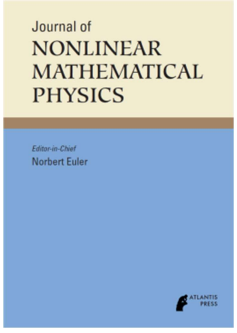


**Journal of Nonlinear Mathematical Physics**

**ISSN (Online): 1776-0852** **ISSN (Print): 1402-9251**
**Journal Home Page: https://www.atlantis-press.com/journals/jnmp**

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**To cite this article**: Luc Vinet, Guo-Fu Yu (2013) On the Discretization of the Coupled Integrable Dispersionless Equations, Journal of Nonlinear Mathematical Physics 20:1, 106–125, DOI: https://doi.org/10.1080/14029251.2013.792476

**To link to this article**: https://doi.org/10.1080/14029251.2013.792476

Published online: 04 January 2021

Journal of Nonlinear Mathematical Physics, Vol. 20, No. 1 (March 2013), 106-125

# On the Discretization of the Coupled Integrable Dispersionless Equations

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Received 4 August 2012

Accepted 18 January 2013

We study the integrable discretization of the coupled integrable dispersionless equations. Two semi-discrete version and one full-discrete version of the system are given via Hirota's bilinear method. Soliton solutions for the derived discrete systems are also presented.

*Keywords*: Soliton solution; Integrable discretization; Hirota bilinear method.

2000 Mathematics Subject Classification: 37K10, 35C05, 37K40

## 1. Introduction

The activity in the field of partial difference integrable equations based on bilinear form was initiated by Hirota [6–9] who proposed discrete analogues of the KdV, the Toda chain, the sine-Gordon (SG) equations, etc. (actually of almost all interesting soliton equations). Another powerful recipe in this aspect is the method based on Lax pair [1,28]. The bilinear integrable discretization method is based on the bilinear formalism and follows three steps. First a given differential equation is transformed into a bilinear equation by a dependent variable transformation. Second the bilinear equation is discretized using the gauge-invariance of the bilinear equation and the integrability of the discretized bilinear equation is determined through soliton solutions. Third the discrete bilinear equation is transformed into a discrete nonlinear equation in the ordinary form by an associated dependent variable transformation. Based on bilinear forms and determinant structure of solutions, Hirota's discretization method has been developed to construct discrete versions of $(2+1)$-dimensional sinh-Gordon equaiton [17], two dimensional Leznov lattice equaiton [18], Camassa-Holm equation [26], short pulse equation [5] and so on.

The already known fully discrete integrable systems are rare, only fully discrete KP (or Hirota-Miwa) equation and fully discrete BKP equation and so on. So it would be meaningful to use

Hirota's discretization method to search for more integrable fully discrete systems and research their integrable properties. These discrete versions may have substantial applications or their solutions have beautiful mathematical structures.

In this paper we consider the coupled integrable dispersionless (CID) equations [21]

$$q_t + 2rr_x = 0, \tag{1.1a}$$

$$r_{xt} - 2qr = 0, \tag{1.1b}$$

where $q$ and $r$ are both functions of $x$ and $t$, the subscripts denote derivatives. The inverse scattering scheme of eqs. (1.1) and its compatibility condition are given as

$$V_x = UV, \qquad V_t = WV, \tag{1.2}$$

and

$$U_t - W_x + [U,W] = 0, \tag{1.3}$$

where

$$U = -\mathrm{i}\lambda \begin{pmatrix} q_x & r_x \\ r_x & -q_x \end{pmatrix}, \tag{1.4}$$

$$W = \begin{pmatrix} 0 & -r \\ r & 0 \end{pmatrix} + \frac{\mathrm{i}}{2\lambda} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.5}$$

A generalized inverse scheme of the CID system was proposed from the group theoretical point of view in [19]. Eqs. (1.1) describe the current-fed string within an external magnetic field. The detailed physical application of the system (1.1) was stated in [20].

It should also be noted that the CID equations are related to the sine-Gordon equation

$$\phi_{xt} = 2q_0 \sin\phi \tag{1.6}$$

through the variable transformation [15]

$$q = q_0 \cos\phi, \quad r = \frac{1}{2}\phi_t. \tag{1.7}$$

In [22], a slightly general result between CID equations and the sine-Gordon equation has been presented. The integrability and hidden symmetry of the CID equations are analyzed in [33]. The $N$ soliton solutions and the infinitely many conservation laws are studied in [29,31]. Painlevé property and Bäcklund transformation are shown in [3,4]. In the following we present two remarks on the CID system (1.1).

**Remark 1.1.** Although several papers called the system the coupled integrable dispersionless equations, this system is not a dispersionless system.

**Remark 1.2.** This system was already recognized by many people before Konno and Oono. As Hirota and Tsujimoto claimed, this system is related to the sine-Gordon equation. This was already known since sine-Gordon equation was derived from this system in the AKNS formulation, see page 13, [2].

Since the physical application of the CID equations, it would be meaningful to consider its integrable discrete versions. The purpose of this paper is to construct discrete analogues of the

CID equations by Hirota's discretization approach. We present two gauge invariant semi-discrete CID equations and one fully discrete CID equations. The soliton solutions are deduced using the perturbation method.

The paper is organized as follows: In section 2, we give a $x$-discrete version of the CID equations and present the soliton solutions. In section 3, we derive a $t$-discrete version of CID equations. In section 4, a fully discrete CID version is worked out. In section 5, discrete analogues and Casorati determinant solutions are given by 2-reduction technique. The section 6 is devoted to conclusion and discussions.

## 2. Integrable semi-discrete version in x direction

Through the dependent variable transformation

$$q = 1 - (\ln F)_{xt}, \qquad r = G/F, \tag{2.1}$$

the CID equations (1.1) can be transformed into the bilinear form

$$D_t^2 F \cdot F - 2G^2 = 0, \tag{2.2}$$
$$(D_x D_t - 2) G \cdot F = 0. \tag{2.3}$$

It is easy to see that this bilinear form is just the 1-component form of the nonlinear coupled Klein-Gordon (sine-Gordon) equation [13]. Bilinear eqs. (2.2) and (2.3) are transformed into

$$D_t^2 F \cdot F = 2G^2, \tag{2.4}$$
$$(D_x D_t - 1) G \cdot F = 0, \tag{2.5}$$

through scaling transformation $x \to \frac{1}{2}x$. Eqs. (2.4)-(2.5) are bilinear forms of the coupled nonlinear Klein-Gordon equation [16]. Let

$$F = f^* f, \tag{2.6}$$
$$G = i D_t f^* \cdot f. \tag{2.7}$$

Then $F$ and $G$ satisfy the bilinear eqs. (2.4)-(2.5) provided that $f$ and $f^*$ satisfy

$$(D_x D_t^2 - D_t) f^* \cdot f = 0, \tag{2.8}$$
$$D_t^2 f^* \cdot f = 0, \tag{2.9}$$

which is a bilinear form of the sine-Gordon equation [16].

The coupled differential-difference CID equations are obtained by discretizing the spacial part of the bilinear eqs. (2.2)-(2.3),

$$D_x G \cdot F \to \frac{1}{\varepsilon}(G_{n+1} F_n - G_n F_{n+1}), \tag{2.10}$$

where $x = n\varepsilon$, $n$ being integers and $\varepsilon$ a spatial-interval. We obtain

$$D_t G_{n+1} \cdot F_n - D_t G_n \cdot F_{n+1} - 2\varepsilon G_n F_n = 0, \tag{2.11}$$
$$D_t^2 F_n \cdot F_n - 2G_n^2 = 0. \tag{2.12}$$

We postulate that the discretized bilinear forms are invariant under the gauge transformation:

$$F_n \to F_n \exp(q_0 n), \tag{2.13}$$
$$G_n \to G_n \exp(q_0 n). \tag{2.14}$$

We find a gauge invariant semi-discrete bilinear CID equations

$$D_t G_{n+1} \cdot F_n - D_t G_n \cdot F_{n+1} - \varepsilon(G_{n+1}F_n + G_n F_{n+1}) = 0, \tag{2.15}$$
$$D_t^2 F_n \cdot F_n - 2G_n^2 = 0. \tag{2.16}$$

Let $G_n = r_n F_n$, $q_n = (\ln F_n)_t$. Then eqs. (2.15)-(2.16) are transformed into

$$r_{n+1,t} - r_{n,t} + (q_{n+1} - q_n - \varepsilon)(r_{n+1} + r_n) = 0, \tag{2.17}$$
$$q_{n,t} = r_n^2. \tag{2.18}$$

When we take the continuum limit $\varepsilon \to 0$, (2.17)-(2.18) reduce to

$$r_{xt} - 2(1 - q_x)r = 0, \tag{2.19}$$
$$q_t = r^2. \tag{2.20}$$

Through transformation $1 - q_x \to q$, the above nonlinear system can be transformed into the CID eqs. (1.1). So we can regard (2.17)-(2.18) as a semi-discrete version of the CID equation. In the following discussion we take $\varepsilon = 1$ in eq. (2.15) for simplicity. According to the Hirota method, in order to construct soliton solutions, we expand $G$ and $F$ in series with a small parameter $\varepsilon$ as

$$F_n = 1 + \varepsilon^2 F_n^{(2)} + \varepsilon^4 F_n^{(4)} + \cdots + \varepsilon^{2k} F_n^{(2k)} + \cdots, \tag{2.21}$$
$$G_n = \varepsilon G_n^{(1)} + \varepsilon^3 G_n^{(3)} + \cdots + \varepsilon^{(2k+1)} G_n^{(2k+1)} + \cdots. \tag{2.22}$$

Substituting the expansion into the above bilinear eqs. (2.15)-(2.16) we find that there are only odd order terms of $\varepsilon$ in the first equation while only even order terms appear in the second one. Comparing the coefficients at each order of $\varepsilon$, we have

$$\varepsilon: \quad G_{n+1,t}^{(1)} - G_{n,t}^{(1)} - G_{n+1}^{(1)} - G_n^{(1)} = 0, \tag{2.23}$$
$$\varepsilon^2: \quad D_t^2(F_n^{(2)} \cdot 1 + 1 \cdot F_n^{(2)}) = 2(G_n^{(1)})^2, \tag{2.24}$$
$$\begin{aligned} \varepsilon^3: \quad & D_t[G_{n+1}^{(1)} \cdot F_n^{(2)} + G_{n+1}^{(3)} \cdot 1 - G_n^{(1)} \cdot F_{n+1}^{(2)} - G_n^{(3)} \cdot 1] \\ & = G_{n+1}^{(3)} + G_n^{(3)} + G_{n+1}^{(1)} F_n^{(2)} + G_n^{(1)} F_{n+1}^{(2)}, \end{aligned} \tag{2.25}$$
$$\varepsilon^4: \quad 2D_t^2 F_n^{(4)} \cdot 1 + D_t^2 F_n^{(2)} \cdot F_n^{(2)} = 4G_n^{(1)} G_n^{(3)}, \tag{2.26}$$
$$\cdots$$

By solving the above linear system, we obtain the one-soliton solution

$$G_n = \varepsilon \exp(\eta_1), \quad F_n = 1 + \varepsilon^2 \frac{1}{4\alpha_1^2} \exp(2\eta_1), \tag{2.27}$$

where $\eta_1 = \alpha_1 t + \beta_1 n + \gamma_1$ and $\alpha_1 = \coth(\beta_1/2)$. $\alpha_1, \gamma_1$ are arbitrary constants, and the two-soliton solution:

$$G_n = \varepsilon[\exp(\eta_1) + \exp(\eta_2)] + \varepsilon^3[A\exp(2\eta_1 + \eta_2) + B\exp(\eta_1 + 2\eta_2)], \tag{2.28}$$

$$F_n = 1 + \varepsilon^2[\frac{1}{4\alpha_1^2}\exp(2\eta_1) + \frac{1}{4\alpha_2^2}\exp(2\eta_2) + \frac{2}{(\alpha_1+\alpha_2)^2}\exp(\eta_1+\eta_2)] + \varepsilon^4 C\exp(2\eta_1 + 2\eta_2), \tag{2.29}$$

where

$$A = \frac{(\alpha_1-\alpha_2)^2}{4\alpha_1^2(\alpha_1+\alpha_2)^2}, \quad B = \frac{(\alpha_1-\alpha_2)^2}{4\alpha_2^2(\alpha_1+\alpha_2)^2}, \quad C = \frac{(\alpha_1-\alpha_2)^4}{16\alpha_1^2\alpha_2^2(\alpha_1+\alpha_2)^4}. \tag{2.30}$$

The corresponding plots are shown in Fig. 1. We can use the following compact expression for the above two-soliton solution,

$$\begin{aligned} F_n = {} & 1 + a(1,1^*)\exp(\eta_1+\eta_1^*) + a(1,2^*)\exp(\eta_1+\eta_2^*) \\ & + a(2,1^*)\exp(\eta_2+\eta_1^*) + a(2,2^*)\exp(\eta_2+\eta_2^*) \\ & + a(1,2,1^*,2^*)\exp(\eta_1+\eta_2+\eta_1^*+\eta_2^*), \end{aligned} \tag{2.31}$$

$$\begin{aligned} G_n = {} & \exp(\eta_1) + \exp(\eta_2) + a(1,2,1^*)\exp(\eta_1+\eta_2+\eta_1^*) \\ & + a(1,2,2^*)\exp(\eta_1+\eta_2+\eta_2^*), \end{aligned} \tag{2.32}$$

where $\eta_i^* = \eta_i, (i = 1,2)$ and

$$a(i,j) = (\alpha_i - \alpha_j)^2, \tag{2.33}$$

$$a(i,j^*) = \frac{1}{(\alpha_i+\alpha_j)^2}, \tag{2.34}$$

$$a(i^*,j^*) = (\alpha_i - \alpha_j)^2, \tag{2.35}$$

$$a(i,j,k^*) = a(i,j)a(i,k^*)a(j,k^*), \tag{2.36}$$

$$a(i,j,k^*,l^*) = a(i,j)a(i,k^*)a(i,l^*)a(j,k^*)a(j,l^*)a(k^*,l^*). \tag{2.37}$$

In the same way, we can construct the three-soliton solution,

$$
\begin{aligned}
G_n = &\exp(\eta_1)+\exp(\eta_2)+\exp(\eta_3)\\
&+a(1,2,1^*)\exp(2\eta_1+\eta_2)+a(1,3,1^*)\exp(2\eta_1+\eta_3)+a(2,3,2^*)\exp(2\eta_2+\eta_3)\\
&+a(1,2,2^*)\exp(\eta_1+2\eta_2)+a(1,3,3^*)\exp(\eta_1+2\eta_3)+a(2,3,3^*)\exp(\eta_2+2\eta_3)\\
&+[a(1,2,3^*)+a(1,3,2^*)+a(2,3,1^*)]\exp(\eta_1+\eta_2+\eta_3)\\
&+a(1,2,3,1^*2^*)\exp(2\eta_1+2\eta_2+\eta_3)+a(1,2,3,1^*3^*)\exp(2\eta_1+\eta_2+2\eta_3)\\
&+a(1,2,3,2^*,3^*)\exp(\eta_1+2\eta_2+2\eta_3), \qquad (2.38)\\
F_n = &1+a(1,1^*)\exp(2\eta_1)+a(2,2^*)\exp(2\eta_2)+a(3,3^*)\exp(2\eta_3)\\
&+[a(1,2^*)+a(2,1^*)]\exp(\eta_1+\eta_2)+[a(2,3^*)+a(3,2^*)]\exp(\eta_2+\eta_3)\\
&+[a(1,3^*)+a(3,1^*)]\exp(\eta_1+\eta_3)\\
&+a(1,2,1^*,2^*)\exp(2\eta_1+2\eta_2)+a(1,3,1^*,3^*)\exp(2\eta_1+2\eta_3)+a(2,3,2^*,3^*)\exp(2\eta_2+2\eta_3)\\
&+[a(1,2,1^*,3^*)+a(1,3,1^*,2^*)]\exp(2\eta_1+\eta_2+\eta_3)\\
&+[a(1,2,2^*,3^*)+a(2,3,1^*,2^*)]\exp(\eta_1+2\eta_2+\eta_3)\\
&+[a(1,3,2^*,3^*)+a(2,3,1^*,3^*)]\exp(\eta_1+\eta_2+2\eta_3)\\
&+a(1,2,3,1^*,2^*,3^*)\exp(2\eta_1+2\eta_2+2\eta_3), \qquad (2.39)
\end{aligned}
$$

where the coefficients are defined as (2.33)-(2.37) and

$$
\begin{aligned}
a(i,j,k,l^*,m^*,n^*) = &a(i,j)a(i,k)a(i,l^*)a(i,m^*)a(i,n^*)\\
&\times a(j,k)a(j,l^*)a(j,m^*)a(j,n^*)\\
&\times a(k,l^*)a(k,m^*)a(k,n^*)\\
&\times a(l^*,m^*)a(l^*,n^*)\\
&\times a(m^*,n^*). \qquad (2.40)
\end{aligned}
$$

The above expression of multi-soliton solutions suggest the exact $N$-soliton solutions of eqs. (2.15)-(2.16) in the following form

$$F_n = \sum_{\mu=0,1}^{(e)} \exp\left[\sum_{j=1}^{2N}\mu_j\eta_j + \sum_{1\le k<l}^{2N}\mu_k\mu_l A_{kl}\right], \qquad (2.41)$$

$$G_n = \sum_{\nu=0,1}^{(o)} \exp\left[\sum_{j=1}^{2N}\nu_j\eta_j + \sum_{1\le k<l}^{2N}\nu_k\nu_l A_{kl}\right], \qquad (2.42)$$

where

$$\eta_j = \eta_{N+j} = \alpha_j t + \beta_j n + \gamma_j, \quad \alpha_j = \coth(\beta_j/2), \quad j = 1,2,\cdots,N, \qquad (2.43)$$

$$\exp(A_{kl}) = (\alpha_k - \alpha_l)^2, \quad k < l = 2,3,\cdots,N \qquad (2.44)$$

$$\exp(A_{k,N+l}) = \frac{1}{(\alpha_k+\alpha_l)^2}, \quad k,l = 1,2,\cdots,N, \qquad (2.45)$$

$$\exp(A_{N+k,N+l}) = (\alpha_k - \alpha_l)^2, \quad k < l = 2,3,\cdots,N. \qquad (2.46)$$

Here $\alpha_j, \gamma_j$ are both real parameters relating respectively to the amplitude and phase of the $i$th soliton. The sum $\sum_{\mu=0,1}^{(e)}$ indicates the summation over all possible combinations of $\mu_1 = 0,1, \mu_2 =$

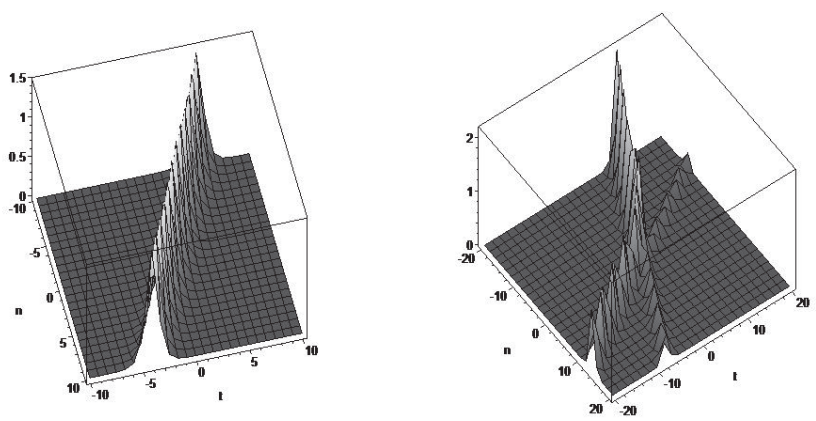
$0,1,\cdots,\mu_{2N}=0,1$, under the condition

$$\sum_{j=1}^{N}\mu_j=\sum_{j=1}^{N}\mu_{N+j}, \tag{2.47}$$

and $\sum_{\nu=0,1}^{(o)}$ indicates the summation over all possible combinations of $\nu_1=0,1,\nu_2=0,1,\cdots,\nu_{2N}=0,1$, under the condition

$$\sum_{j=1}^{N}\nu_j=1+\sum_{j=1}^{N}\nu_{N+j}. \tag{2.48}$$

The form of the $N$-soliton solution (2.41)-(2.42) is the same as that of the combined Schrödinger-mKdV equation in the Ref. [10]. The proof of the $N$-soliton solution here can be completed by induction and it is similar as that in [10]. One can check the details therein.




$(a)$ $(b)$

Fig. 1. soliton solution of $r$ ($\beta_1=1,\beta_2=2,\gamma_1=\gamma_2=0$): $(a)$ one-soliton, $(b)$ two-soliton

## 3. Integrable semi-discrete version in t direction

A coupled differential-difference CID equations are obtained by discretizing the time part of the bilinear eqs. (2.2)-(2.3),

$$D_tG\cdot F\to\frac{1}{\delta}(G_{m+1}F_m-G_mF_{m+1}), \tag{3.1}$$

$$D_t^2F\cdot F\to\frac{2}{\delta^2}\left(F_{m+1}F_{m-1}-(F_m)^2\right), \tag{3.2}$$

where $t=m\delta$, $m$ is an integer and $\delta$ a time-interval. We find a gauge invariant set of semi-discrete bilinear CID equations

$$D_xG_{m+1}\cdot F_m-D_xG_m\cdot F_{m+1}=\delta(F_mG_{m+1}+F_{m+1}G_m), \tag{3.3}$$

$$F_{m+1}F_{m-1}-F_m^2=\delta^2G_m^2. \tag{3.4}$$

Let $G_m = r_m F_m$, $u_m = \ln F_m$. Eqs. (3.3)-(3.4) are then transformed into the ordinary nonlinear form

$$r_{m+1,x} - r_{m,x} + (r_{m+1} + r_m)(u_{m+1,x} - u_{m,x}) = \delta(r_m + r_{m+1}), \tag{3.5}$$
$$e^{u_{m+1}+u_{m-1}-2u_m} - 1 = \delta^2 r_m^2. \tag{3.6}$$

When we take the continuum limit $\delta \to 0$, eqs. (3.5)-(3.6) reduce to

$$r_{xt} + 2ru_{xt} = 2r, \tag{3.7}$$
$$u_{tt} = r^2. \tag{3.8}$$

Performing the variable transformation $q = 1 - u_{xt}$ and differentiating eq. (3.8) with $x$, we obtain the original nonlinear eqs. (1.1). As before we take $\delta = 1$ for simplicity in the following. Using the perturbation method, we can construct soliton solutions of the semi-discrete CID eqs. (3.3)-(3.4). For the one-soliton solution we have

$$G_m = \exp(\xi_1), \quad F_m = 1 + \frac{(\alpha_1^2 - 1)^2}{16\alpha_1^2}\exp(2\xi_1), \tag{3.9}$$

where $\xi_i = \alpha_i x + \beta_i m + \gamma_i$ and $\alpha_i = \coth(\beta_i/2)$. $\alpha_i, \gamma_i$ are arbitrary constants. The two-soliton solution reads

$$G_m = \exp(\xi_1) + \exp(\xi_2) + A\exp(2\xi_1 + \xi_2) + B\exp(\xi_1 + 2\xi_2), \tag{3.10}$$
$$F_m = 1 + \frac{(\alpha_1^2 - 1)^2}{16\alpha_1^2}\exp(2\xi_1) + \frac{(\alpha_2^2 - 1)^2}{16\alpha_2^2}\exp(2\xi_2)$$
$$+ \frac{\alpha_1^2\alpha_2^2 - \alpha_1^2 - \alpha_2^2 + 1}{2(\alpha_1 + \alpha_2)^2}\exp(\xi_1 + \xi_2) + C\exp(2\xi_1 + 2\xi_2), \tag{3.11}$$

where

$$A = \frac{(\alpha_1^2 - 1)^2(\alpha_1 - \alpha_2)^2}{16\alpha_1^2(\alpha_1 + \alpha_2)^2}, \quad B = \frac{(\alpha_2^2 - 1)^2(\alpha_1 - \alpha_2)^2}{16\alpha_2^2(\alpha_1 + \alpha_2)^2},$$
$$C = \frac{(\alpha_1^2 - 1)^2(\alpha_2^2 - 1)^2(\alpha_1 - \alpha_2)^4}{256\alpha_1^2\alpha_2^2(\alpha_1 + \alpha_2)^4}. \tag{3.12}$$

Similar as the previous section, we can rewrite the two-soliton solution as the following compact form

$$G_m = \exp(\xi_1) + \exp(\xi_2) + b(1,2,1^*)\exp(2\xi_1 + \xi_2) + b(1,2,2^*)\exp(\xi_1 + 2\xi_2), \tag{3.13}$$
$$F_m = 1 + b(1,1^*)\exp(2\xi_1) + b(2,2^*)\exp(2\xi_2)$$
$$+ [b(1,2^*) + b(2,1^*)]\exp(\xi_1 + \xi_2) + b(1,2,1^*,2^*)\exp(2\xi_1 + 2\xi_2), \tag{3.14}$$

where

$$b(k,l) = \frac{4(\alpha_k - \alpha_l)^2}{\alpha_k^2\alpha_l^2 - \alpha_k^2 - \alpha_l^2 + 1}, \tag{3.15}$$

$$b(k,l^*) = \frac{\alpha_k^2\alpha_l^2 - \alpha_k^2 - \alpha_l^2 + 1}{4(\alpha_k + \alpha_l)^2}, \tag{3.16}$$

$$b(k^*,l^*) = \frac{4(\alpha_k - \alpha_l)^2}{\alpha_k^2\alpha_l^2 - \alpha_k^2 - \alpha_l^2 + 1}, \tag{3.17}$$

$$b(i,j,k^*) = b(i,j)b(i,k^*)b(j,k^*), \tag{3.18}$$

$$b(i,j,k^*,l^*) = b(i,j)b(i,k^*)b(i,l^*)b(j,k^*)b(j,l^*)b(k^*,l^*). \tag{3.19}$$

Three soliton solution is presented by the expression (2.38)-(2.39) with the coefficients $a(\cdots)$ substituted by $b(\cdots)$ which obey (3.15)-(3.19) and operation rule (2.37). Generally, the $N$-soliton solution is given by

$$F_m = \sum_{\mu=0,1}^{(e)} \exp\left[\sum_{j=1}^{2N} \mu_j\xi_j + \sum_{1\le k<l}^{2N} \mu_k\mu_l A_{kl}\right], \tag{3.20}$$

$$G_m = \sum_{\nu=0,1}^{(o)} \exp\left[\sum_{j=1}^{2N} \nu_j\xi_j + \sum_{1\le k<l}^{2N} \nu_k\nu_l A_{kl}\right], \tag{3.21}$$

where

$$\eta_j = \xi_{N+j} = \alpha_j x + \beta_j m + \gamma_j, \quad \exp(\beta_j) = \frac{\alpha_j + 1}{\alpha_j - 1}, \quad j = 1,2,\cdots,N, \tag{3.22}$$

$$\exp(A_{kl}) = \frac{4(\alpha_k - \alpha_l)^2}{\alpha_k^2\alpha_l^2 - \alpha_k^2 - \alpha_l^2 + 1}, k < l = 2,3,\cdots,N \tag{3.23}$$

$$\exp(A_{k,N+l}) = \frac{\alpha_k^2\alpha_l^2 - \alpha_k^2 - \alpha_l^2 + 1}{4(\alpha_k + \alpha_l)^2}, k,l = 1,2,\cdots,N, \tag{3.24}$$

$$\exp(A_{N+k,N+l}) = \frac{4(\alpha_k - \alpha_l)^2}{\alpha_k^2\alpha_l^2 - \alpha_k^2 - \alpha_l^2 + 1}, k < l = 2,3,\cdots,N. \tag{3.25}$$

Here $\alpha_j, \beta_j, \gamma_j$ are all real parameters. The summations $\sum_{\mu=0,1}^{(e)}$ and $\sum_{\nu=0,1}^{(o)}$ satisfy the condition (2.47) and (2.48) respectively. The proof of $N$-soliton solution is similar as that in [10].

## 4. Integrable fully discrete version of the CID equations

The fully discrete CID equations are obtained by discretizing the bilinear eqs. (2.2)-(2.3)

$$D_t^2 F \cdot F \to \frac{2}{\varepsilon^2}\left(F_n^{m+1}F_n^{m-1} - (F_n^m)^2\right), \tag{4.1}$$

$$D_x D_t G \cdot F \to \frac{1}{\delta\varepsilon}(G_{n+1}^{m+1}F_n^m - G_n^{m+1}F_{n+1}^m - G_{n+1}^m F_n^{m+1} + G_n^m F_{n+1}^{m+1}). \tag{4.2}$$

In the following, we also postulate that the discretized bilinear forms are invariant under the gauge transformation:

$$F_n^m \to F_n^m \exp(q_0 n + p_0 m), \tag{4.3}$$

$$G_n^m \to G_n^m \exp(q_0 n + p_0 m). \tag{4.4}$$

We rewrite $GF$ as

$$GF \to (1/4)\left(G_{n+1}^{m+1}F_n^m + G_n^m F_{n+1}^{m+1} + G_n^{m+1}F_{n+1}^m + G_{n+1}^m F_n^{m+1}\right), \tag{4.5}$$

and find the gauge invariant fully discrete CID equations

$$\begin{aligned} &G_{n+1}^{m+1}F_n^m - G_n^{m+1}F_{n+1}^m - G_{n+1}^m F_n^{m+1} + G_n^m F_{n+1}^{m+1} \\ &\quad = (\delta\varepsilon/2)\left(G_{n+1}^{m+1}F_n^m + G_n^m F_{n+1}^{m+1} + G_n^{m+1}F_{n+1}^m + G_{n+1}^m F_n^{m+1}\right), \end{aligned} \tag{4.6}$$

$$(1/\varepsilon^2)\left(F_n^{m+1}F_n^{m-1} - (F_n^m)^2\right) = G_n^m G_n^m. \tag{4.7}$$

We introduce the auxiliary variable $\Gamma_n^m$ defined by

$$\Gamma_n^m = \frac{F_n^{m+1}F_{n+1}^m}{F_n^m F_{n+1}^{m+1}}. \tag{4.8}$$

It is easy to check that $\Gamma_n^m$ satisfies the identity

$$\Gamma_n^m = \frac{F_n^{m+1}F_n^{m-1}}{(F_n^m)^2}\frac{(F_{n+1}^m)^2}{F_{n+1}^{m+1}F_{n+1}^{m-1}}\Gamma_n^{m-1}. \tag{4.9}$$

Let $G_n^m = r_n^m F_n^m$. The eqs. (4.6)-(4.7) are then transformed into the ordinary nonlinear form

$$r_{n+1}^{m+1} + r_n^m - (r_n^{m+1} + r_{n+1}^m)\Gamma_n^m = \frac{\delta\varepsilon}{2}\left[r_{n+1}^{m+1} + r_n^m + (r_n^{m+1} + r_{n+1}^m)\Gamma_n^{m+1}\right], \tag{4.10}$$

$$\Gamma_n^m = \frac{1+(\varepsilon r_n^m)^2}{1+(\varepsilon r_{n+1}^m)^2}\Gamma_n^{m-1}. \tag{4.11}$$

The fully discrete CID equations are the 1-component form of the coupled discrete Klein-Gordon (sine-Gordon) equation [13]. Let $\varepsilon = \delta = 1$. The one-soliton solution for fully discrete CID Eqs. (4.6)-(4.7) is given by

$$G_n^m = c p^m q^n, \quad F_n^m = 1 + \frac{p^{2m}q^{2n}}{(p-p^{-1})^2}, \tag{4.12}$$

where $p, q$ and $c$ are arbitrary constants and satisfy the dispersion relation $q = \frac{3p-1}{p-3}$. The two-soliton solution reads

$$\begin{aligned} F_n^m &= 1 + a(1,1^*)\exp(2\eta_1) + a(2,2^*)\exp(2\eta_2) \\ &\quad + [a(1,2^*) + a(2,1^*)]\exp(\eta_1+\eta_2) + a(1,2,1^*,2^*)\exp(2\eta_1+2\eta_2), \end{aligned} \tag{4.13}$$

$$G_n^m = \exp(\eta_1) + \exp(\eta_2) + a(1,2,1^*)\exp(2\eta_1+\eta_2) + a(1,2,2^*)\exp(\eta_1+2\eta_2), \tag{4.14}$$

where $\exp(\eta_i)$ and the coefficients are defined by

$$\exp(\eta_i) = c_i p_i^m q_i^n, \quad q_i = \frac{3p_i - 1}{p_i - 3}, \tag{4.15}$$

$$a(i,j) = \frac{(p_i - p_j)^2}{p_i p_j}, \tag{4.16}$$

$$a(i,j^*) = \frac{p_i p_j}{(p_i p_j - 1)^2}, \tag{4.17}$$

$$a(i^*,j^*) = \frac{(p_i - p_j)^2}{p_i p_j}. \tag{4.18}$$

It is easy to verify the three-soliton solutions for the fully discrete CID eqs. (4.6)-(4.7) in the form (2.38)-(2.39) with the coefficients (4.16)-(4.18). Similar as the discussion in the previous semi-discrete cases, we present here the $N$-soliton solutions

$$F_n^m = \sum_{\mu=0,1}^{(e)} \exp\left[\sum_{j=1}^{2N} \mu_j \eta_j + \sum_{1\le k<l}^{2N} \mu_k \mu_l A_{kl}\right], \tag{4.19}$$

$$G_n^m = \sum_{\nu=0,1}^{(o)} \exp\left[\sum_{j=1}^{2N} \nu_j \eta_j + \sum_{1\le k<l}^{2N} \nu_k \nu_l A_{kl}\right], \tag{4.20}$$

where

$$\exp(\eta_j) = \exp(\eta_{N+j}) = c_j p_j^m q_j^n, \quad q_j = \frac{3p_j - 1}{p_j - 3}, \quad j = 1, 2, \cdots, N, \tag{4.21}$$

$$\exp(A_{kl}) = \frac{(p_k - p_l)^2}{p_k p_l}, k < l = 2, 3, \cdots, N \tag{4.22}$$

$$\exp(A_{k,N+l}) = \frac{p_k p_l}{(p_k p_l - 1)^2}, k, l = 1, 2, \cdots, N, \tag{4.23}$$

$$\exp(A_{N+k,N+l}) = \frac{(p_k - p_l)^2}{p_k p_l}, k < l = 2, 3, \cdots, N. \tag{4.24}$$

Here $c_j$, $p_j$, $q_j$ are all real parameters. The summations $\sum_{\mu=0,1}^{(e)}$ and $\sum_{\nu=0,1}^{(o)}$ satisfy the condition (2.47) and (2.48) respectively.

## 5. Discrete analogues by 2-reduction

Since the relationship between the sine-Gordon equation and the CID equations, in semi- and fully discrete analogues of the CID equations, there are also connections with semi- and fully discrete sine-Gordon equations. In ref. [5], authors derived the discrete analogues and Casorati determinant solutions of short pulse equation by 2-reduction of the bilinear two-dimensional Toda lattice (2DTL) equations. Following the idea [5], in this section we use the reduction technique to deduce discrete analogues and Casorati determinant solutions of the CID equations.

It is known that by setting $q = q_0 \cos\phi$, $r = \frac{1}{2}\phi_t$, the CID equations

$$q_t + 2rr_x = 0, \tag{5.1}$$

$$r_{xt} - 2qr = 0, \tag{5.2}$$

are transformed into the sine-Gordon equation

$$\phi_{xt} = 2q_0 \sin\phi. \tag{5.3}$$

We take $q_0 = 1/2$ for simplicity in the following discussion. As is shown [11, 12], upon the dependent variable transformation

$$\phi(x,t) = 2\mathrm{i} \ln \frac{F^*(x,t)}{F(x,t)},$$

the sine-Gordon equation (5.3) leads to the bilinear form:

$$FF_{xt} - F_xF_t = \frac{1}{4}(F^2 - F^{*2}), \tag{5.4}$$

$$F^*F^*_{xt} - F^*_xF^*_t = \frac{1}{4}(F^{*2} - F^2), \tag{5.5}$$

where $F^*$ is the complex conjugate of $F$. Henceforth, the solutions of the CID equations are obtained by $F$ and $F^*$ through the dependent variable transformation

$$q = \frac{1}{2}\cos\left(2\mathrm{i}\ln\frac{F^*(x,t)}{F(x,t)}\right), \quad r = \frac{1}{2}\frac{\partial}{\partial t}\phi = \left(\mathrm{i}\ln\frac{F^*(x,t)}{F(x,t)}\right)_t. \tag{5.6}$$

It has been shown in [5] that the bilinear equations (5.4) and (5.5) are actually obtained as the $2-$reduction of the 2DTL equations:

$$\frac{1}{2}D_XD_T\tau_n\cdot\tau_n = \tau_n^2 - \tau_{n+1}\tau_{n-1}, \tag{5.7}$$

i.e.,

$$\tau_n\frac{\partial^2\tau_n}{\partial X\partial T} - \frac{\partial\tau_n}{\partial X}\frac{\partial\tau_n}{\partial T} = \tau_n^2 - \tau_{n+1}\tau_{n-1}. \tag{5.8}$$

Applying the 2-reduction $\tau_{n-1} = \alpha\tau_{n+1}$ ($\alpha$ is a constant), we get

$$\tau_n\frac{\partial^2\tau_n}{\partial X\partial T} - \frac{\partial\tau_n}{\partial X}\frac{\partial\tau_n}{\partial T} = \tau_n^2 - \tau_{n+1}^2, \tag{5.9}$$

where the gauge transformation $\tau_n \to \alpha^{\frac{n}{2}}\tau_n$ is used. Letting $f = \tau_0$ and $\bar{f} = \tau_1$, we have

$$ff_{XT} - f_Xf_T = f^2 - \bar{f}^2, \tag{5.10}$$

$$\bar{f}\bar{f}_{XT} - \bar{f}_X\bar{f}_T = \bar{f}^2 - f^2, \tag{5.11}$$

Under the independent variable transformation $x = 2X$, $t = 2T$, we obtain

$$ff_{xt} - f_xf_t = \frac{1}{4}(f^2 - \bar{f}^2), \tag{5.12}$$

$$\bar{f}\bar{f}_{xt} - \bar{f}_x\bar{f}_t = \frac{1}{4}(\bar{f}^2 - f^2), \tag{5.13}$$

which are bilinear form of the CID equations.

Next we give the Casorati determinant ($N$-soliton) solution of the CID equation. The Casorati determinant solution of the 2TDL equation is of the form

$$\tau_n(X,T) = |\psi_i^{(n+j-1)}(X,T)|_{1\le i,j\le N}, \tag{5.14}$$

where $\psi_i^{(n)}(X,T)$ satisfies linear dispersion relations

$$\frac{\partial\psi_i^{(n)}}{\partial X} = \psi_i^{(n+1)}, \quad \frac{\partial\psi_i^{(n)}}{\partial T} = \psi_i^{(n-1)}. \tag{5.15}$$

For example, a particular choice of $\psi_i^{(n)}(X,T)$

$$\psi_i^{(n)}(X,T) = c_ip_i^ne^{p_iX+\frac{1}{p_i}T+\eta_{0i}} + d_iq_i^ne^{q_iX+\frac{1}{q_i}T+\eta'_{0i}}, \tag{5.16}$$

with $c_i$ and $d_i$ being constants, satisfies the linear dispersion relations and gives the $N$-soliton solutions.

Applying the 2-reduction $q_i = -p_i$ and the change of variables $x = 2X$ and $t = 2T$, we obtain the determinant solution of bilinear equations (5.12)-(5.13):

$$f(x,t) = \tau_0(x,t), \quad \bar{f}(x,t) = \tau_1(x,t), \tag{5.17}$$

$$\tau_n(x,t) = |\psi_i^{(n+j-1)}(x,t)|_{1\le i,j\le N}, \tag{5.18}$$

where

$$\psi_i^{(n)}(x,t) = c_i p_i^n e^{\frac{1}{2}p_i x + \frac{1}{2p_i}t + \eta_{0i}} + d_i(-p_i)^n e^{-\frac{1}{2}p_i x - \frac{1}{2p_i}t + \eta'_{0i}}. \tag{5.19}$$

Let us introduce $\alpha$ and $\beta$ such that $F^* = \alpha\bar{f}$ and $F = \beta f$, where $F$ and $F^*$ are complex conjugate of each other. Note that a set of $F$ and $F^*$ gives solutions of the CID equations as well as a set of $f$ and $\bar{f}$ because of

$$r = \frac{1}{2}\frac{\partial}{\partial t}\left(2\mathrm{i}\ln\frac{F^*}{F}\right) = \frac{1}{2}\frac{\partial}{\partial t}\left(2\mathrm{i}\ln\frac{\alpha\bar{f}}{\beta f}\right) = \frac{1}{2}\frac{\partial}{\partial t}\left(2\mathrm{i}\ln\frac{\bar{f}}{f}\right). \tag{5.20}$$

By choosing phase constants appropriately, the functions $f$ and $\bar{f}$ can be made to be complex conjugate of each other to keep the reality and regularity of $r$. For example, the following choice

$$\psi_i^{(n)}(x,t) = p_i^n e^{\frac{1}{2}p_i x + \frac{1}{2p_i}t + \eta_{0i} - \mathrm{i}\pi/4} + (-p_i)^n e^{-\frac{1}{2}p_i x - \frac{1}{2p_i}t + \eta'_{0i} + \mathrm{i}\pi/4} \tag{5.21}$$

guarantees the reality and regularity of the solution.

Summarizing the above results, the determinant solution of the CID equation is given by

$$\begin{aligned} &q = \frac{1}{2}\cos\left(2\mathrm{i}\ln\frac{\bar{f}(x,t)}{f(x,t)}\right), \quad r = \frac{1}{2}\frac{\partial}{\partial t}\phi = \frac{\partial}{\partial t}\left(\mathrm{i}\ln\frac{\bar{f}(x,t)}{f(x,t)}\right), \\ &f(x,t) = \tau_0(x,t), \quad \bar{f}(x,t) = \tau_1(x,t), \\ &\tau_n(x,t) = |\psi_i^{(n+j-1)}(x,t)|_{1\le i,j\le N}, \end{aligned} \tag{5.22}$$

where

$$\psi_i^{(n)}(x,t) = p_i^n e^{\frac{1}{2}p_i x + \frac{1}{2p_i}t + \eta_{0i} - \mathrm{i}\pi/4} + (-p_i)^n e^{-\frac{1}{2}p_i x - \frac{1}{2p_i}t + \eta'_{0i} + \mathrm{i}\pi/4}.$$

Consider the following Casorati determinant:

$$\tau_n(k,T) = |\psi_i^{(n+j-1)}(k,T)|_{1\le i,j\le N}, \tag{5.23}$$

where $\psi_i^{(n)}$ satisfies the dispersion relations

$$\triangle_k \psi_i^{(n)} = \psi_i^{(n+1)}, \tag{5.24}$$

$$\partial_T \psi_i^{(n)} = \psi_i^{(n-1)}, \tag{5.25}$$

where $\triangle_k$ is the backward difference operator with the spacing constant $a$:

$$\triangle_k f(k) = \frac{f(k) - f(k-1)}{a}. \tag{5.26}$$

One can choose

$$\psi_i^{(n)}(k,T) = c_i p_i^n (1-ap_i)^{-k} e^{\frac{1}{p_i}T+\xi_{i0}} + d_i q_i^n (1-aq_i)^{-k} e^{\frac{1}{q_i}T+\eta_{i0}}, \tag{5.27}$$

which automatically satisfies the dispersion relation (5.24) and (5.25). The above Casorati determinant satisfies the bilinear form of the semi-discrete 2DTL equation

$$\left(\frac{1}{a}D_T - 1\right)\tau_n(k+1)\cdot\tau_n(k) + \tau_{n+1}(k+1)\cdot\tau_{n-1}(k) = 0. \tag{5.28}$$

Applying the 2-reduction

$$q_i = -p_i,$$

and letting

$$f_k = \tau_0(k), \quad \bar{f}_k = \tau_1(k) = \left(\prod_{i=1}^{N} p_i^2\right)\tau_{-1}(k),$$

we obtain

$$\frac{1}{a}D_T f_{k+1}\cdot f_k - f_{k+1}f_k + \bar{f}_{k+1}\bar{f}_k = 0, \tag{5.29}$$

$$\frac{1}{a}D_T \bar{f}_{k+1}\cdot \bar{f}_k - \bar{f}_{k+1}\bar{f}_k + f_{k+1}f_k = 0, \tag{5.30}$$

where the gauge transformation $\tau_n \to (\prod_{i=1}^{N} p_i)^n \tau_n$ is used. Under the change of independent variable $t = 2T$, from (5.29)-(5.30) we can obtain

$$\frac{2}{a}\left(\left(\ln\frac{\bar{f}_{k+1}}{\bar{f}_k}\right)_t - \left(\ln\frac{f_{k+1}}{f_k}\right)_t\right) = \frac{\bar{f}_{k+1}\bar{f}_k}{f_{k+1}f_k} - \frac{f_{k+1}f_k}{\bar{f}_{k+1}\bar{f}_k}. \tag{5.31}$$

One can check the deducing details in [5]. Introducing the dependent variable transformation $\phi_k(t) = 2\mathrm{i}\ln\frac{\bar{f}_k(s)}{f_k(s)}$, one arrives at

$$\frac{1}{2a}\left(\frac{\mathrm{d}\phi_{k+1}}{\mathrm{d}t} - \frac{\mathrm{d}\phi_k}{\mathrm{d}t}\right) = \sin\left(\frac{\phi_{k+1}+\phi_k}{2}\right), \tag{5.32}$$

which is nothing but an integrable semi-discretization of the sine-Gordon equation. By introducing the variable transformation

$$r_k = \frac{1}{2}\frac{\mathrm{d}\phi_k}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{i}\ln\frac{\bar{f}_k(s)}{f_k(s)}\right), \tag{5.33}$$

$$q_k = \cos\left(\frac{\phi_{k+1}+\phi_k}{2}\right) = \frac{1}{2}\left(\frac{\bar{f}_{k+1}\bar{f}_k}{f_{k+1}f_k} + \frac{f_{k+1}f_k}{\bar{f}_{k+1}\bar{f}_k}\right), \tag{5.34}$$

and together with the fact

$$\cos^2\left(\frac{\phi_{k+1}+\phi_k}{2}\right) + \sin^2\left(\frac{\phi_{k+1}+\phi_k}{2}\right) = 1, \tag{5.35}$$

we can obtain an integrable semi-discrete analogue of the CID equations

$$(r_{k+1}-r_k)^2 = a^2(1-q_k^2), \tag{5.36}$$

$$\frac{\mathrm{d}q_k}{\mathrm{d}t} = \frac{r_k^2 - r_{k+1}^2}{a}. \tag{5.37}$$

From the construction of the semi-discrete analogue, we know that the semi-discrete analogue of the CID equations has the following Casorati determinant solution:

$$r_k(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{i}\ln\frac{\bar{f}_k}{f_k}\right), \tag{5.38}$$

$$f_k(t) = \tau_0(k,t), \quad \bar{f}_k(t) = \tau_1(k,t), \tag{5.39}$$

$$\tau_n(k,t) = |\psi_i^{(n+j-1)}(k,t)|_{1\le i,j\le N}, \tag{5.40}$$

with

$$\psi_i^{(n)}(k,t) = p_i^n(1-ap_i)^{-k}e^{\frac{1}{2p_i}t+\xi_{i0}-\mathrm{i}\pi/4} + (-p_i)^n(1+ap_i)^{-k}e^{-\frac{1}{2p_i}t+\eta_{i0}+\mathrm{i}\pi/4}.$$

We can rewrite the semi-discrete CID equations in an alternative version which converges to the CID equations in the continuous limit $a\to 0$. Multiplying eq. (5.37) by $2q_k$, we have

$$\frac{\mathrm{d}q_k^2}{\mathrm{d}t} = \frac{2q_k(r_k^2 - r_{k+1}^2)}{a}. \tag{5.41}$$

A substitution of $q_k^2$ from (5.36) into equation (5.41) results to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{r_{k+1}-r_k}{a} = q_k(r_{k+1}+r_k). \tag{5.42}$$

In the continuous limit $a\to 0$, (5.37) and (5.42) converge to the CID equations

$$q_t + 2rr_x = 0,$$
$$r_{xt} - 2qr = 0.$$

By the dependent variable transformation $1-\frac{q_{n+1}-q_n}{\varepsilon}\to q_n$, the previous semi-discrete CID equations (2.17) and (2.18) turn into (5.37) and (5.42). Considering the symmetry of $x$ and $t$ in the sine-Gordon equation, semi-discrete analogue of $t$ part can be completed similarly.

To construct a full-discrete analogue of the CID equations, we introduce one more discrete variable $l$ which corresponds to the discrete time variable. It is known that the bilinear equations

$$\left(\frac{2}{a}D_s - 1\right)\tau_n(k+1,l)\cdot\tau_n(k,l) + \tau_{n+1}(k+1,l)\tau_{n-1}(k,l) = 0, \tag{5.43}$$

and

$$(2bD_s - 1)\tau_n(k,l+1)\cdot\tau_{n+1}(k,l) + \tau_n(k,l)\tau_{n+1}(k,l+1) = 0, \tag{5.44}$$

have the Casorati determinant solution

$$\tau_n(k,l) = |\psi_i^{(n+j-1)}(k,l)|_{1\le i,j\le N}, \tag{5.45}$$

with

$$\psi_i^{(n)} = c_i p_i^n (1-ap_i)^{-k}(1-\frac{b}{p_i})^{-l} e^{\frac{1}{2p_i}s+\xi_{i0}} + d_i q_i^n (1-aq_i)^{-k}(1-\frac{b}{q_i})^{-l} e^{\frac{1}{2q_i}s+\eta_{i0}}. \tag{5.46}$$

Applying the 2-reduction $\tau_{n-1} = (\prod_{i=1}^N p_i^2)^{-1}\tau_{n+1}$,i.e. adding constraint $q_i = -p_i$ to the $N$-soliton solution, one obtains

$$\left(\frac{2}{a}D_s - 1\right)\tau_n(k+1,l)\cdot\tau_n(k,l) + \tau_{n+1}(k+1,l)\tau_{n+1}(k,l) = 0, \tag{5.47}$$

and

$$(2bD_s - 1)\tau_n(k,l+1)\cdot\tau_{n+1}(k,l) + \tau_n(k,l)\tau_{n+1}(k,l+1) = 0, \tag{5.48}$$

where the gauge transformation $\tau_n \to (\prod_{i=1}^N p_i)^n \tau_n$ is used. Letting

$$f_{k,l} = \tau_0(k,l), \quad \bar{f}_{k,l} = \tau_1(k,l),$$

and introducing

$$r_{k,l} = \left(\mathrm{i}\ln\frac{\bar{f}_{k,l}}{f_{k,l}}\right)_s, \tag{5.49}$$

together with auxiliary variable

$$v_{k,l} = ka - (\ln \bar{f}_{k,l} f_{k,l})_s, \tag{5.50}$$

from (5.47) and (5.48), we find the following relations

$$(r_{k+1,l} - r_{k,l})^2 = a^2(1-q_{k,l}^2), \tag{5.51}$$

$$(r_{k,l+1} + r_{k,l})^2 = \frac{1}{b^2} - \left(v_{k,l+1} - v_{k,l} + \frac{1}{b}\right)^2, \tag{5.52}$$

where $q_{k,l} = \frac{v_{k+1,l}-v_{k,l}}{a}$. Substituting $k = k+1$ into (5.52) and subtracting each other, we obtain

$$\begin{aligned}&(r_{k,l+1} + r_{k,l})^2 - (r_{k+1,l+1} + r_{k+1,l})^2\\ =&(v_{k+1,l+1} - v_{k,l})^2 - (v_{k,l+1} - v_{k,l})^2 + \frac{2a}{b}(q_{k,l+1} - q_{k,l}),\end{aligned} \tag{5.53}$$

which converges to

$$\frac{\mathrm{d}q_k}{\mathrm{d}t} = \frac{r_k^2 - r_{k+1}^2}{a}, \tag{5.54}$$

in the continuous limit $b \to 0$. Hence eqs. (5.51) and (5.52) constitute an integrable full-discretization of the CID equation.

From the construction above, the determinant solution of the full-discrete CID equation is

$$r_{k,l} = \mathrm{i}\left(\frac{\bar{g}_{k,l}}{\bar{f}_{k,l}} - \frac{g_{k,l}}{f_{k,l}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{i}\ln\frac{\bar{f}_{k,l}}{\bar{f}_{k,l}}\right), \tag{5.55}$$

$$q_{k,l} = \frac{1}{2}\left(\frac{\bar{f}_{k+1,l}\bar{f}_{k,l}}{f_{k+1,l}f_{k,l}} + \frac{f_{k+1,l}f_{k,l}}{\bar{f}_{k+1,l}\bar{f}_{k,l}}\right), \tag{5.56}$$

$$f_{k,l} = \tau_0(k,l), \quad \bar{f}_{k,l} = \tau_1(k,l),$$
$$g_{k,l} = \rho_0(k,l), \quad \bar{g}_{k,l} = \rho_1(k,l),$$

$$\tau_n(k,l) = \begin{vmatrix} \psi_1^{(n)}(k,l) & \psi_1^{(n+1)}(k,l) & \cdots & \psi_1^{(n+N-1)}(k,l) \\ \psi_2^{(n)}(k,l) & \psi_2^{(n+1)}(k,l) & \cdots & \psi_2^{(n+N-1)}(k,l) \\ \vdots & \vdots & & \vdots \\ \psi_N^{(n)}(k,l) & \psi_N^{(n+1)}(k,l) & \cdots & \psi_N^{(n+N-1)}(k,l) \end{vmatrix} \quad \text{and}$$

$$\rho_n(k,l) = \begin{vmatrix} \psi_1^{(n-1)}(k,l) & \psi_1^{(n+1)}(k,l) & \cdots & \psi_1^{(n+N-1)}(k,l) \\ \psi_2^{(n-1)}(k,l) & \psi_2^{(n+1)}(k,l) & \cdots & \psi_2^{(n+N-1)}(k,l) \\ \vdots & \vdots & & \vdots \\ \psi_N^{(n-1)}(k,l) & \psi_N^{(n+1)}(k,l) & \cdots & \psi_N^{(n+N-1)}(k,l) \end{vmatrix},$$

with

$$\psi_i^{(n)} = p_i^n(1-ap_i)^{-k}(1-\frac{b}{p_i})^{-l}e^{\frac{1}{2p_i}s+\xi_{i0}-\mathrm{i}\pi/4} + (-p_i)^n(1+ap_i)^{-k}(1+\frac{b}{p_i})^{-l}e^{-\frac{1}{2p_i}s+\eta_{i0}+\mathrm{i}\pi/4}, \tag{5.57}$$

where the phase constants $\pm\mathrm{i}\pi/4$ play the role of keeping the reality of the solution and $s$ is an auxiliary parameter.

## 6. Conclusion and discussion

Integrability and $N$-soliton solution of the semi-discrete sine-Gordon equation

$$\phi_{n+1,t} + \phi_{n,t} = \sin\phi_{n+1} - \sin\phi_n \tag{6.1}$$

have been shown in [23, 27] and [25] respectively. Since the CID system shares the bilinear form with sine-Gordon equation, it is natural to consider the relation between the semi-discrete CID eqs. (2.17)-(2.18) (or (5.37) and (5.42)) and the semi-discrete sine-Gordon eq. (6.1).

It is known that coupled forms of integrable models can be obtained from the bilinear equations. See for example the KdV equation [30], sine-Gordon equation [16], modified KdV equation [16], vector Ito equation [14], nonlinear Schrödinger equation [24], NNV equation [32]. The coupled form of the CID equations can hence also be given

$$D_t^2 F\cdot F = 2\sum_{k=1}^{N} G_k^2, \tag{6.2}$$

$$(D_x D_t - 2)G_k\cdot F = 0. \tag{6.3}$$

They can be transformed into the nonlinear form

$$u_t = \sum_{k=1}^{N} r_k^2, \tag{6.4}$$

$$(r_k)_{xt} + 2r_k(u_x - 1) = 0, \tag{6.5}$$

by the dependent variable transformation $u = (\ln F)_t, G_k = r_k F$. A further extension of the coupled form is

$$D_t^2 F \cdot F = 2 \sum_{j,k=1}^{N} c_{j,k} G_j G_k, \tag{6.6}$$

$$(D_x D_t - 2) G_j \cdot F = 0, \tag{6.7}$$

where $c_{j,k}$ are arbitrary coupling constants. With the transformation $G_j = r_j F, u = 2(\ln F)_t$, the general coupled CID eqs. (6.6)-(6.7) are mapped into the ordinary nonlinear form

$$u_t = \sum_{j,k=1}^{N} c_{j,k} r_j r_k, \tag{6.8}$$

$$(r_j)_{xt} + 2r_j(u_x - 1) = 0. \tag{6.9}$$

From the bilinear integrable discretization approach, discrete version of these coupled CID equations can also be obtained [13].

To summarize, we have presented here two semi-discrete integrable versions and one fully discrete integrable version for the CID equations and have derived their soliton solutions. Based on the results obtained, it would be natural to further consider other integrable properties such as the determinant structure of the $N$-soliton solutions for these semi-discrete and fully discrete integrable lattice.

## Acknowledgements

Authors are grateful to anonymous referee for pointing out relevant literature and helpful suggestion in improving the original manuscript. The work of L.V. is supported by a grant from the National Science and Engineering Research Council (NSERC) of Canada. G. Yu acknowledges a postdoctoral fellowship from the Mathematical Physics Laboratory of the CRM. He is also supported by the National Natural Science Foundation of China (Grant no.10901105) and Chenguang Program (No.09CG08) sponsored by Shanghai Municipal Education Commission and Shanghai Educational Development Foundation.

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