Riemann theta functions according to the asymptotic properties and the algebro-geometric characters of the meromorphic function on the hyperelliptic curve.

2. The Relativistic Toda Hierarchy

In this section, we construct the relativistic Toda hierarchy. Throughout this paper we suppose the following hypothesis.

Hypothesis 2.1. Assume that a and b satisfy

$$a(\cdot, t), b(\cdot, t) \in \mathbb{C}^{\mathbb{Z}}, \quad t \in \mathbb{R}, \quad a(n, \cdot), b(n, \cdot) \in C^1(\mathbb{R}), \quad n \in \mathbb{Z},$$
$$a(n, t)b(n, t) \neq 0, \quad (n, t) \in \mathbb{Z} \times \mathbb{R},$$

where $\mathbb{C}^{\mathbb{Z}}$ denotes the set of all complex-valued sequences indexed by $\mathbb{Z}$. For the sake of convenience, we denote by $E^{\pm}$ the shift operators acting on complex-valued sequences $f = \{f(n)\}_{n\in\mathbb{Z}}$ according to

$$(E^{\pm}f)(n) = f(n \pm 1), \quad n \in \mathbb{Z}$$

and define difference operators by $\triangle = E - 1$, $\triangle^* = E^- - 1$. Moreover, we will frequently use the notation

$$f^{\pm} = E^{\pm}f, \quad f \in \mathbb{C}^{\mathbb{Z}}.$$