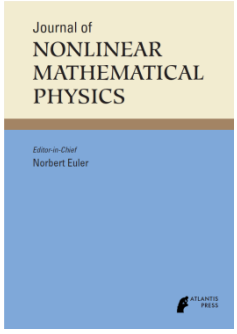

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LIE THEOREM VIA RANK 2 DISTRIBUTIONS (INTEGRATION OF PDE OF CLASS $\omega = 1$)

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In this paper we investigate compatible overdetermined systems of PDEs on the plane with one common characteristic. Lie's theorem states that its integration is equivalent to a system of ODEs, and we give a new proof by relating it to the geometry of rank 2 distributions. We find a criterion for integration in quadratures and in closed form, and discuss nonlinear Laplace transformations and symmetric PDE models.

Keywords: Lie's class 1; Darboux integrability; system of PDEs; characteristic; integral; Goursat flag; symbols; compatibility; Spencer cohomology.

0. Introduction

Consider a nonlinear system $\mathcal{E}$ of partial differential equations of orders $\leq k$, which we treat geometrically as a submanifold in k-jets. We will study its integration micro-locally near a regular point (in a neighborhood $U \subset \mathcal{E}$).

0.1. *Formulation of the problem*

We assume the system $\mathcal{E}$ is overdetermined and compatible (formally integrable).

Whenever the above assumptions on $\mathcal{E}$ are fulfilled we can evaluate algebraically formal dimension and rank of the system, namely we can determine on how many functions of how many variables a general solution formally depends.

For a particular case, when $\mathcal{E}$ is an (over)determined scalar system on the plane the complex characteristic variety consists of a finite number of points (characteristics). In general the complex characteristic variety can have positive dimension, but in this paper we assume it is discrete. Let ω be the total number of points counted with multiplicity.

If $\omega = 0$, i.e. the system $\mathcal{E}$ has finite type, its local solution space is finite-dimensional. Then integration of $\mathcal{E}$ can be reduced to a system of ODEs; we call such systems Frobenius.

If $\omega > 0$, then additional assumptions must be imposed to guarantee existence of solutions. Counting of solutions can still be carried on the formal level and the general stratum of the solution space of $\mathcal{E}$ is parametrized by ω functions of one variable. Sophus Lie called the number ω class of the system; for Ellie Cartan this is the character s_1 (provided the Cartan number is 1: $s_i = 0$ for $i > 1$).

We adapt here this definition of ω based on the Cartan test [6] (and omit discussion of the formal definition via characteristics, see [16, 17, 23] for details). Alternatively this number ω can be described so: Since the characteristic variety is discrete, the symbol g_k of the system $\mathcal{E}$ stabilizes and $\omega = \lim_{k\to\infty} \dim g_k$ (more details are in Sec. 1.1).

In this paper we restrict to the case $\omega = 1$, which is the next simplest after the finite type case $\omega = 0$.

0.2. *Main results*

For $\omega = 1$ Sophus Lie obtained in 1893 a theorem, which states that this case can be reduced to ODEs as well.

Theorem A. *A compatible regular overdetermined system $\mathcal{E}$ of class $\omega = 1$ can be locally integrated via ODEs.*

The proof in [20] is rather sketchy. The result was later obtained in [8] without reference to Lie. We will demonstrate the claim via geometry of rank 2 distributions and relate it to other important results.[a]

Two remarks are of order. First is that we have changed formal integrability to local smooth integrability (this is indeed missing in the classical papers). Generally it is wrong due to Levi and similar examples. The essential feature is regularity of the characteristic variety and peculiarity of $\omega = 1$ (i.e. for $\omega > 1$ the passage from formal integrability to local integrability is not generally correct).

Second, and more important, is that the reduction procedure can be made explicit and this allows not only to claim the reduction theoretically, but also to develop a practical algorithm for integrability.

In this paper we discuss the geometry behind integrability and concentrate on the problem of effective solution of PDE systems. So we are especially interested in quadrature and integration in closed form (an ideal case of Darboux integrability [2, 7, 13]).

Recall that closed form means possibility to represent the general solution parametrically through arbitrary functions, their derivatives, free parameters and constants (but no quadratures). For a PDE systems of class $\omega = 1$ on scalar function $u = u(x, y)$ this writes as

$$(x, y, u) = \Psi(\varsigma, f(\tau), f'(\tau), \ldots, f^{(q)}(\tau), c_1, \ldots, c_m). \tag{1}$$

For an underdetermined ODE on $u = u(x)$ we shall remove independent variable y to the left and the free variable ς to the right.

[a]Let us mention that paper [27] discusses another reduction to ODEs for the involutive PDE systems of the 2nd order. This family meets ours by class $\omega = 1$ systems of type $2E_2$ in terminology of Sec. 2.1, which were studied by Cartan in [4].

Theorem B. (i) *A system $\mathcal{E}$ of class $\omega = 1$ is integrable in quadratures if it has a transitive solvable Lie group of internal symmetries.*[b]

(ii) *A system $\mathcal{E}$ of class $\omega = 1$ is integrable in closed form if and only if it is linearizable by an internal transformation.*

Of course, one is interested in algorithmic integration, so that an effective linearization is important. Then a sequence of generalized Laplace transformations (these are the external transformations introduced for class $\omega = 1$ in [16]) finishes the job.

As we shall explain, determining both linearization and quadrature is related to investigation of the rank 2 distributions internally related to the system $\mathcal{E}$. Linearizable systems correspond to Goursat distributions, i.e. canonical Cartan distributions on the jet spaces for ODEs (in general nonlinear situation the growth vector is unrestricted). Rank 2 distributions for the systems integrable in quadratures have the structure of integrable extensions, which can be decoded starting from its Tanaka algebra.

Thus we can model types of reduction, based on the normal forms of rank 2 distributions. In particular, the simplest among exactly solvable (i.e. closed form with a quadrature) class $\omega = 1$ compatible nonlinearizable PDE systems will be those that can be reduced to Hilbert–Cartan equation (the algebra of its non-characteristic symmetries coincides with the exceptional Lie group G_2), see Sec. 3.5. More complicated examples will be presented at the end of the paper.

0.3. *Structure of the paper*

We will exploit the geometric theory of PDE, jet-geometry and the basics of Spencer formal theory. We will also use the geometry of vector distributions. The reader is invited to consult [14, 23, 26] for details.

Notations are different from source to source, and we adapt those of [17]. Since this paper is a continuation of [16], an acquaintance with the latter will be useful (but not mandatory).

The paper is organized as follows.

In Sec. 1 we recast the class $\omega = 1$ systems into the language of the geometry of differential equations and provide a new modern proof of Theorem A. Reduction to rank 2 distributions is the crucial ingredient. We then discuss an algorithmic method to integrate such systems and prove part (i) of Theorem B.

In Sec. 2 we discuss another more general method of integration of PDEs via integrable extensions (coverings), and relate this to the generalized symmetries. Notice that integrable extensions for rank 2 distributions were classified in [3], so their description in the symmetric cases reduces to purely algebraic questions.

In Sec. 3 we formulate the main invariants of compatible systems $\mathcal{E}$ of class $\omega = 1$, and we discuss transformations of such systems in linear and nonlinear cases. We investigate linear system from the viewpoint of internal geometry (complimentary to the external point of view in [16]), obtain the linearization criterion and finish the proof of Theorem B.

[b]Internal symmetries are transformations of the equation $\mathcal{E}$ considered as a manifold preserving the induced Cartan distribution. They are more general than the classical Lie symmetries, but can differ from the higher (Lie–Bäcklund) symmetries.

Depending on the type of the system and its reduced rank 2 distribution we can describe the structure of the general solution and a method of its integration. To be specific in this section we restrict to the case of two independent and one dependent variables, though the case of more independent variables makes no fundamental difference (non-scalar systems can be treated similarly or can be re-written via Drah's trick [24]).

Section 4 is devoted to various examples of compatible PDE systems of class $\omega = 1$. We will perform integration via the method of integrable extensions, generalized nonlinear Laplace transformations and discuss their relation to Darboux integrability. Explicit reduction of overdetermined PDEs to underdetermined Monge equations will be shown. Some of the most symmetric examples are coverings of the overdetermined involutive system of order 2 on the plane investigated by Cartan.

1. Around Sophus Lie Theorems

In this section we give a modern proof of Theorem A. Sophus Lie's original approach is indirect and hard to implement. We present a geometric method, which is the base of our approach to integration of class $\omega = 1$ systems. Furthermore we will elaborate this theorem to get the constructive Theorem B.

1.1. *The geometric setup*

Consider the space of jets $J^k(W, N)$, where $W = \mathbb{R}^n$ is the space of independent variables $x = (x^i)$, $N = \mathbb{R}^m$ is the space of dependent variables $u = (u^j)$. These coordinates on $J^0(W, N) = W \times N$ induce the canonical coordinates u^j_σ on the space of jets, $\sigma = (i_1, \dots, i_l)$ being multi-indices, $l \le k$, $1 \le i_s \le n$, $1 \le j \le m$, as follows: For the k-jet of a map $h : W \to N$ at the point a, $a_k = [h]^k_a$, we let $u^j_\sigma(a_k) = \frac{\partial^l h^j}{\partial x^{i_1} \cdots \partial x^{i_l}}(a)$. Let $\pi_k : J^k \to W$, $\pi_{k,k-1} : J^k \to J^{k-1}$ denote the natural projections.

This jet-space is equipped with the canonical Cartan distribution $\mathcal{C} = \mathrm{Ann}(\theta^j_\sigma : |\sigma| < k) \subset TJ^k$, where $\theta^j_\sigma = du^j_\sigma - \sum u^j_{\sigma+1_i} dx^i$ in canonical coordinates. The total derivatives $\mathcal{D}_{x^i} = \partial_{x^i} + \sum u^j_{\sigma+1_i} \partial_{u^j_\sigma}$ are vector fields on J^∞, but being truncated they can be considered as sections of the Cartan distribution.

The fiber $\pi^{-1}_{k,k-1}(\cdot)$ has standard identification with $S^k T^*W \otimes TN$ [14]. It is a subbundle of $\mathcal{C}$, and together with the truncated total derivatives it spans the Cartan distribution.

Consider a compatible overdetermined system of PDEs as a submanifold in the space of jets $\mathcal{E} \subset J^k(W, N)$ (assuming $\mathcal{E}$ to be of pure order k is not crucial), which is regular with respect to all projections. Its symbol is the subbundle $g_k = \mathrm{Ker}(d\pi_{k,k-1} : T\mathcal{E} \to TJ^{k-1}) \subset S^k T^*W \otimes TN$.

Prolongations of the equation $\mathcal{E}_s \subset J^s$, $s > k$, are defined as the zero loci of the differential corollaries of the PDEs defining $\mathcal{E}$, and we assume the projection maps $\pi_{s,s-1} : \mathcal{E}_s \to \mathcal{E}_{s-1}$ are regular and submersive (this involves vanishing of the compatibility conditions, and so constitutes the formal integrability assumption). Its symbol $g_s = \mathrm{Ker}(d\pi_{s,s-1} : T\mathcal{E}_s \to T\mathcal{E}_{s-1})$ can be calculated algebraically as $g_s = (g_k \otimes S^{s-k}W^*) \cap (S^s W^* \otimes N)$.

As mentioned in the Introduction, the condition $\omega = 1$ translates to $\dim g_s = 1$ for large $s \ge k$. This happens from the level $\mathcal{E}$ becomes involutive (see the discussion about relation of this with compatibility in [16, 17]), and without loss of generality we will assume in this section that this is true for $s = k$.

Then by the standard arguments from formal theory of differential equations [23], we conclude that the symbol g_k is generated by $p^k \otimes v$ for the (unique up to scale) characteristic covector $p \in T^*W$ and some $v \in TN$ (if the characteristic variety consists of one covector it is automatically real, so we can skip the traditional complexification).

The induced Cartan distribution on the equation $\mathcal{C}_\mathcal{E} = \mathcal{C} \cap T\mathcal{E}$ has rank $n+1$. Indeed, it is generated by the vertical vector $p^k \otimes v$ and n total derivatives $\mathcal{D}_{x^i}$ restricted to the equation, which are defined mod g_k. Denoting the (non-canonical) horizontal space by H we get

$$\mathcal{C}_\mathcal{E} = H \oplus g_k.$$

Lemma 1.1. *Let $p \in PT^*W$ be the (unique) characteristic covector. There is a unique $(n-1)$-dimensional subdistribution $\Pi \subset \mathcal{C}_\mathcal{E}$ such that $d\pi_k(\Pi) = p^\perp \subset TW$ and Π consists of Cauchy characteristics of $\mathcal{C}_\mathcal{E}$.*

Proof. Let H be some choice of horizontal space, $\Pi \subset H$ the lift of $p^\perp$ and η a vertical vector field (section of g_k).

The Lie bracket of sections of $\mathcal{C}_\mathcal{E}$ induces the pointwise bracket on the vectors from $\mathcal{C}_\mathcal{E}$ with values in the normal bundle. Its restriction to the horizontal and vertical vectors is the map $H \otimes g_k \to T\mathcal{E}/\mathcal{C}_\mathcal{E}$.

With identification $H \simeq TW$ this latter is the restriction of the natural pairing $TW \otimes S^kT^*W \otimes TN \to S^{k-1}T^*W \otimes TN$, see [17]. Thus the bracket-map is $H \otimes g_k \to \nu_{k-1} = \langle p^{k-1} \otimes v\rangle \subset S^{k-1}T^*W \otimes TN$, and it follows that $[\xi, \eta] = 0$ for all $\xi \in \Pi$.

Next consider the restriction of the bracket to two horizontal fields $\Lambda^2 H \to \nu_{k-1}$. Since the total derivative operators commute, and the sections of H can be taken as truncated total derivatives modulo g_k, then two vector fields from $\Gamma(\Pi)$ commute modulo $\mathcal{C}_\mathcal{E}$. This implies that $[\xi, \xi'] = 0\ \forall\, \xi, \xi' \in \Pi$.

It remains to choose an additional vector $\zeta \in H \backslash \Pi$ and consider the induced bracket $\tau : \langle\zeta\rangle \otimes \Pi \to \nu_{k-1}$. It can be nontrivial since H is defined up to g_k and $(\zeta, p^k \otimes \theta) \mapsto k\, p^{k-1} p(\zeta) \otimes \theta \neq 0$.

Let us change $H = \langle\zeta\rangle \oplus \Pi$ by modifying Π as the graph of the map $-\tau(\zeta, \cdot) \in \Pi^* \otimes \nu_{k-1} \simeq \Pi^* \otimes g_k$, where we use identification $[\zeta, \cdot] : g_k \xrightarrow{\sim} \nu_{k-1}$. Then the new space Π is still involutive with respect to the induced bracket and it commutes with both ζ and η mod $\mathcal{C}_\mathcal{E}$. This means that the sections of Π are Cauchy characteristics.

Uniqueness of Π follows from the fact that the above (bracket) pairing $\langle\zeta\rangle \otimes g_k \to \nu_{k-1}$ is nonzero. □

Remark 1.2. For $n = 2$ characteristic vectors are dual to characteristic covectors. It is not however true that the former can be lifted to Cauchy characteristics of $\mathcal{C}_\mathcal{E}$. This is peculiarity of the case $\omega = 1$.

1.2. *Reduction to rank 2 distributions*

Due to Lemma 1.1 internal geometry of the distribution $\mathcal{C}_\mathcal{E}$ is equivalent to that of the rank 2 distribution $\mathcal{C}_\mathcal{E}/\Pi$. This implies Sophus Lie theorem.

Proof of Theorem A. Consider the pair $(\mathcal{E}, \mathcal{C}_{\mathcal{E}})$. Solutions of the system are n-dimensional integral submanifolds of the distribution, whose projection to the base are submersive.

It follows from the proof of Lemma 1.1 that an n-dimensional subspace of $\mathcal{C}_{\mathcal{E}}$ is involutive with respect to the (bracket) pairing $\Lambda^2\mathcal{C}_{\mathcal{E}} \to \nu_{k-1}$ if and only if it contains Π. In other words, a solution must be tangent to Π, and also it must be transversal to g_k (this latter requirement can be omitted for generalized solutions).

It is the standard fact, that the sub-distribution Π generated by Cauchy characteristics is integrable and shifts along it are symmetries for $\mathcal{C}_{\mathcal{E}}$. Taking the (local) quotient we arrive to the manifold $M = \mathcal{E}/\Pi$ (quotient by the leaves) equipped with a rank 2 distribution $\Delta = \mathcal{C}_{\mathcal{E}}/\Pi$ without characteristics.

Maximal integral manifolds of such a distribution are integral curves (it follows from the proof of Lemma 1.1 that the distribution $\mathcal{C}_{\mathcal{E}}$ is not integrable, so Δ is non-integrable as well), which can be found by solving underdetermined ODEs.

The space of integral curves of Δ is locally parametrized by one arbitrary function of one variable. Indeed, we can add an arbitrary constraint determining the underdetermined equation for the curves. For instance, this is given by a choice of a curve γ in the image of any submersion $\rho : M \to \mathbb{R}^2$ with fibers transversal to Δ, because $d\rho^{-1}(T\gamma) \cap \Delta$ is a line field in $\rho^{-1}(\gamma)$.

The inverse of the quotient map $\mathcal{E} \to M$ sends any of the integral curves of Δ to an n-dimensional integral surface, which is a generalized solution of $\mathcal{E}$ (and classical solutions are dense among the local generalized solutions). □

Note that this proof, as well as the arguments from the previous subsection, uses integration of ODE systems twice: first to solve the Frobenius system, corresponding to Cauchy characteristics Π, and then to find the integral curves of $\Delta = \mathcal{C}_{\mathcal{E}}/\Pi$.

The latter integration can be split in turn into integration of the bracket-closure of the distribution $\Delta_\infty = \Delta + [\Delta, \Delta] + \cdots$, which is Frobenius and then integrating Δ in the leaves.

In the first case the order of the system is $\dim \mathcal{E} - (n-1) = \operatorname{codim} \Pi$. In the second it is split into an ODE of order equal to the number of first integrals for Δ in M $(= \operatorname{codim} \Delta_\infty)$ and an ODE of order $\dim \Delta_\infty - 2$.

Remark 1.3. The shift along Cauchy characteristic is a characteristic symmetry for $\mathcal{E}$ (its flow is trivial on the space of solutions). The quotient of all symmetries by these is the algebra of non-characteristic symmetries. It is isomorphic to the Lie algebra $\operatorname{Sym}(M, \Delta)$.

1.3. *Constructive integration methods*

A theorem of Sophus Lie states that ODEs with a transitive solvable Lie group of symmetries are integrable in quadratures. This is equivalent to the claim that if a holonomic distribution[c] Δ on a manifold M admits a solvable symmetry Lie group of complimentary dimension with orbits transversal to it, then the integral leaves of Δ can be expressed in quadratures [19].

[c]This means it satisfies the Frobenius condition $[\Gamma(\Delta), \Gamma(\Delta)] \subset \Gamma(\Delta)$.

We extend this theorem to non-holonomic distributions Δ. We assume at first the distributions are completely non-holonomic, i.e. the bracket closure Δ_∞ equals TM and so Δ has no first integrals. Generic such distributions have no integral surfaces, and integral curves (which always exist) are the maximal integral manifolds.

Theorem 1.4. *Let Δ be a completely non-holonomic distribution of rank r on a manifold M^m. Suppose a solvable Lie group G of dimension $m-r$ acts by symmetries with orbits everywhere transversal to Δ. Then local integral curves of Δ can be found by quadratures.*

Proof. Denote by $\pi : M \to L^r = M/G$ the local quotient by the orbits (the space of G-invariants[d]). Notice that π_* maps Δ onto TL.

Choose a curve $\gamma \subset L$ and restrict the distribution Δ to $\pi^{-1}(\gamma)$. This is a line field and G acts transitively by symmetries on $\pi^{-1}(\gamma)$. By the classical Sophus Lie theorem the integral curves of this line field can be found by quadratures. Thus these restricted integral curves are parametrized by $m-r$ integration constants in $\pi^{-1}(\gamma)$, while the curves $\gamma \subset L$ are parametrized by $r-1$ function of 1 variable. The integral curves of Δ in M are given through these by quadratures. □

In particular, for our case $r = 2$ we get dependence on one function of one variable. Thus for general class $\omega = 1$ compatible PDE system we need three solvable Lie groups to integrate it in quadratures: one group G_1 of dimension equal to corank of the characteristic space Π to perform the reduction $(\mathcal{E}, \mathcal{C}_\mathcal{E}) \to (M, \Delta)$, the second group G_2 of dimension equal to corank of Δ_∞ in M, and finally the third group G_3 of dimension $\operatorname{rank}(\Delta_\infty) - \operatorname{rank}(\Delta)$ (all actions should be transversal).

Remark 1.5. A more general result is this: Consider a solvable Lie group G acting as transversal symmetries of Δ in M. Denote by $\bar{\Delta}$ in $\bar{M}$ the quotient distribution. Then integral curves of Δ can be found from integral curves of $\bar{\Delta}$ by quadratures. The number of involved integrals is the dimension of G, while the maximal number of iterated integrals is equal to the length of the derived series of G.

Let us consider an example from [25] of a Monge equation $\mathcal{E}$ on $y = y(x)$, $z = z(x)$ with three-dimensional solvable symmetry group:

$$z' = z^2 + \psi(z) + (y'' + y)^2. \tag{2}$$

The Lie algebra $\mathfrak{g} = \operatorname{Sym}(\mathcal{E})$ is generated by the (prolongations of) vector fields $\partial_x, \cos x\, \partial_y, \sin x\, \partial_y$ on $J^0(\mathbb{R}, \mathbb{R}^2) = \mathbb{R}^3(x, y, z)$.

If one (naively) substitutes $y = h(x)$, then $z(x)$ satisfies a Riccati equation, and so its solution cannot be found by quadratures (a similar problem occurs for general $\omega = 1$ class PDEs, so general reduction to ODEs from Theorem A does not necessarily yield an exact solution).

The correct approach of Theorem 1.4 is to consider the quotient, i.e. to pass to the space of G-invariants $\mathbb{R}^2(z, y'' + y)$. A curve in this space is given by an equation $y'' + y = f(z)$.

[d]Passage to this space is the only place where we have to use the Lie group action, at others we can relax the assumption to the Lie algebra action.

Substituting this back into (2) we find the autonomous first order equation

$$z' = z^2 + \psi(z) + f(z)^2,$$

which is easily integrable in quadratures, and afterwards the second order equation for y is integrated in quadratures too.

Remark 1.6. The previous naive argument uncovers as follows. The curve in the plane $L^2 = \mathbb{R}^2(z, y'' + y)$ is specified via a parameter x: $y'' + y = \tilde{h}(x)$ and $z(x)$ is given by $z' = z^2 + \psi(z) + \tilde{h}(x)^2$. Since the last equation cannot be integrated in quadratures, the initial data (a curve in L^2 and 3 integration constants) is not given explicitly, and so the result ceases to be given via an explicit formula.

2. Integrable Extensions and Generalized Symmetries

Integrable extensions or coverings [15] are mappings of PDEs $\mathcal{E} \to \bar{\mathcal{E}}$ such that solutions of $\mathcal{E}$ are obtained from those of $\bar{\mathcal{E}}$ by solving ODEs. For (underdetermined) ODEs a covering is a submersion $\pi : (M, \Delta) \to (\bar{M}, \bar{\Delta})$, i.e. $d_x\pi : \Delta_x \to \bar{\Delta}_{\pi(x)}$ is an isomorphism $\forall\, x \in M$.

We will call passage from (M, Δ) to $(\bar{M}, \bar{\Delta})$ integrable extension, and the inverse — integral deprolongation.

These coverings of systems of ODEs (or distributions) were studied in [3] as they are useful in solving the system. Indeed a sequence of integrable extensions can decompose a given system into a sequence of 1st order scalar ODEs.

It is easy to see that quotient by the Cauchy characteristic of class $\omega = 1$ systems, which is basic for Theorem A, commutes with integral deprolongation (since, by Sec. 3 and [3], it is deprolongation in the sense of distributions, i.e. quotient by the Cauchy characteristic for the derived distribution). Thus it is enough to study integrable extensions of rank 2 distributions. We will relate them to the symmetry approach of the previous section.

For instance, we can write the symmetry reduction of Theorem 1.4 and Remark 1.5 via integrable extensions. Let $\bar{\rho} : \bar{M} \to \mathbb{R}^2(x)$ be a submersion giving the independence condition (as in the proof of Theorem A, Sec. 1.2, the map $d_{\bar{a}}\bar{\rho}|_{\Delta}$ is an isomorphism at any point $\bar{a} \in \bar{M}$), i.e. with the coordinate u along fibers of $\bar{\rho}$ the integral curves of $\bar{\Delta}$ write $u = u(x)$. The equation for these integral curves has the form $F[x, u] = 0$, where the latter is an underdetermined ordinary (nonlinear) differential operator and both F and u are multi-dimensional.

Then provided that the Lie group G of symmetries has derived series $G = G_l \supset G_{l-1} \supset \cdots \supset G_0 = 0$ with Abelian quotients $G_i/G_{i+1} = V_i$ we can choose coordinates v_i on V_i and have the equation for integral curves of Δ in this form:

$$\begin{gathered} F[x, u] = 0, \quad v_1' = H_1(x, u), \quad v_2' = H_2(x, u, v_1), \ldots, \\ v_{l-1}' = H_{l-1}(x, u, v_1, \ldots, v_{l-2}), \quad v_l' = H_l(x, u, v_1, \ldots, v_{l-1}). \end{gathered} \tag{3}$$

This is a *special integrable extension*, related to the symmetry methods. The most general form of an integrable extension is given by

$$\begin{gathered} F[x, u] = 0, \quad v_1' = H_1(x, u, v_1), \quad v_2' = H_2(x, u, v_1, v_2), \ldots, \\ v_{l-1}' = H_{l-1}(x, u, v_1, \ldots, v_{l-1}), \quad v_l' = H_l(x, u, v_1, \ldots, v_{l-1}, v_l). \end{gathered} \tag{4}$$

2.1. *Existence of integral deprolongations for* $(2,5)$ *distributions*

Due to existence of normal forms $(2,m)$ distributions have the structure of integrable extensions for $m<5$. This holds true also in the first nontrivial case $m=5$, where such distributions have moduli.

Theorem 2.1. *A regular* 2*-distribution* Δ *on a manifold* M^5 *admits local submersion onto a* 2*-distribution in a four-dimensional manifold* $(\bar{M}^4,\bar{\Delta})$.

The claim follows from (is equivalent to) a result due to Goursat.

Theorem 2.2 ([12, Sec. 76]). *A regular rank* 2 *distribution in a five-dimensional manifold can be locally represented as the canonical distribution of the Monge equation* $\mathcal{E}: v'=f(x,u,u',u'',v)$.

Indeed, the distribution of this equation $\mathcal{E}\subset J^{1,2}(\mathbb{R},\mathbb{R}^2)$ is

$$\Delta=\langle\partial_x+u_1\partial_u+u_2\partial_{u_1}+f\partial_v,\partial_{u_2}\rangle, \tag{5}$$

which has the structure of integrable extension over $J^2(\mathbb{R},\mathbb{R})$, equipped with the canonical Cartan distribution $\langle\partial_x+u_1\partial_u+u_2\partial_{u_1},\partial_{u_2}\rangle$; the projection $\pi:\mathcal{E}\to J^2$ is $(x,u,u_1,u_2,v)\mapsto(x,u,u_1,u_2)$.

For completeness we give an alternative proof of Goursat's theorem (using vector fields approach instead of EDS methods).

Define the *commutator of two distributions* $[\Delta,\Delta']$ by the formula: $\Gamma([\Delta,\Delta'])=[\Gamma(\Delta),\Gamma(\Delta')]$, where $\Gamma(\Delta)$ denotes the module of sections of the distribution Δ.

Proof of Theorem 2.2. We consider the general situation when $[\Delta,[\Delta,\Delta]]=TM$ (elsewise the distributions have normal forms and the statement follows).

Consider the maps $\Upsilon:\Gamma(\Delta)\times\Gamma(\Delta)\to\Gamma(\Lambda^4TM)$ and $\Theta_i:\Gamma(\Delta)\times\Gamma(\Delta)\to\Gamma(\Lambda^5TM)$ given by

$$\Upsilon(\zeta,\eta)=\zeta\wedge\eta\wedge[\zeta,\eta]\wedge[\zeta,[\zeta,\eta]],$$

$$\Theta_0(\zeta,\eta)=\Upsilon\wedge[\eta,[\zeta,\eta]],\quad \Theta_1(\zeta,\eta)=\Upsilon\wedge[\zeta,[\zeta,[\zeta,\eta]]],$$

$$\Theta_2(\zeta,\eta)=\Upsilon\wedge[\zeta,[\eta,[\zeta,\eta]]],\quad \Theta_3(\zeta,\eta)=\Upsilon\wedge[\eta,[\eta,[\zeta,\eta]]].$$

A change of frame $\tilde{\zeta}=a\zeta+b\eta$, $\tilde{\eta}=c\zeta+d\eta$ induces the changes:

$$\Theta_0(\tilde{\zeta},\tilde{\eta})=\delta^5\Theta_0(\zeta,\eta),\quad \delta=\begin{vmatrix}a & b\\ c & d\end{vmatrix}$$

$$\begin{aligned}\delta^{-4}\Theta_1(\tilde{\zeta},\tilde{\eta})&=a^3\,\Theta_1(\zeta,\eta)+a^2b\,(2\Theta_2(\zeta,\eta)+\Theta_3(\eta,\zeta))\\&\quad+ab^2\,(2\Theta_2(\eta,\zeta)+\Theta_3(\zeta,\eta))+b^3\,\Theta_1(\eta,\zeta)+\sigma\Theta_0(\eta,\zeta).\end{aligned}$$

where $\sigma=a\cdot(a\eta+b\zeta)(b)-b\cdot(a\eta+b\zeta)(a)$. This implies the existence of a solution $\frac{a}{b}\in C^\infty(M,\mathbb{R}P^1)$ to $\Theta_1(\zeta,\eta)=0$.

Let us straighten $\zeta=\partial_{u_2}$ in a local chart $\mathbb{R}^5\hookrightarrow M$, and denote the quotient by $\mathbb{R}^4=\mathbb{R}^5/\zeta$ (i.e. $u_2=\text{const}$). Then the distribution becomes a u_2-dependent vector field $\eta=\Delta/\zeta$ in $\mathbb{R}^4$. The Lie derivative L_ζ corresponds to the derivative by u_2, which we denote by the prime.

Condition $\Theta_1(\zeta,\eta) = 0$ reads $\eta \wedge \eta' \wedge \eta'' \wedge \eta''' = 0$, and we can assume the highest derivative is resolved:

$$\eta''' = a_2\eta'' + a_1\eta' + a_0\eta.$$

By reparametrization of time u_2 and scaling of η we can achieve $a_0 = a_1 = 0$ (in contrast the Laguerre–Forsyth canonical form). Then the equation is $\eta''' = a_2\eta''$, and the solution is $\eta = \xi_0 + u_2\xi_1 + f\partial_v$, where $f_{u_2u_2} \neq 0$, $\eta''\|\partial_v$ and ξ_0, ξ_1 are u_2-independent fields on $\mathbb{R}^3 = \mathbb{R}^4/\partial_v$.

Now in our general case the distribution $\langle\xi_0,\xi_1\rangle$ in $\mathbb{R}^3$ is contact, so in the Darboux coordinates $\xi_0 = \partial_x + u_1\partial_u$, $\xi_1 = \partial_{u_1}$. Thus we obtain local coordinates on M such that Δ has form (5). □

2.2. *Non-existence of integral deprolongations for* $(2,m)$ *distributions with* $m > 5$

Dimensional count: generic rank 2 distribution in M^m depends on $2(m-2) - m = m - 4$ functions of m variables (quotient of sections of $(2,m)$-Grassmanian by the pseudogroup of local diffeomorphisms), while integrable extension depends on one function of m variables (in both cases: and some number of functions of fewer variables). Thus for $m > 5$ there are obstructions to existence of the structure of integrable extension over a lower-dimensional manifold.

These obstructions are important relative differential invariants of the distribution. For example, in dimension six there are two relative invariants vanishing of which characterizes possibility to represent the distribution as the Monge equation $v' = f(x,u,u',u'',u''',v)$.

It is interesting to notice that for $m = 5$ we get seemingly determined system (the same functional dimension). And indeed, the one-dimensional distribution V has the property of deprolongation (the vertical distribution of the projection π) if and only if $[V,\Delta] \subset V + \Delta$. This can be written as four equations on four functions specifying V (these latter can be taken as the first integrals, but then the system has order 2; it is better to write V via a generating vector field $\partial_v + F_1\partial_x + F_2\partial_u + F_3\partial_p + F_4\partial_q$ and take the components F_i as the unknowns).

This 4×4 system is not however determined since every covector is characteristic (direct calculation or this observation: in the normal form from Sec. 2.1 the totality of integrable deprolongations has functional moduli — the general solution depends on a function of five variables).

Remark 2.3. It is also interesting to try to deprolong by rank 2 foliation, which indeed exists as a generic (2,4) distribution has Engel normal form, and so integrally deprolongs to the contact (2,3) distribution. The conditions for existence of such rank 2 distribution V are:

$$[V,V] \subset V, \quad [V,\Delta] \subset V + \Delta. \tag{6}$$

This is a system of four equations on three unknowns (1st integrals of V), but it is not overdetermined: again all covectors are characteristic!

Finally notice that even though a generic rank 2 distribution in $M = \mathbb{R}^5$ has locally the structure of an integrable extension of type (4) (with $l = 1$, $\dim V_1 = 1$) over the Engel

distribution in $\bar{M} = \mathbb{R}^4$, it does not have a form of special integrable extension of type (3). This can be again verified by dimensional calculus.

2.3. *Generalized symmetries*

A space $\mathcal{G}$ of vector fields is a Lie symmetry algebra of Δ if and only if

$$[\mathcal{G}, \mathcal{G}] \subset \mathcal{G}, \quad [\mathcal{G}, \Delta] \subset \Delta.$$

Remark 2.4. Let us recall that a collection of differential operators $\langle F_i \rangle$ form a symmetry algebra for the PDE system $\mathcal{E} = \langle H_j = 0 \rangle$ if and only if

$$\{F_\alpha, F_\beta\} = 0 \quad \mathrm{mod}\langle F_i \rangle, \quad \{F_\alpha, H_\beta\} = 0 \quad \mathrm{mod}\langle H_j \rangle,$$

where $\{,\}$ is the Jacobi bracket (we write the condition for simplicity in the case of scalar or square matrix equations), see [14].

If we are interested in compatibility of the systems $\langle F_i \rangle$ and $\langle H_j \rangle$, then the last condition changes to more general

$$\{F_\alpha, H_\beta\} = 0 \quad \mathrm{mod}\langle F_i, H_j \rangle,$$

see [17]. Such F are called generalized symmetries, conditional symmetries or auxiliary integrals.

Based on this remark we can treat distributions V satisfying condition (6) as generalized symmetries. It allows the following symmetry reduction: if $L \subset M$ is an integral curve of Δ, then in the union of V-leaves meeting L the integral curves of Δ can be found via lower-dimensional determined ODE. If, in addition, V is the orbit of a solvable Lie group action, the solutions can be found in quadratures. The generalized symmetries are more common than the classical ones.

Example 2.5. Consider the symmetries of the Engel distribution, which is the Cartan (higher contact) distribution on $J^2(\mathbb{R}, \mathbb{R}) = \mathbb{R}^4(x, y, y_1, y_2)$. In canonical coordinates it is $\Delta^2 = \langle \xi_1 = \partial_x + y_1\partial_y + y_2\partial_{y_1}, \xi_2 = \partial_{y_2} \rangle$.

By Lie–Bäcklund theorem the symmetries are lifts of contact fields on $J^1(\mathbb{R}, \mathbb{R})$, and so are defined by one function of three arguments.

The generalized symmetries $\eta = \partial_y + \lambda_1\partial_x + \lambda_2\partial_{y_1} + \lambda_3\partial_{y_2}$ (here unlike for symmetries we can normalize one of the coefficients by scaling) are defined by $[\xi_i, \eta] = 0 \bmod \Delta + \langle \eta \rangle$ which is equivalent to

$$\lambda_{2y_2} = \frac{y_1\lambda_2 - y_2}{y_1\lambda_1 - 1}\lambda_{1y_2}, \quad \lambda_3 = \frac{y_1\lambda_2 - y_2}{1 - y_1\lambda_1}\xi_1(\lambda_1) + \xi_1(\lambda_2) + \lambda_2\frac{\lambda_2 - y_2\lambda_1}{1 - y_1\lambda_1}.$$

So the generalized symmetries depend on one function of four arguments λ_1.

It is often the case that a system (distribution) has no symmetries, but it admits generalized symmetries that can (partially) integrate Δ.

In particular, if these generalized symmetries have the structure of special integrable extension (3) over $\mathbb{R}^2(x)$ (i.e. no underlying constraint $F[x, u] = 0$),[e] then the integral

[e]Notice that truncation of the formulae in (3) on the level i yields symmetries ∂_{v_i}, but they need not to extend when we increase i.

curves of Δ can be found in quadratures. This is a generalization of Theorem B, part (i), and in such a form it can be inverted.

Theorem 2.6. *If integral curves of Δ are given in quadratures, then this distribution integrally deprolongs and has the structure of special integrable extension.*

Proof. By the assumption the general form of the integral curve in proper coordinates is

$$x^i = \psi^i(t), \quad u^1 = \int \sum \Phi_i^1(\psi(t))(\psi^i(t))' dt,$$

$$u^2 = \int \sum \Phi_i^2(\psi(t), u^1)(\psi^i(t))' dt, \quad \text{etc.}$$

(linearity in ψ' under the integral encodes the fact that the family of curves is integral for a vector distribution).

In other words, the distribution is given through one-forms

$$\omega^j = du^j - \sum \Phi_i^j \, dx^i$$

over the curve $\gamma \subset \mathbb{R}^r(x)$ given by $x = \psi(t)$. Since Φ_i^j depends only on ψ and $u^1, \ldots, u^{j-1}$, this is equivalent to the claim. □

For instance, integral curves of a generic rank 2 distribution in $\mathbb{R}^5$ cannot be found in quadratures — such Δ does not have the structure of special integrable extension (3). Some cases, when the integral curves are given by quadratures, are discussed in [4, 12].

Corollary 2.7. *If a system $\mathcal{E}$ of class $\omega = 1$ is integrable in quadratures, then it has the structure of special integrable extension.*

The latter can be seen as a multi-dimensional analog of (3), but it is equivalent to the same property for the reduction along Cauchy characteristics — the rank 2 distribution Δ.

3. Integration of Class $\omega = 1$ Systems

In this section we split the totality of $\omega = 1$ systems into classes, and discuss transformations between them as a method of integration. $r = \dim H^{*,1}(\mathcal{E})$ will be the total amount of PDEs in the system.[f]

3.1. *Type and complexity*

We introduce the following rule for a choice of generators of the system $\mathcal{E}$ of class $\omega = 1$. Consider the orders of the PDEs in the system: $k_{\min} = k_1 \leq \cdots \leq k_r = k_{\max}$, which are taken with multiplicities $m_i = \{\#j : k_j = i\} = \dim H^{i-1,1}(\mathcal{E})$.

So the system $\mathcal{E}$ is given by m_{k_1} equations $F_{1,1}, \ldots, F_{1,m_1}$ of order k_1, $\ldots$, m_{k_s} equations $F_{s,1}, \ldots, F_{s,m_s}$ of orders $k_s = k_{\max}$ $(s = r - m_{k_r} + 1)$.

We write $\mathcal{E}$ symbolically as $\sum_{i=1}^r E_{k_i} = \sum m_i E_i$, and call the latter the type of $\mathcal{E}$. See [16] for the table of class $\omega = 1$ systems of order $k_{\max} \leq 5$ (this table works equally well for general nonlinear systems).

[f]Starting from this section we restrict to base dimension $n = 2$. Familiarity with the Spencer cohomology $H^{i,j}(\mathcal{E})$ [23] is not crucial.

Let g_i denote the symbols of $\mathcal{E}$. Starting from some jet-level t the dimensions of these subspaces stabilize: $\dim g_i = 1$ for $i \geq t$. This is equivalent to involutivity of the prolongation $\mathcal{E}^{(t-k_{\max})}$.

Definition 3.1. Complexity of $\mathcal{E}$ is the number $\varkappa = \sum_{i=0}^{\infty}(\dim g_i - 1)$.

This number measures the amount of Cauchy data needed to specify a solution. It gives a partial order on the totality of class $\omega = 1$ systems. All our reductions will decrease the order.

By definition all systems of class $\omega = 0$ (for ODEs the relevant complexity is the dimension of the solutions space) are taken to be of lower complexity than the systems of class $\omega = 1$.

Lemma 3.2. *Denote by $\hat{\mathcal{E}}$ the equation prolonged to the jet-level $t = \min\{i \colon \dim g_i = 1\}$, where it is involutive. Then* $\dim \hat{\mathcal{E}} = \varkappa + t + 3$.

Proof. Since the base is two-dimensional, we get $\dim \hat{\mathcal{E}} = 2 + \sum_{i=0}^{t} \dim g_i$, whence the claim. □

The Cartan distribution $\mathcal{C}_{\hat{\mathcal{E}}}$ of $\hat{\mathcal{E}}$ (by Sec. 1.1) has rank 3. According to Lemma 1.1 it contains a unique (up to scale) Cauchy characteristic field, the (local) quotient by which we denote (M, Δ). This Δ is a rank 2 distribution describing the internal geometry of $(\hat{\mathcal{E}}, \mathcal{C}_{\hat{\mathcal{E}}})$. By Lemma 3.2 the manifold M has dimension $\mu = \varkappa + t + 2$.

3.2. *Derived flags of a rank 2 distribution*

The strong derived flag of Δ is defined by $\nabla_1 = \Delta$, $\nabla_{i+1} = [\nabla_i, \nabla_i]$. Its growth vector is the finite sequence $(\dim \nabla_i)_{i=1}^{\tau}$, where τ is the stabilization level (in the regularity assumptions, we adopt, all the ranks are constant).

The weak derived flag is given by $\Delta_1 = \Delta$, $\Delta_{i+1} = [\Delta_i, \Delta]$. Notations $\partial^{i-1}\Delta = \Delta_i$ are also used. The following cases are possible.

I. The growth vector is $(2, 3, 4, \ldots)$. In this case by Cartan theorem [3, 5] the system can be deprolonged,[g] i.e. there exists another manifold $\bar{M}$ of dimension $\bar{\mu} = \mu - 1$ equipped with rank 2 distribution $\bar{\Delta}$ such that $\Delta = \mathbb{P}(\bar{\Delta})$ is the prolongation.

The symmetries of Δ are preserved under passage to $\bar{\Delta}$, and the solutions are mapped forward in such a way that to any solution of $\bar{\Delta}$ there corresponds a one-dimensional family of integral curves of Δ.

Thus passage to deprolongation is a nice reduction of the system, for which the complexity $\varkappa$ (it exists on both ODE and PDE levels) decreases. For linear class $\omega = 1$ systems this corresponds to the (generalized) Laplace transformation, see [16] and the next section.

II. The distribution Δ is not completely non-holonomic, i.e. $\nabla_\tau \neq TM$. In this case[h] there are $p = \mu - \operatorname{rank} \nabla_\tau$ first integrals $I_1, \ldots, I_p$ that pull-back to first integrals of the system $\mathcal{E}$. We can fix the values of I_j and reduce the complexity of the system.

[g]In [3] the growth vector of the weak derived flag was considered. However, when $\dim \Delta = 2$, this makes no difference at the first three elements of the sequence $(2, 3, x, \ldots)$, where $x = \dim \Delta_3 = \dim \nabla_3$ can be $3, 4, 5$.
[h]Again here it makes no difference if we consider weak or strong derived flag, only the length τ can change.

For linear systems existence of such integrals means that the sequence of Laplace transformations does not reduce $\mathcal{E}$ to E_1 but stops on a finite type (class $\omega = 0$) system [16]. For nonlinear systems the relative invariants that control existence of intermediate integrals can be calculated as generalized Laplace invariants of the linearization.

III. The general case: the distribution is totally non-holonomic and not deprolongable. Thus the growth vector is $(2, 3, 5, \dots, \mu)$.

To find integral curves of Δ one can use integrable extension idea of Sec. 2 to decrease the complexity. Of course, due to Sec. 2.2 a generic rank 2 distribution on a manifold of dimension > 5 has no integrable extensions, so only general Theorem 1.4 can be applied. But distributions with symmetries do have such extensions, as the symmetry reduction gives integral deprolongation. Thus search of integrable extension (generalized symmetries) is an integration method.

Recall also that one has to prolong the system to the level of involutivity in order to make quotient by Cauchy characteristic. The following calculation shows importance of this.

Example 3.3. Consider a compatible system $\mathcal{E} \subset J^3(M)$ of type $2E_3$: $u_{xxx} = F, u_{xyy} = G$, with $F, G \in C^\infty(J^2M)$). The system is not involutive on the level of 3rd jets, since it has nonzero Spencer cohomology $H^{3,2}(\mathcal{E}) = \mathbb{R}$ and the symbol is not stable: $\dim g_3 = 2$, $\dim g_{3+i} = 1$. Thus if we do not prolong the system, then the growth vector of the weak derived flag of the Cartan distribution $\mathcal{C}_\mathcal{E}$ is $(4, 7, 9, 10)$. In addition $\mathcal{C}_\mathcal{E}$ has no Cauchy characteristics, while its derived $\partial\mathcal{C}_\mathcal{E}$ has three Cauchy characteristics, so that the pattern is wrong.

The prolonged system $\mathcal{E}^{(1)} \subset J^4(M)$ is involutive and the reduction of Theorem A works — there is one Cauchy characteristic for the original distribution and one more for the derived.

Consider for instance the system with $F = \frac{1}{4}u^4_{yyy}$, $G = \frac{1}{2}u^2_{yyy}$. The weak derived flag of the reduced (by Cauchy characteristic) system has growth $(2, 3, 4, 5, 6, 7, 8, 9)$, while the strong growth vector is $(2, 3, 4, 6, 9)$. Thus there is one intermediate integral $u_{xxy} - \frac{1}{3}u^3_{yyy} = c$, and after deprolongation both growth vectors are $(2, 3, 5, 8)$ — the corresponding graded nilpotent Lie algebra [3, 26] is free truncated.

3.3. *Internal geometry of linear systems*

Let us briefly summarize the results of [16], where linear compatible PDE systems of class $\omega = 1$ were studied.

It was shown that such systems $\mathcal{E}$ (with dependent variable u) can be integrated via generalized Laplace transformation, which is a first order differential operator $L : u \mapsto v = Xu$, with X having the same symbol as the characteristic vector field.

Denote the system we obtain on the variable v by $\tilde{\mathcal{E}}$. It is also linear and compatible. Denote the inverse operator by $L^{-1} : v \mapsto u$. As proven in [16] only three different situations are possible:

(1) $\tilde{\mathcal{E}}$ has class $\omega = 1$ and L^{-1} is a differential operator.
(2) $\tilde{\mathcal{E}}$ has class $\omega = 1$ but L^{-1} is given by a finite type system.
(3) $\tilde{\mathcal{E}}$ has class $\omega = 0$ and L^{-1} is an integral operator.

Case (1) is generic. If the itinerary of the transformations for $\mathcal{E}$ meets only such equations, then a sequence of Laplace transformations provides complete integration of the PDE system $\mathcal{E}$.

Moreover, under generalized Laplace transformation the complexity strictly decreases. Generically it decreases only by 1: $\varkappa \mapsto \varkappa - 1$.

These results were obtained using the external geometry of $\mathcal{E}$. Let us reformulate them in the internal language.

Proposition 3.4. *A generalized Laplace transformation for $\omega = 1$ linear systems is composed from the following maps in subsequent stages: some number of prolongations, a diffeomorphism, some number of deprolongations. For an involutive system only two last steps are required.*

Proof. Indeed, from internal viewpoint the rank 2 distribution Δ, obtained from $\mathcal{E}$ via reduction by Cauchy characteristic, is a Goursat distribution (or Goursat in the leaves of the first integrals if the distribution is not totally non-holonomic). Since the Goursat distribution has the canonical normal form (see [18, 22], we neglect singularities) the claim follows. □

Let us show how this works. We start with generic linear $3E_3$ of class $\omega = 1$. Then in three Laplace transformations it becomes equation of type E_1 (we refer to [16] for particular examples). We indicate the growth vector consisting of ascending by one integers, and indicate the internal coordinates on the equation: p, q, r, s, t are the classical notations for the 1st and 2nd derivatives, and ϱ is one of the 3rd derivatives).

$$
\begin{tikzcd}
\mathcal{E} = 3E_3 \arrow[r, squiggly] & \tilde{\mathcal{E}} = E_2 + E_3 \arrow[r, squiggly] & \breve{\mathcal{E}} = 2E_2 \arrow[r, squiggly] & \bar{\mathcal{E}} = E_1 \\
\boxed{\begin{matrix} x, y, u, p, q \\ r, s, t, \varrho \end{matrix}} \arrow[r] & \boxed{\begin{matrix} x, y, u, p \\ q, r, t, \varrho \end{matrix}} \arrow[r] & \boxed{\begin{matrix} x, y, u \\ p, q, t \end{matrix}} \arrow[r] & \boxed{\begin{matrix} x, y \\ u, q \end{matrix}} \\
T\mathcal{E} \supset \Delta & T\tilde{\mathcal{E}} \supset \tilde{\Delta} & T\breve{\mathcal{E}} \supset \breve{\Delta} & T\bar{\mathcal{E}} \supset \bar{\Delta} \\
(3,4,\ldots,9) & (3,4,\ldots,8) & (3,4,\ldots,6) & (3,4) \\
 & & \cdots & \cdots \\
 & T\tilde{\mathcal{E}}^{(1)} \supset \tilde{\Delta}^{(1)} \arrow[uu] \arrow[ur] & T\breve{\mathcal{E}}^{(2)} \supset \breve{\Delta}^{(2)} \arrow[uu] \arrow[ur] & T\bar{\mathcal{E}}^{(2)} \supset \bar{\Delta}^{(2)} \arrow[uu] \arrow[ur] \\
 & (3,4,\ldots,9) & (3,4,\ldots,8) & (3,4,\ldots,6)
\end{tikzcd}
$$

The first transformation is a diffeomorphism followed by a deprolongation, the next one is a diffeomorphism followed by two deprolongations and the third one is of the same kind.

If we choose non-generic $3E_3$, then the route could be $3E_3 \rightsquigarrow 2E_2 \rightsquigarrow E_1$, so that the first stage contains more deprolongations.

Starting with $2E_3$ one needs to prolong once to follow the scheme.

Remark 3.5. Now we can explain decrease of complexity $\varkappa$ via internal geometry. Since $\varkappa = \dim \hat{\mathcal{E}} - t - 3$, where t is the order of involutivity, we see that $\varkappa$ is defined correctly even if we prolong above the involutivity level (increase $\dim \mathcal{E}$ and t equally). The diffeomorphism

in the above proposition does not change the dimension, but it increases the order. Whence the claim.

3.4. *Closed form of the general solution*

By the results of Sec. 1.2 $\mathcal{E}$ has closed form of the general solution if and only if the same is true for the reduced underdetermined ODE (encoded by Δ).

For rank 2 distributions the criterion for closed form description of the integral curves (without constants) is known since Cartan [5, 18]: this is equivalent to Δ being Goursat, i.e. the canonical distribution $\mathcal{C}$ on the jet-space $J^d(\mathbb{R},\mathbb{R})$, $d = \mu - 2 = \varkappa + t$.

On the other hand we have demonstrated in [16] that systems that are internally equivalent to $(J^d(\mathbb{R},\mathbb{R}) \times \mathbb{R}, \mathcal{C} \times \mathbb{R})$ are internally linearizable and have no first integrals (Δ completely non-holonomic).

Intermediate integrals correspond to constants in the form of the general solution (1), and the respective normal form is Goursat–Frobenius, namely $(J^{d-m}(\mathbb{R},\mathbb{R}) \times \mathbb{R}^m \times \mathbb{R}, \mathcal{C} \times 0 \times \mathbb{R})$, where $m = \operatorname{codim}(\Delta_\infty)$ is the codimension of the bracket-closure of Δ.

Linearizability is not hampered by the additional constants. Thus we get the following statement equivalent to Theorem B(ii).

Theorem 3.6. *General solution of a compatible system $\mathcal{E}$ of class $\omega = 1$ and complexity $\varkappa$ can be expressed in a closed form via a function f, its $q \le \varkappa$ derivatives and $\varkappa - q$ constants if and only if $\mathcal{E}$ is linearizable by an internal transformation.*

Proof. Possibility to express solutions of a linear compatible systems of class $\omega = 1$ in closed form is proved in [16], so we need only to demonstrate that $q + m = \varkappa$, where m is the amount of the first integrals (constants).

The amount of derivatives to express all internal coordinates u_σ, $|\sigma| \le t$, on $\mathcal{E}$ is $d - m$. However the derivatives u_σ are obtained from (x, y, u) via $|\sigma|$ differentiations, so u shall be expressed in $d - m - t$ derivatives of f only, and this number equals to $\varkappa - m$. □

3.5. *Transformations of nonlinear systems*

Let us discuss some features of the transformations theory in the nonlinear case.

I. Quasi-linear systems allow some deprolongations, but generic pure order k systems (kE_k) have none — after quotient by Cauchy characteristics the growth vector is $(2, 3, 5, \ldots)$. However if $\mathcal{E}$ is involutive ($t = k_{\max}$) with different orders ($k_{\min} < k_{\max}$), then we claim: *The top equations are quasi-linear, and this implies the existence of at least one deprolongation.*

Indeed, provided that the characteristic is $\partial_x - \lambda\,\partial_y$, where λ is a function on the jets of order $k_{\min}$, the top derivatives on the level $k = k_{\max}$ must satisfy $u_{i,k-i} = \lambda^i u_{0,k}$ (this is due to the fact that ξ is the characteristic for all PDEs of $\mathcal{E}$). Thus the PDE of order $k_{\max}$ in $\mathcal{E}$ can be chosen linear in top-derivatives.

Consequently $\mathcal{D}_x - \lambda\mathcal{D}_y + \rho\partial_{u_{0,k}}$ is the Cauchy characteristic of $\mathcal{C}_\mathcal{E}$ for some function ρ on k-jets, and the two other generators of $\mathcal{C}_\mathcal{E}$ are $\mathcal{D}_y$ and $\partial_{u_{0,k}}$. A straightforward calculation yields that the latter field is a Cauchy characteristic for the derived distribution $\partial\mathcal{C}_\mathcal{E}$, so the system can be deprolonged.

II. Re-covering the first integrals is the same as for the linear systems. These restrictions introduce constants to the form of the general solution of $\mathcal{E}$, similar as deprolongations add derivatives to the form of the general solution.

III. If $\mathcal{E}$ allows the structure of $\bar{d}$-dimensional integrable extension $\mathcal{E} \to \bar{\mathcal{E}}$, then its solutions can be expressed via those of $\bar{\mathcal{E}}$ as $u = L_{\bar{d}}(\bar{u})$, where $L_{\bar{d}}$ is the resolution operator of a scalar ODE of order $\bar{d}$.

Provided a solvable Lie group of dimension $\bar{d}$ acts by transversal symmetries, the operator $L_{\bar{d}}$ can be expressed via the $\bar{d}$-multiple quadrature $\mathcal{D}_\tau^{-\bar{d}}$ ($\mathcal{D}_\tau^{-1} = \int \Box\, d\tau$ being quadrature by the parameter τ).

For instance, if Δ is maximally symmetric non-Goursat distribution, then its deprolongation is flat in the sense of Tanaka [26], and so has the structure of successive integral extensions over the rank 2 distribution in $\mathbb{R}^5$ with G_2 symmetry [3]. So we get the following theorem.

Theorem 3.7. *If $\mathcal{E}$ has reduction Δ, which deprolongs to a Tanaka-flat rank 2 distribution, then $\mathcal{E}$ can be solved in closed form and quadratures.*

Indeed, deprolongations can be interpreted as nonlinear Laplace transformations with differential inverses (this yields a closed form over the solutions of the reduced system $\bar{\Delta}$), while Tanaka flat rank 2 distributions $\bar{\Delta}$ project via integrable extensions to the Hilbert–Cartan equation [3] (so its integral curves are given in quadratures).

Thus we get the next easy case (after linearizable systems) of exactly solvable class $\omega = 1$ systems, which are reduced to symmetric Monge systems (these latter were classified in [3]).

4. Examples of Symmetric PDEs

4.1. *Model reductions to ODEs*

Consider the following compatible class $\omega = 1$ systems of the type kE_k:

$$2E_2 : u_{xx} = \lambda, \quad u_{xy} = \frac{\lambda^2}{2}, \quad u_{yy} = \frac{\lambda^3}{3};$$

$$3E_3 : u_{xxx} = \lambda, \quad u_{xxy} = \frac{\lambda^2}{2}, \quad u_{xyy} = \frac{\lambda^3}{3}, \quad u_{yyy} = \frac{\lambda^4}{4};$$

$$4E_4 : u_{xxxx} = \lambda, \quad u_{xxxy} = \frac{\lambda^2}{2}, \quad u_{xxyy} = \frac{\lambda^3}{3}, \quad u_{xyyy} = \frac{\lambda^4}{4}, \quad u_{yyyy} = \frac{\lambda^5}{5} \quad \text{etc.}$$

The reduced growth vectors[i] and the generators of the weak derived flags are the following:

$$(2,1,2) : (e_1, e_1', e_2, e_3, e_3'),$$

$$(2,1,2,3) : (e_1, e_1', e_2, e_3, e_3', e_4, e_4', e_4''),$$

$$(2,1,2,3,4) : (e_1, e_1', e_2, e_3, e_3', e_4, e_4', e_4'', e_5, e_5', e_5'', e_5''') \quad \text{etc.}$$

[i]These are the ones we have used in [3]: we pass from the usual growth vector $(n_1, n_2, n_3, \ldots)$ to $(n_1, n_2 - n_1, n_3 - n_2, \ldots)$.

Commutators are given by $[e_1, e_1'] = e_2$, $[e_1, e_2] = e_3$, $[e_1', e_2] = e_3'$, $[e_1, e_3] = e_4$, $[e_1, e_3'] = [e_1', e_3] = e_4'$, $[e_1', e_3'] = e_4''$ etc. (the commutators of e_2 and e_3 and others are zero) — these yields the structure of graded nilpotent Carnot algebra associated to the weak derived flag [3, 26].

The corresponding Monge underdetermined systems of ODEs are:

$$(2,1,2) : y' = \frac{1}{2}(z'')^2,$$

$$(2,1,2,3) : y'' = \frac{1}{2}(z''')^2, \quad u' = \frac{1}{3}(z''')^3,$$

$$(2,1,2,3,4) : y''' = \frac{1}{2}(z^{iv})^2, \quad u'' = \frac{1}{3}(z^{iv})^3, \quad v' = \frac{1}{4}(z^{iv})^4 \quad \text{etc.}$$

This follows from the explicit form of the generators. Indeed, let us demonstrate this, for simplicity, in the case $3E_3$.

$$e_1 = -\mathcal{D}_x = -\left(\partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{xy}\partial_{u_y} + \lambda\partial_{u_{xx}} + \frac{\lambda^2}{2}\partial_{u_{xy}} + \frac{\lambda^3}{3}\partial_{u_{yy}}\right),$$

$$e_1' = \partial_\lambda, \quad e_2 = \partial_{u_{xx}} + \lambda\partial_{u_{xy}} + \lambda^2\partial_{u_{yy}},$$

$$e_3 = \partial_{u_x} + \lambda\partial_{u_y}, \quad e_3' = \partial_{u_{xy}} + 2\lambda\partial_{u_{yy}},$$

$$e_4 = \partial_u, \; e_4' = \partial_{u_y}, \quad e_4'' = 2\partial_{u_{yy}}.$$

In this list we have omitted the generator of (rank 3) Cartan distribution $\mathcal{D}_y$ because the Cauchy characteristic equals $\mathcal{D}_y - \lambda\mathcal{D}_x$ and we need to quotient by it.

Now to perform the quotient one has either to pass to invariants, or to restrict to the transversal of the action. We choose the second approach: Change the notations $z := u$, $z' := u_x$, $z'' := u_{xx}$, $z''' := \lambda = u_{xxx}$, $y := u_y$, $y' := u_{xy}$, $u := u_{yy}$ and we are done.

Remark 4.1. To recover the solutions of $\mathcal{E}$ from the solutions of the ODE system, the first method must be used. For example, the first of the equations — system $2E_2$ from [4] — has the following invariants of shifts along Cauchy characteristic ($u_{xx} = \lambda$)

$$t = -\lambda, \quad w = -2u + 2yu_y + 2xu_x - x^2\lambda - xy\lambda^2 - \frac{1}{3}y^2\lambda^3,$$

$$z = u_y - u_x\lambda + \frac{1}{2}x\lambda^2 + \frac{1}{6}y\lambda^3, \quad z_1 = u_x - x\lambda - \frac{1}{2}y\lambda^2, \quad z_2 = x + y\lambda.$$

Thus we can express the general solution parametrically as

$$x = z''(t) + st, \quad y = s, \quad u = sz + z'z'' - \frac{1}{2}w - \frac{1}{2}t\,z''^2 - \frac{1}{2}t^2 s\,z'' - \frac{1}{6}t^3 s^2,$$

where the two functions $z = z(t), w = w(t)$ are related by the Hilbert–Cartan equation $w' = (z'')^2$.

Another interesting sequence of equations, considered in [10], is provided by

$$u_{xy} = \frac{2n}{x+y}\sqrt{u_x u_y}. \tag{7}$$

These PDEs are Darboux integrable with intermediate integrals of order $(n+1)$, see [1].

In order to integrate (7) let us linearize it via Goursat substitution [11] $p = \sqrt{u_x}$, $q = \sqrt{u_y}$, which leads to the system

$$p_y = \frac{n}{x+y} q, \quad q_x = \frac{n}{x+y} p. \tag{8}$$

Then $\mathcal{D}_y$-intermediate integral can be found as an ODE on q of the form ($\mathcal{D}_x$-intermediate integral is obtained similarly via an ODE on p)

$$L = \sum_{i=0}^{n} \frac{\alpha_i}{(x+y)^{n-i}} q_i = 0, \tag{9}$$

where $q_i = \mathcal{D}_y^i q$. The condition of intermediate integral, $L_x = 0$ on (8), is an overdetermined linear system on constants α_i. With normalization $\alpha_n = 1$ its unique solution is given by

$$\alpha_{n-i} = \frac{n^2(n-1)^2 \cdots (n-i+1)^2}{i!}.$$

Consider now the overdetermined compatible system (7)+(9). It has class $\omega = 1$ and type $E_2 + E_m$, $m = n+1$. The Cauchy characteristic is $\mathcal{D}_y + \varphi_m \partial_{q_m}$ for a properly chosen function φ_m.

Reduction along Cauchy characteristic (which again can be interpreted as intersection with the level of the $\mathcal{D}_x$-intermediate integral) yields a rank 2 distribution on a manifold $\bar{\mathcal{E}}$ of dimension $(2n+3)$. This system has n deprolongations, and so it reduces to a rank 2 distribution on $(n+3)$-dimensional manifold.

The symmetry analysis (done by I. Anderson) coupled together with unique symmetry model for rank 2 distributions [3, 9] implies that this rank 2 distribution corresponds to the Cartan distribution of the Monge equation

$$y' = (z^{(n)})^2.$$

In particular, we recover the result (known to Goursat) that the solutions of equation $(7)_{n=2}$ are expressed via the solutions of Hilbert–Cartan equation.

Remark 4.2. Thus we realize two boundary lines from the Zoo of types in [16] — the bottom and the diagonal — as the most symmetric PDEs in its class (both types of PDE and the reduction are fixed).

Here we refer to the group of contact symmetries (internal symmetry group of class $\omega = 1$ systems is infinite-dimensional), which turns out to be isomorphic to the internal symmetry group of the reduction by Cauchy characteristic in most cases. This Lie–Bäcklund type theorem will be discussed in details elsewhere.[j]

4.2. *Representation of solutions*

Linear class $\omega = 1$ compatible systems $\mathcal{E}$ are solvable in Moutard form, when u is expressed directly as a function of (x, y). In general, the closed form solution force all of the variables (x, y, u) to be expressed via parameters (ζ, τ), as in (1).

[j]Added in proof: arXiv: 1205.2914.

Quotient by the Cauchy characteristics and integral deprolongations (as well as the linearization map) usually do not preserve the fibers of the submersion $\mathcal{E} \to \mathbb{R}^2(x,y)$ giving the independence condition. This is the reason that many exactly solvable systems do not possess Moutard form. We illustrate this with two 2nd order examples.

Example 4.3. Consider the system $(7)+(9)_{n=1}$:

$$u_{xx} = -2\frac{u_x}{x+y}, \quad u_{xy} = 2\frac{\sqrt{u_x u_y}}{x+y}. \tag{10}$$

The reduced growth vector is $(3,1,1,1)$, so $\mathcal{E}$ is internally linearizable and is solvable via generalized Laplace transformations [16]. It is however non-Moutard.

To see this let us describe the general solution. Goursat substitution

$$u_x = p^2, \quad u_y = q^2, \tag{11}$$

linearizes the equation

$$p_x = -\frac{p}{x+y}, \quad q_x = \frac{p}{x+y}, \quad p_y = \frac{q}{x+y}. \tag{12}$$

Notice that this has vector growth $(3,1,1)$, and so one could suggest it serves as deprolongation of (10), but it does not. The reason (as shown below) is that it is impossible to deprolong preserving the base coordinates x, y. In fact, (10) is an integrable extension of (12) via (11).

The next step is to observe that the last equation of (12) can be used as a definition of q and the second equation is then the differential corollary of the first. Thus we can restrict to the equation E_1:

$$p_x = -\frac{p}{x+y} \tag{13}$$

with the reduced growth vector $(3,1)$. Inverse transformations are: integrable extension

$$u_x = p^2, \quad u_y = (x+y)^2 p_y^2$$

to the space $\mathbb{R}^5(x,y,u,u_x,u_y)$ and then prolongation to the original equation $\mathcal{E} = \mathbb{R}^6(x,y,u,u_x,u_y,u_{yy})$.

Let us return to the closed form of the solution. Solving (13) we get $p = \psi(y)/(x+y)$. Integrating the Frobenius system

$$u_x = \frac{\psi(y)^2}{(x+y)^2}, \quad u_y = \left(\psi'(y) - \frac{\psi(y)}{x+y}\right)^2,$$

yields the solution

$$u = \phi(y) - \frac{\psi(y)^2}{x+y} \quad \text{with } \phi'(y) = \psi'(y)^2. \tag{14}$$

This latter constraint is internally equivalent to the Engel distribution (or to $J^2(\mathbb{R},\mathbb{R})$) and the equivalence is given explicitly by

$$y = \sigma''(\tau), \quad \psi = \tau\sigma''(\tau) - \sigma'(\tau), \quad \psi' = \tau, \quad \phi = \tau^2\sigma''(\tau) - 2\tau\sigma'(\tau) + 2\sigma(\tau).$$

Thus alignment of the equation $\mathcal{E}$ (10) to the jet-space $J^{0,3}(\mathbb{R},\mathbb{R}^2)=\mathbb{R}^6(x,\tau,\sigma,\sigma_1,\sigma_2,\sigma_3)$ is the following:

$$x=x,\quad y=\sigma_2,\quad u=\tau^2\sigma_2-2\tau\sigma_1+2\sigma-\frac{(\tau\sigma_2-\sigma_1)^2}{x+\sigma_2},$$

$$u_x=\frac{(\tau\sigma_2-\sigma_1)^2}{(x+\sigma_2)^2},\quad u_y=\frac{(\tau x+\sigma_1)^2}{(x+\sigma_2)^2},$$

$$u_{yy}=2(\tau x+\sigma_1)\frac{((x+\sigma_2)^2-(\tau x+\sigma_1)\sigma_3)}{(x+\sigma_2)^3\sigma_3}.$$

This determines a diffeomorphism $J^{0,3}(\mathbb{R},\mathbb{R}^2)\to\mathcal{E}$. The inverse map $\mathcal{E}\to J^{0,3}(\mathbb{R},\mathbb{R}^2)$ is also given in differential-algebraic form

$$x=x,\quad \tau=\sqrt{u_x}+\sqrt{u_y},\quad \sigma=\frac{1}{2}(u-xu_x+yu_y-2x\sqrt{u_xu_y}),$$

$$\sigma_1=y\sqrt{u_y}-x\sqrt{u_x},\quad \sigma_2=y,\quad \sigma_3=\frac{2(x+y)\sqrt{u_y}}{2u_y+(x+y)u_{yy}}.$$

Thus Laplace transformation $\mathcal{E}=2E_2\mapsto E_1=\{\frac{\partial\sigma}{\partial\tau}=0\}$ has the form

$$x=x,\quad \tau=\sqrt{u_x}+\sqrt{u_y},\quad \sigma=\frac{1}{2}(u-xu_x+yu_y-2x\sqrt{u_xu_y})$$

(σ_1 corresponds to σ_x) and it decomposes internally into the composition of the diffeomorphism $\mathcal{E}\to J^{0,3}(\mathbb{R},\mathbb{R}^2)$ followed by the double deprolongation (we do not change the original Cauchy characteristic, which is the direction of the first factor in $J^{0,3}(\mathbb{R},\mathbb{R}^2)=\mathbb{R}\times J^3(\mathbb{R},\mathbb{R})$):

$$J^{0,3}(\mathbb{R},\mathbb{R}^2)\to J^{0,1}(\mathbb{R},\mathbb{R}^2)=\mathbb{R}^4(x,\tau,\sigma,\sigma_1).$$

Let us explain why $2E_2$ (10) cannot be transformed to E_1 by a Laplace transformation preserving the (x,y)-base (Moutard type).

We wish to find a relation on x,y,u,u_x,u_y excluding the above functions $\phi(y),\psi(y)$. But this is impossible since

$$\sqrt{u_x}=\frac{\psi(y)}{x+y},\quad \sqrt{u_y}=\psi'(y)-\frac{\psi(y)}{x+y}$$

and so u,u_x,u_y are algebraically independent.

Another approach is to show that the constraint $\phi'(y)=\psi'(y)^2$ in the form (14) is equivalent to the standard Engel distribution on $J^2(\mathbb{R},\mathbb{R})$ internally, but not externally. Indeed, no point transformation can map the above constraint to the equation $\phi'(y)=0$, since their point symmetry groups have dimensions 10 and ∞ respectively.

Example 4.4. Another interesting system, discussed in Sec. 4.1, is the Cartan involutive $2E_2$ model

$$u_{xx}=\lambda,\quad u_{xy}=\frac{1}{2}\lambda^2,\quad u_{yy}=\frac{1}{3}\lambda^3. \tag{15}$$

The reduced growth vector of its (five-dimensional) reduction $\bar{\mathcal{E}}$ is $(2,1,2)$, so Laplace transformation in the sense of linear theory does not exist. But $\mathcal{E}$ has the structure of

integrable extension over E_1, namely over the gas dynamics equation

$$v_y = vv_x. \tag{16}$$

Indeed, this latter is just the compatibility condition on the parameter $\lambda = v$ along the Cauchy characteristic. The transformation from (16) to (15) is a composition of a three-dimensional integrable extension and the deprolongation:

$$\mathbb{R}^6(x, y, u, u_x, u_y, u_{yy}) \overset{\text{prol}}{\dashrightarrow} \mathbb{R}^7(x, y, u, u_x, u_y, u_{yy}, u_{yyy}) \overset{\int\text{ext}}{\dashleftarrow} \mathbb{R}^4(x, y, v, v_y).$$

Since (16) is clearly not of Moutard type, this explains that (15) is not of Moutard type. Its solutions are though expressible via the solutions of Hilbert–Cartan equation.

This makes (15) internally equivalent to deprolongation of the equation $(7)+(9)_{n=2}$ of type $E_2 + E_3$ considered in Sec. 4.1. Another equation equivalent to (15) is the following compatible $E_2 + E_3$:

$$w_{xx} = 0, \quad w_{xyy}^2 + xw_{xy} - w_y = 0.$$

4.3. *Other symmetric models*

In [4] Cartan considers also submaximal symmetric systems $\mathcal{E}$:

$$2E_2 : u_{xx} = \lambda, \quad u_{xy} = \lambda^m, \quad u_{yy} = \frac{m^2}{2m-1}\lambda^{2m-1}. \tag{17}$$

Its contact symmetry algebra has dimension seven, and this is the next possible number after the maximal finite value 14 for $\dim \mathrm{Sym}(\mathcal{E})$.

The Cauchy characteristic is $\xi = \mathcal{D}_y - m\lambda^{m-1}\mathcal{D}_x$, and the reduced system $\bar{\mathcal{E}}$ can be found by restricting to the transversal $y = \text{const}$ to ξ. In other words, the rank 2 distribution of $\bar{\mathcal{E}}$ is given by its generators

$$\Delta = \langle \mathcal{D}_x = \partial_x + p\partial_u + r\partial_p + r^m\partial_q, \partial_r \rangle,$$

which after a change of coordinates is identical with the Cartan distribution of the Monge equation ($m \neq -1, \frac{1}{3}, \frac{2}{3}, 2$ — the exceptional cases corresponding to $\dim \mathrm{Sym}(\Delta) = 14$; $m \neq 0, 1$ — the exceptional cases corresponding to $\dim \mathrm{Sym}(\Delta) = \infty$)

$$w' = (v'')^m. \tag{18}$$

The higher analogs of (17) are straightforward:

$$3E_3 : u_{xxx} = \alpha, \quad u_{xxy} = \alpha^m, \quad u_{xyy} = \frac{m^2}{2m-1}\alpha^{2m-1}, \quad u_{yyy} = \frac{m^3}{3m-2}\alpha^{3m-2}.$$

Dimension of the contact symmetry algebra here is 10 (for generic m; this is readily checked with the help of *Differential Geometry* package of Maple), while dimension of the maximal symmetric nonlinear model (system $3E_3$ from Sec. 4.1) is 12.

The reduction by Cauchy characteristic is the following underdetermined ODE system:

$$z'' = (v''')^m, \quad w' = \frac{m^2}{2m-1}(v''')^{2m-1}.$$

It is a (three-dimensional) integrable extension of Monge equation (18).

Similarly we construct equations $4E_4$ etc. The submaximal symmetric class $\omega = 1$ system of type nE_n is

$$\left\{u_{n-i,i} = \frac{m^i}{im-i+1} u_{n,0}^{im-i+1} : 1 \le i \le n\right\}.$$

Its contact symmetry algebra has dimension $\frac{1}{2}n(n+1)+4$ (vs. the maximal dimension $\frac{1}{2}n(n+1)+6$ — the details on this calculation will be presented elsewhere[k]).

It is an interesting open problem what are the sub-maximal symmetric PDE systems of the type $E_2 + E_n$ and what are (sub-)maximal models for the other types from the Zoo of [16].

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