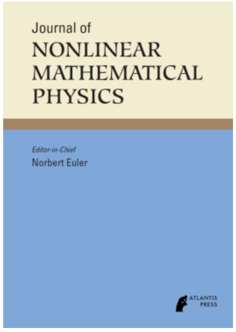

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A NEW (γ_n, σ_k)-KP HIERARCHY AND GENERALIZED DRESSING METHOD

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A new (γ_n, σ_k)-KP hierarchy (KPH) with two new time series γ_n and σ_k, which consists of γ_n-flow, σ_k-flow and mixed γ_n and σ_k evolution equations of eigenfunctions, is proposed. Two reductions and constrained flows of (γ_n, σ_k)-KPH are studied. The dressing method is generalized to the (γ_n, σ_k)-KPH and some solutions are presented.

Keywords: (γ_n, σ_k)-KP hierarchy; constrained flows; Lax representation; generalized dressing method.

1. Introduction

Generalizations of KP hierarchy (KPH) attract a lot of interests from both physical and mathematical points of view [2–4, 7, 8, 15–17, 19, 21, 22]. One kind of generalization is the multi-component KPH [2], which contains many physical relevant nonlinear integrable systems such as Davey–Stewartson equation, two-dimensional Toda lattice and three-wave resonant integrable equations. Another kind of generalization of KP equation is the so-called KP equation with self-consistent sources (KPESCS) [15, 16]. For example, the first type and second type of KPESCS consists of KP equation with some additional terms and eigenvalue problem or time evolution equations of eigenfunctions of KP equation, respectively [5, 6, 15–17, 23].

Denote the time series of KPH by $\{t_n\}$. Recently, we proposed an approach to construct an extended KPH (exKPH) by introducing another time series $\{\tau_k\}$ [10, 13, 24]. The exKPH consists of t_n-flow of KPH, τ_k-flow and the t_n-evolution equations of eigenfunctions. To make difference, we may call the exKPH as (t_n, τ_k)-KPH. The (t_n, τ_k)-KPH contains the first type and second type of KPESCS. Also we developed the dressing method to solve the (t_n, τ_k)-KPH [12]. The paper [14] generalized the (t_n, τ_k)-KPH to the (τ_n, τ_k)-KPH which consists of τ_n-flow, τ_k-flow and the τ_n-evolution equations of eigenfunctions and τ_k-evolutions of eigenfunctions. However, [14] did not find the solution of (τ_n, τ_k)-KPH. In contrast to one t_n-evolution equation of eigenfunctions as coupling equation in our (t_n, τ_k)-KPH, there are two coupling equations: τ_n-evolution and τ_k-evolution equations of eigenfunctions in (τ_n, τ_k)-KPH. Our generalized dressing method cannot be applied to the (τ_n, τ_k)-KPH due to too many coupling equations.

In this paper, we generalize the (t_n, τ_k)-KPH to (γ_n, σ_k)-KPH by introducing two new time series γ_n and σ_k with two parameters α_n and β_k. The (γ_n, σ_k)-KPH consists of γ_n-flow, σ_k-flow and one mixed γ_n and σ_k evolution equation of eigenfunctions. The (γ_n, σ_k)-KPH can be reduced to the KPH and (t_n, τ_k)-KPH, and contains first type and second type as well as mixed type of KPESCS as special cases. The constrained flows of the (γ_n, σ_k)-KPH can be regarded as generalization of Gelfand–Dickey hierarchy (GDH), which contains the first type, second type as well as mixed type of GDH with self-consistent sources in special cases. We also develop the dressing method to solve the (γ_n, σ_k)-KPH. Comparing with the multi-component generalization, we generalize the KPH by means of introducing two new time series and adding eigenfunctions as components.

The paper is organized as follows: In Sec. 2, we propose a new (γ_n, σ_k)-KPH. Section 3 presents the constrained flows of the (γ_n, σ_k)-KPH. Section 4 devotes to develop the generalized dressing method for solving the (γ_n, σ_k)-KPH. Section 5 presents the N-soliton solutions and a conclusion is given in the Sec. 6.

2. A New (γ_n, σ_k)-KP Hierarchy

2.1. *The KP hierarchy and extended KP hierarchy*

Let us first recall the construction of the KPH [2–4, 7, 21] and the exKPH [12, 13]. It is well known that the pseudo-differential operator L with potential functions u_i is defined as

$$L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \cdots.$$

The KPH is given by [4]

$$L_{t_n} = [B_n, L], \tag{1}$$

where $B_n = L^n_+$ stands for the differential part of L^n. The compatibility of the t_n-flow and t_k-flow of (1) leads to the zero-curvature representation of KPH

$$B_{n,t_k} - B_{k,t_n} + [B_n, B_k] = 0. \tag{2}$$

In particular, $B_2 = \partial^2 + u_1$, $B_3 = \partial^3 + 3u_1\partial + 3(u_{1x} + u_2)$ and (2) by setting $t_2 = y$, $t_3 = t$ and $u_1 = u$ yields the KP equation

$$(4u_t - 12uu_x - u_{xxx})_x - 3u_{yy} = 0.$$

Based on the observation that the squared eigenfunction symmetry constraint given by

$$L^k = B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i,$$

$$q_{i,t_n} = B_n(q_i), \quad r_{i,t_n} = -B_n^*(r_i)$$

is compatible with KPH [1, 9], we proposed the exKPH as follows in [13]

$$L_{t_n} = [B_n, L], \tag{3a}$$

$$L_{\tau_k} = \left[B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i, L\right], \tag{3b}$$

$$q_{i,t_n} = B_n(q_i), \quad r_{i,t_n} = -B_n^*(r_i), \quad i = 1, \ldots, N. \tag{3c}$$

The commutativity of (3a) and (3b) under (3c) gives rise to the following zero-curvature representation for exKPH (3)

$$B_{n,\tau_k} - B_{k,t_n} + [B_n, B_k] + \left[B_n, \sum_{i=1}^{N} q_i \partial^{-1} r_i\right]_+ = 0, \tag{4a}$$

$$q_{i,t_n} = B_n(q_i), \quad r_{i,t_n} = -B_n^*(r_i), \quad i = 1, \ldots, N. \tag{4b}$$

To different (3) from (1) and the generalized KPH presented in this paper, we may denote (3) or (4) by (t_n, τ_k)-KPH. We developed the dressing method to solve the (t_n, τ_k)-KPH and obtained its solutions in [12]. The paper [14] generalized the (t_n, τ_k)-KPH to (τ_n, τ_k)-KPH as follows

$$B_{n,\tau_k} - B_{k,\tau_n} + [B_n, B_k] + \left[B_n, \sum_{i=1}^{N} q_i \partial^{-1} r_i\right]_+ + \left[\sum_{i=1}^{N} q_i \partial^{-1} r_i, B_k\right]_+ = 0, \tag{5a}$$

$$q_{i,\tau_n} = B_n(q_i), \quad r_{i,\tau_n} = -B_n^*(r_i), \tag{5b}$$

$$q_{i,\tau_k} = B_k(q_i), \quad r_{i,\tau_k} = -B_k^*(r_i), \quad i = 1, \ldots, N. \tag{5c}$$

But [14] did not find the solutions for the (τ_n, τ_k)-KPH (5). In contrast to one pair of coupling equations (3c) (or (4b)) in (t_n, τ_k)-KPH, there are two pairs of coupling equations (5b) and (5c) in (τ_n, τ_k)-KPH. In fact, the dressing method developed in our paper [12] cannot be applied to the (τ_n, τ_k)-KPH (5) since there are too many (two) coupling systems (5b) and (5c).

2.2. *A new* (γ_n, σ_k)*-KP hierarchy*

Stimulated by the (t_n, τ_k)-KPH (3) and (4), we propose the following generalized KPH with two generalized time series γ_n and σ_k:

$$L_{\gamma_n} = \left[B_n + \alpha_n \sum_{i=1}^{N} q_i \partial^{-1} r_i, L\right], \tag{6a}$$

$$L_{\sigma_k} = \left[B_k + \beta_k \sum_{i=1}^{N} q_i \partial^{-1} r_i, L \right], \tag{6b}$$

$$\begin{aligned} &\alpha_n(q_{i,\sigma_k} - B_k(q_i)) - \beta_k(q_{i,\gamma_n} - B_n(q_i)) = 0, \\ &\alpha_n(r_{i,\sigma_k} + B_k^*(r_i)) - \beta_k(r_{i,\gamma_n} + B_n^*(r_i)) = 0, \quad i = 1, 2, \ldots, N. \end{aligned} \tag{6c}$$

We will prove the compatibility of (6a) and (6b) under (6c) in the following theorem. First we need the following Lemma presented in [13]

$$\left[B_n, \sum_{i=1}^{N} q_i \partial^{-1} r_i \right]_- = \sum_{i=1}^{N} B_n(q_i) \partial^{-1} r_i - \sum_{i=1}^{N} q_i \partial^{-1} B_n^*(r_i). \tag{7}$$

Theorem 1. *The* γ_n*-flow* (6a) *and* σ_k*-flow* (6b) *under* (6c) *are compatible.*

Proof. Denote

$$\tilde{B}_n = B_n + \alpha_n \sum_{i=1}^{N} q_i \partial^{-1} r_i,$$

$$\tilde{B}_k = B_k + \beta_k \sum_{i=1}^{N} q_i \partial^{-1} r_i.$$

In order to prove $L_{\gamma_n,\sigma_k} = L_{\sigma_k,\gamma_n}$, i.e.

$$[\tilde{B}_{n,\sigma_k} - \tilde{B}_{k,\gamma_n} + [\tilde{B}_n, \tilde{B}_k], L] = 0$$

we only need to prove

$$\tilde{B}_{n,\sigma_k} - \tilde{B}_{k,\gamma_n} + [\tilde{B}_n, \tilde{B}_k] = 0. \tag{8}$$

For convenience, we omit $\sum$. We can find that

$$\begin{aligned} \tilde{B}_{n,\sigma_k} &= B_{n,\sigma_k} + \alpha_n(q\partial^{-1}r)_{\sigma_k} = [B_k + \beta_k(q\partial^{-1}r), L^n]_+ + \alpha_n(q\partial^{-1}r)_{\sigma_k} \\ &= [B_k, L^n]_+ + \beta_k[q\partial^{-1}r, L^n]_+ + \alpha_n q_{\sigma_k}\partial^{-1}r + \alpha_n q\partial^{-1}r_{\sigma_k}, \end{aligned} \tag{9}$$

and similarly,

$$\tilde{B}_{k,\gamma_n} = [B_n, L^k]_+ + \alpha_n[q\partial^{-1}r, L^k]_+ + \beta_k q_{\gamma_n}\partial^{-1}r + \beta_k q\partial^{-1}r_{\gamma_n}. \tag{10}$$

Making use of the basic Lemma (7), we have

$$\begin{aligned} [\tilde{B}_n, \tilde{B}_k] &= [B_n, B_k] + [B_n, \beta_k q\partial^{-1}r] + [\alpha_n q\partial^{-1}r, B_k] \\ &= [L^n - (L^n)_-, L^k - (L^k)_-]_+ + \beta_k[B_n, q\partial^{-1}r]_+ + \alpha_n[q\partial^{-1}r, B_k]_+ \\ &\quad + \beta_k[B_n, q\partial^{-1}r]_- + \alpha_n[q\partial^{-1}r, B_k]_- \\ &= [B_n, L^k]_+ + [L^n, B_k]_+ - [(L^n)_-, (L^k)_-]_+ \\ &\quad + \beta_k[B_n, q\partial^{-1}r]_+ + \alpha_n[q\partial^{-1}r, B_k]_+ \\ &\quad + \beta_k B_n(q)\partial^{-1}r - \beta_k q\partial^{-1}B_n^*(r) \\ &\quad - \alpha_n B_k(q)\partial^{-1}r + \alpha_n q\partial^{-1}B_k^*(r). \end{aligned} \tag{11}$$

Then (9), (10) and (11) under (6c) yields

$$\tilde{B}_{n,\sigma_k} - \tilde{B}_{k,\gamma_n} + [\tilde{B}_n, \tilde{B}_k] = [\alpha_n(q_{\sigma_k} - B_k(q)) - \beta_k(q_{\gamma_n} - B_n(q))]\partial^{-1}r + q\partial^{-1}[\alpha_n(r_{\sigma_k} + B_k^*(r)) - \beta_k(r_{\gamma_n} + B_n^*(r))] = 0. \qquad \square$$

Then the compatibility of γ_n-flow (6a) and σ_k-flow (6b) under (6c) gives rise to the zero-curvature representation for (6)

$$\left(B_n + \alpha_n \sum_{i=1}^{N} q_i\partial^{-1}r_i\right)_{\sigma_k} - \left(B_k + \beta_k \sum_{i=1}^{N} q_i\partial^{-1}r_i\right)_{\gamma_n} + \left[B_n + \alpha_n \sum_{i=1}^{N} q_i\partial^{-1}r_i, B_k + \beta_k \sum_{i=1}^{N} q_i\partial^{-1}r_i\right] = 0$$

which under (6c) can be simplified as follows. Then we have the following theorem.

Theorem 2. *The commutativity of* (6a) *and* (6b) *under* (6c) *gives rise to the zero-curvature equation for the generalized KPH with two generalized time series*

$$B_{n,\sigma_k} - B_{k,\gamma_n} + [B_n, B_k] + \beta_k \left[B_n, \sum_{i=1}^{N} q_i\partial^{-1}r_i\right]_+ + \alpha_n \left[\sum_{i=1}^{N} q_i\partial^{-1}r_i, B_k\right]_+ = 0, \tag{12a}$$

$$\begin{aligned} &\alpha_n(q_{i,\sigma_k} - B_k(q_i)) - \beta_k(q_{i,\gamma_n} - B_n(q_i)) = 0, \\ &\alpha_n(r_{i,\sigma_k} + B_k^*(r_i)) - \beta_k(r_{i,\gamma_n} + B_n^*(r_i)) = 0, \quad i = 1, 2, \ldots, N, \end{aligned} \tag{12b}$$

with the Lax representation

$$\psi_{\gamma_n} = \left(B_n + \alpha_n \sum_{i=1}^{N} q_i\partial^{-1}r_i\right)(\psi), \quad \psi_{\sigma_k} = \left(B_k + \beta_k \sum_{i=1}^{N} q_i\partial^{-1}r_i\right)(\psi). \tag{13}$$

We briefly call (6) and (12) as (γ_n, σ_k)-KPH. It is easy to see that (γ_n, σ_k)-KPH (6) and (12) for $\alpha_n = \beta_k = 0$ reduces to KPH (1) and (2), (γ_n, τ_k)-KPH for $\alpha_n = 0$, $\beta_k = 1$ reduces to (t_n, τ_k)-KPH (3) and (4). So (γ_n, σ_k)-KPH (6) and (12) present a more generalized KPH which contains the KPH and (t_n, τ_k)-KPH as the special cases.

Example 1. Let us take $n = 2$ and $k = 3$, and set $\gamma_2 = y$, $\sigma_3 = t$, $u_1 = u$. Then Eqs. (12) becomes

$$B_{2,t} - B_{3,y} + [B_2, B_3] + \beta_3 \left[B_2, \sum_{i=1}^{N} q_i\partial^{-1}r_i\right]_+ + \alpha_2 \left[\sum_{i=1}^{N} q_i\partial^{-1}r_i, B_3\right]_+ = 0, \tag{14a}$$

$$\begin{aligned}&\alpha_2(q_{i,t}-B_3(q_i))-\beta_3(q_{i,y}-B_2(q_i))=0,\\&\alpha_2(r_{i,t}+B_3^*(r_i))-\beta_3(r_{i,y}+B_2^*(r_i))=0,\quad i=1,2,\ldots,N,\end{aligned}\tag{14b}$$

which gives the following nonlinear equation

$$\begin{aligned}&4u_t-3\partial^{-1}u_{yy}-12uu_x-u_{xxx}-3\alpha_2\sum_{i=1}^N(q_ir_i)_y+4\beta_3\sum_{i=1}^N(q_ir_i)_x\\&\quad+3\alpha_2\sum_{i=1}^N(q_ir_{i,xx}-q_{i,xx}r_i)=0,\end{aligned}\tag{15a}$$

$$\begin{aligned}&\alpha_2\left(q_{i,t}-q_{i,xxx}-3uq_{i,x}-\frac{3}{2}q_i\partial^{-1}u_y-\frac{3}{2}q_iu_x-\frac{3}{2}q_i\sum_{j=1}^Nq_jr_j\right)\\&\quad-\beta_3(q_{i,y}-q_{i,xx}-2uq_i)=0,\\&\alpha_2\left(r_{i,t}-r_{i,xxx}-3ur_{i,x}+\frac{3}{2}r_i\partial^{-1}u_y-\frac{3}{2}r_iu_x+\frac{3}{2}r_i\sum_{j=1}^Nq_jr_j\right)\\&\quad-\beta_3(r_{i,y}+r_{i,xx}+2ur_i)=0,\quad i=1,2,\ldots,N,\end{aligned}\tag{15b}$$

with the Lax representation as follows

$$\begin{aligned}\psi_y&=\left(\partial^2+2u+\alpha_2\sum_{i=1}^Nq_i\partial^{-1}r_i\right)(\psi),\\\psi_t&=\left(\partial^3+3u\partial+\frac{3}{2}\partial^{-1}u_y+\frac{3}{2}u_x+\frac{3}{2}\beta_3\sum_{i=1}^Nq_i\partial^{-1}r_i\right)(\psi).\end{aligned}\tag{16}$$

Specially, when take $\alpha_2=\beta_3=0$; $\alpha_2=0$, $\beta_3=1$; $\alpha_2=1$, $\beta_3=0$ and $\alpha_2=1$, $\beta_3=1$, respectively, (15) and (16) reduces to the KP equation [4], the first type of KPESCS [15, 16, 23], the second type of KPESCS [6, 13, 15] and the mixed type of KPESCS [6] and their Lax representations, respectively.

3. Reduction

Consider the constraint given by

$$L^k=B_k+\beta_k\sum_{i=1}^Nq_i\partial^{-1}r_i.\tag{17}$$

Then (6b) yields

$$\begin{aligned}(L^k)_{\sigma_k}&=\left[B_k+\beta_k\sum_{i=1}^Nq_i\partial^{-1}r_i,L^k\right]=0,\\B_{k,\sigma_k}&=(L^k_{\sigma_k})_+=0,\\\left(\sum_{i=1}^Nq_i\partial^{-1}r_i\right)_{\sigma_k}&=(L^k_{\sigma_k})_-=0,\end{aligned}\tag{18}$$

which imply that L, B_k, q_i and r_i under (17) are independent of σ_k. Subsequently, q_{i,σ_k} and r_{i,σ_k} in (6c) should be replaced by $\lambda_i q_i$ and $-\lambda_i r_i$ as in the case of constrained flow of KP [1, 9], namely (6c) under the constraint (17) should be replaced by

$$\begin{aligned} &\alpha_n(\lambda_i q_i - B_k(q_i)) - \beta_k(q_{i,\gamma_n} - B_n(q_i)) = 0,\\ &\alpha_n(-\lambda_i r_i + B_k^*(r_i)) - \beta_k(r_{i,\gamma_n} + B_n^*(r_i)) = 0. \end{aligned} \tag{19}$$

We will show that the constraint (17) is invariant under the γ_n-flow (6a) and (19). In fact, making use of (6a), (7) and (19), we have

$$(L^k - B_k)_{\gamma_n} = (L^k_{\gamma_n})_- = [\tilde{B}_n, L^k]_-,$$

$$\begin{aligned} \left(\beta_k \sum_{i=1}^N q_i \partial^{-1} r_i\right)_{\gamma_n} &= \beta_k \sum_{i=1}^N (q_{i,\gamma_n}\partial^{-1} r_i + q_i \partial^{-1} r_{i,\gamma_n})\\ &= \sum_{i=1}^N [\beta_k B_n(q_i)\partial^{-1} r_i + \alpha_n(\lambda_i q_i - B_k(q_i))\partial^{-1} r_i\\ &\quad - \beta_k q_i \partial^{-1} B_n^*(r_i) + \alpha_n q_i \partial^{-1}(-\lambda_i r_i + B_k^*(r_i))]\\ &= \left[B_n, \beta_k \sum_{i=1}^N q_i\partial^{-1} r_i\right]_- - \left[B_k, \alpha_n \sum_{i=1}^N q_i \partial^{-1} r_i\right]_-\\ &= \left[\tilde{B}_n, \beta_k \sum_{i=1}^N q_i\partial^{-1} r_i\right]_- - \left[B_k, \alpha_n \sum_{i=1}^N q_i \partial^{-1} r_i\right]_-\\ &= [\tilde{B}_n, L^k]_- - [\tilde{B}_n, B_k]_- - [B_k, \tilde{B}_n]_- + [B_k, B_n]_-\\ &= [\tilde{B}_n, L^k]_-. \end{aligned}$$

Then

$$\left(L^k - B_k - \beta_k \sum_{i=1}^N q_i \partial^{-1} r_i\right)_{\gamma_n} = 0.$$

This means that the sub-manifold determined by the k-constraint (17) is invariant under the γ_n-flow (6a) and (19).

Therefore, the constrained flow of (γ_n, σ_k)-KPH (6) and (12) under (17) reads

$$B_{k,\gamma_n} + [B_k, B_n] + \beta_k \left[\sum_{i=1}^N q_i \partial^{-1} r_i, B_n\right]_+ + \alpha_n \left[B_k, \sum_{i=1}^N q_i\partial^{-1} r_i\right]_+ = 0, \tag{20a}$$

$$\begin{aligned} &\alpha_n(\lambda_i q_i - B_k(q_i)) - \beta_k(q_{i,\gamma_n} - B_n(q_i)) = 0,\\ &\alpha_n(-\lambda_i r_i + B_k^*(r_i)) - \beta_k(r_{i,\gamma_n} + B_n^*(r_i)) = 0, \quad i = 1, 2, \ldots, N. \end{aligned} \tag{20b}$$

with

$$B_n = \left(B_k + \beta_k \sum_{i=1}^{N} q_i \partial^{-1} r_i \right)_+^{\frac{n}{k}}. \tag{20c}$$

The system (20) can be regarded as the generalized GDH. When $\alpha_n = \beta_k = 0$, (20) reduces to the GDH. When $\alpha_n = 1$, $\beta_k = 0$, (20) is just the first type of GDH with self-consistent sources. When $\alpha_n = 0$, $\beta_k = 1$, (20) represent the second type of GDH with self-consistent sources.

Example 2. When $k = 2$, $n = 3$, $\gamma_3 = t$, $u_1 = u$, (20) gives

$$u_t - \frac{1}{4}u_{xxx} - 3uu_x + \alpha_3 \sum_{i=1}^{N}(q_i r_i)_x + \frac{3}{4}\beta_2 \sum_{i=1}^{N}(q_i r_{i,xx} - q_{i,xx} r_i) = 0, \tag{21a}$$

$$-\beta_2 \left(q_{i,t} - q_{i,xxx} - 3uq_{i,x} - \frac{3}{2}u_x q_i - \frac{3}{2} q_i \sum_{j=1}^{N} q_j r_j \right) + \alpha_3(\lambda_i q_i - q_{i,xx} - 2uq_i) = 0, \tag{21b}$$

$$\beta_2 \left(r_{i,t} - r_{i,xxx} - 3ur_{i,x} - \frac{3}{2}u_x r_i + \frac{3}{2} r_i \sum_{j=1}^{N} q_j r_j \right) - \alpha_3(-\lambda_i r_i + r_{i,xx} + 2ur_i) = 0, \quad i = 1, 2, \ldots, N. \tag{21c}$$

Which is just the mixed type of KdV equation with self-consistent sources. Equation (21) with $\alpha_3 = 1$, $\beta_2 = 0$ gives the first type of KdV equation with sources [11, 18]. Equation (21) with $\alpha_3 = 0$, $\beta_2 = 1$ gives the second type of KdV equation with sources [13].

Example 3. When $k = 3$, $n = 2$ and $\gamma_2 = t$, $u_1 = u$, (20) gives rise to the mixed type of Boussinesq equation with self-consistent sources

$$\frac{1}{3}u_{xxxx} + 2(u^2)_{xx} + u_{tt} + \sum_{i=1}^{N} \left[-\frac{4}{3}\beta_3 (q_i r_i)_{xx} + \alpha_2 (q_i r_i)_{xt} + \alpha_2 (q_{i,xx} r_i - q_i r_{i,xx})_x \right] = 0, \tag{22a}$$

$$\begin{aligned} &\alpha_2 \left[\lambda_i q_i - q_{i,xxx} - 3uq_{i,x} - q_i \left(\frac{3}{2}\partial^{-1}u_y + \frac{3}{2}u_x + \frac{3}{2}\sum_{i=1}^{N} q_j r_j \right) \right] - \beta_3(q_{i,t} - q_{i,xx} - 2uq_i) = 0, \\ &\alpha_2 \left[-\lambda_i r_i - r_{i,xxx} - 3ur_{i,x} + r_i \left(\frac{3}{2}\partial^{-1}u_y - \frac{3}{2}u_x + \frac{3}{2}\sum_{i=1}^{N} q_j r_j \right) \right] - \beta_3(r_{i,t} - r_{i,xx} - 2ur_i) = 0, \quad i = 1, 2, \ldots, N. \end{aligned} \tag{22b}$$

Equation (22) with $\alpha_2 = 1$, $\beta_3 = 0$ gives the first type of Boussinesq equation with sources. Equation (21) with $\alpha_2 = 0$, $\beta_3 = 1$ gives the second type of Boussinesq equation with sources [13].

4. Dressing Approach for (γ_n, σ_k)-KPH

Inspired by [4, 20], we consider the generalized dressing approach for (γ_n, σ_k)-KPH. Assume that operator L of (γ_n, σ_k)-KPH can be written as a dressing form

$$L = W\partial W^{-1}, \tag{23}$$

$$W = 1 + w_1\partial^{-1} + w_2\partial^{-2} + \cdots. \tag{24}$$

Proposition 1. *If W defined by* (24) *satisfies*

$$W_{\gamma_n} = -L_-^n W + \alpha_n \sum_{i=1}^{N} q_i\partial^{-1} r_i W, \tag{25a}$$

$$W_{\sigma_k} = -L_-^k W + \beta_k \sum_{i=1}^{N} q_i\partial^{-1} r_i W \tag{25b}$$

then L satisfies (6a) *and* (6b).

Proof. Based on (23) and (25a), we have

$$\begin{aligned} L_{\gamma_n} &= W_{\gamma_n}\partial W - W\partial W^{-1} W_{\gamma_n} W^{-1} \\ &= \left(L_+^n + \alpha_n \sum_{i=1}^{N} q_i\partial^{-1} r_i\right) L - L\left(L_+^n + \alpha_n \sum_{i=1}^{N} q_i\partial^{-1} r_i\right) \\ &= \left[B_n + \alpha_n \sum_{i=1}^{N} q_i\partial^{-1} r_i, L\right]. \end{aligned}$$

Similarly, we can prove that L satisfies (6b). □

It is well known that the Wronskian determinant [4]

$$Wr(h_1, \ldots, h_N) = \begin{vmatrix} h_1 & h_2 & \cdots & h_N \\ h_1' & h_2' & \cdots & h_N' \\ \vdots & \vdots & \vdots & \vdots \\ h_1^{N-1} & h_2^{N-1} & \cdots & h_N^{N-1} \end{vmatrix}$$

is a τ-function of the KPH and the Nth order differential operator given by

$$W = \frac{1}{Wr(h_1, \ldots, h_N)} \begin{vmatrix} h_1 & h_2 & \cdots & h_N & 1 \\ h_1' & h_2' & \cdots & h_N' & \partial \\ \vdots & \vdots & \vdots & \vdots & \\ h_1^N & h_2^N & \cdots & h_N^N & \partial^N \end{vmatrix} \tag{26}$$

provides the dressing operator, where $h_1, h_2, \ldots, h_N$ are N independent functions and satisfy $W(h_i) = 0$.

This dressing operator W is constructed as follows: Let f_i, g_i satisfy

$$f_{i,\gamma_n} = \partial^n(f_i), \quad f_{i,\sigma_k} = \partial^k(f_i), \tag{27a}$$

$$g_{i,\gamma_n} = \partial^n(g_i), \quad g_{i,\sigma_k} = \partial^k(g_i), \quad i = 1, \ldots, N, \tag{27b}$$

and let h_i be the linear combination of f_i and g_i

$$h_i = f_i + F_i(\alpha_n\gamma_n + \beta_k\sigma_k)g_i, \quad i = 1, \ldots, N, \tag{28}$$

with $F_i(X)$ being a differentiable function of X, $X = \alpha_n\gamma_n + \beta_k\sigma_k$.

Define

$$q_i = -\dot{F}_i W(g_i), \quad r_i = (-1)^{N-i}\frac{Wr(h_1, \ldots, \hat{h}_i, \ldots, h_N)}{Wr(h_1, \ldots, h_N)}, \quad i = 1, \ldots, N, \tag{29}$$

where the hat $\hat{}$ means rule out this term from the Wronskian determinant, $\dot{F}_i = \frac{d\alpha_i}{dX}$. We have the following theorem.

Theorem 3. *Let W be defined by* (26) *and* (28), $L = W\partial W^{-1}$, q_i *and* r_i *be given by* (29), *then* W, L, q_i, r_i *satisfy* (25) *and* (γ_n, σ_k)*-KPH* (6) *and* (12).

To prove Theorem 3, we need several lemmas. The first one is given by Oevel and Strampp [20].

Lemma 1. $W^{-1} = \sum_{i=1}^N h_i\partial^{-1}r_i$.

Lemma 2 ([13]). *The operator $\partial^{-1}r_iW$ is a non-negative differential operator and*

$$(\partial^{-1}r_iW)(h_j) = \delta_{ij}, \quad 1 \le i, \ j \le N. \tag{30}$$

Proof of Theorem 3. For (25a), taking ∂_{γ_n} to the identity $W(h_i) = 0$, using (27), (28), the definition (29) and Lemma 2, we find

$$\begin{aligned}
0 &= (W_{\gamma_n})(h_i) + (W\partial^n)(f_i) + \alpha_n\dot{F}_iW(g_i) + F_i(W\partial^n)(g_i)\\
&= (W_{\gamma_n})(h_i) + (W\partial^n)(h_i) - \alpha_n q_i\\
&= (W_{\gamma_n})(h_i) + (L^nW)(h_i) - \alpha_n\sum_{j=1}^N q_j\delta_{ji}\\
&= \left(W_{\gamma_n} + L^n_-W - \alpha_n\sum_{j=1}^N q_j\partial^{-1}r_jW\right)(h_i).
\end{aligned}$$

Since the non-negative difference operator acting on h_i in the last expression has degree $<$ N, it cannot annihilate N independent functions unless the operator itself vanishes. Hence (25a) is proved. Similarly, we can prove (25b). Then Proposition 1 leads to (6a) and (6b).

The first equation in (6c) is easy to be verified by a direct calculation, so it remains to prove the second equation in (6c). First, we see that

$$\begin{aligned}(W^{-1})_{\gamma_n} &= -W^{-1}W_{\gamma_n}W^{-1} = -W^{-1}\left(L_+^n - L^n + \alpha_n \sum_{j=1}^{N} q_j \partial^{-1} r_j\right)\\ &= \partial^n W^{-1} - W^{-1}B_n - \alpha_n W^{-1}\sum_{j=1}^{N} q_j\partial^{-1}r_j, \end{aligned} \tag{31}$$

$$(W^{-1})_{\sigma_k} = \partial^k W^{-1} - W^{-1}B_k - \beta_k W^{-1}\sum_{j=1}^{N} q_j\partial^{-1}r_j. \tag{32}$$

On the other hand, from $W^{-1} = \sum h_i \partial^{-1} r_i$ we have

$$(W^{-1})_{\gamma_n} = \sum \partial^n (h_i)\partial^{-1} r_i + \sum h_i \partial^{-1} r_{i,\gamma_n}, \tag{33}$$

$$(W^{-1})_{\sigma_k} = \sum \partial^k (h_i)\partial^{-1} r_i + \sum h_i \partial^{-1} r_{i,\sigma_k}. \tag{34}$$

It is obviously that $\alpha_n(32) - \beta_k(31) = \alpha_n(34) - \beta_k(33)$, i.e.

$$\begin{aligned} -\beta_k \sum \partial^n(h_i)\partial^{-1}r_i &- \beta_k \sum h_i\partial^{-1}r_{i,\gamma_n} + \alpha_n \sum \partial^k(h_i)\partial^{-1}r_i\\ &+ \alpha_n \sum h_i \partial^{-1} r_{i,\sigma_k}\\ &= -\beta_k(\partial^n W^{-1} - W^{-1}B_n)_- + \alpha_n(\partial^k W^{-1} - W^{-1}B_k)_-\\ &= -\beta_k \sum \partial^n(h_i)\partial^{-1}r_i + \beta_k\sum h_i\partial^{-1}B_n^*(r_i) + \alpha_n \sum \partial^k(h_i)\partial^{-1}r_i\\ &- \alpha_n \sum h_i \partial^{-1} B_k^*(r_i). \end{aligned}$$

The above equations gives

$$\beta_k \sum h_i \partial^{-1}(r_{i,\gamma_n} + B_n^*(r_i)) - \alpha_n \sum h_i\partial^{-1}(r_{i,\sigma_k} + B_k^*(r_i)) = 0,$$

which implies the second equation in (6c) holds. □

5. N-Soliton Solutions for (γ_n, σ_k)-KPH

Using Theorem 3, we can find N-soliton solutions to every equations in the (γ_n, σ_k)-KPH (6) and (12). Let us illustrate it by solving (15). We take the solution of (27) as follows

$$\begin{aligned} f_i &:= \exp(\lambda_i x + \lambda_i^2 y + \lambda_i^3 t) = e^{\xi_i}, \quad g_i := \exp(\mu_i x + \mu_i^2 y + \mu_i^3 t) = e^{\eta_i},\\ h_i &:= f_i + F_i(\alpha_2 y + \beta_3 t) g_i = 2\sqrt{F_i}\exp\left(\frac{\xi_i + \eta_i}{2}\right)\cosh(\Omega_i), \quad \Omega_i = \frac{1}{2}(\xi_i - \eta_i - \ln F_i). \end{aligned} \tag{35}$$

For example, when $N = 1$, $W = \partial - \frac{h'}{h}$,

$$L = W\partial^{-1}W = \partial + \frac{(\lambda_1 - \mu_1)^2}{4}\operatorname{sech}^2\Omega_1\partial^{-1} + \cdots.$$

The one-soliton solution for (15) with $N = 1$ as follows

$$u = \frac{(\lambda_1 - \mu_1)^2}{4} \operatorname{sech}^2 \Omega_1,$$

$$q_1 = \sqrt{\alpha_2 F_{1y} + \beta_3 F_{1t}}(\lambda_1 - \mu_1) e^{\xi_1 + \eta_1} \operatorname{sech} \Omega_1,$$

$$r_1 = \frac{1}{\sqrt{F_1}} e^{-(\xi_1 + \eta_1)} \operatorname{sech} \Omega_1.$$

In the case of $N = 2$, the two-soliton solution for (15) is given

$$u = \partial^2 \ln \Theta,$$

$$q_1 = (\alpha_2 F_{1y} + \beta_3 F_{1t}) \frac{(\lambda_1 - \mu_1)(\lambda_2 - \mu_1)}{\Theta} \left(1 + F_2 \frac{(\lambda_1 - \mu_2)(\mu_2 - \mu_1)}{(\lambda_1 - \lambda_2)(\lambda_2 - \mu_1)} e^{\eta_2 - \xi_1} \right) e^{\eta_1},$$

$$q_2 = (\alpha_2 F_{2y} + \beta_3 F_{2t}) \frac{(\lambda_2 - \mu_2)(\lambda_1 - \mu_2)}{\Theta} \left(1 + F_1 \frac{(\lambda_2 - \mu_1)(\mu_1 - \mu_2)}{(\lambda_2 - \lambda_1)(\lambda_1 - \mu_2)} e^{\eta_1 - \xi_2} \right) e^{\eta_2},$$

$$r_1 = \frac{1 + F_2 e^{\eta_2 - \xi_2}}{\lambda_2 - \mu_1} e^{-\xi_1}, \quad r_2 = \frac{1 + F_1 e^{\eta_1 - \xi_1}}{\lambda_2 - \mu_1} e^{-\xi_2},$$

where

$$\Theta = 1 + F_1 \frac{\lambda_2 - \mu_1}{\lambda_2 - \lambda_1} e^{\eta_1 - \xi_1} + F_2 \frac{\mu_2 - \lambda_1}{\lambda_2 - \lambda_1} e^{\eta_2 - \xi_2} + F_1 F_2 \frac{\mu_2 - \mu_1}{\lambda_2 - \lambda_1} e^{\eta_1 + \eta_2 - \xi_1 - \xi_2}.$$

6. Conclusion

In contrast to the multi-component generalization of KPH, we generalize KPH by introducing new time series γ_n and σ_k and adding eigenfunctions as components. The (γ_n, σ_k)-KPH includes KPH and exKPH, and contains first type and second type as well as mixed type of KPESCS as special cases. The constrained flows of (γ_n, σ_k)-KPH can be regarded as the generalized GDH. We develop the dressing method for solving the (γ_n, σ_k)-KPH and present its N-soliton solutions.

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