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## ON THE RECURSION OPERATOR FOR THE NONCOMMUTATIVE BURGERS HIERARCHY

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The noncommutative Burgers recursion operator is constructed via the Cole–Hopf transformation, and its structural properties are studied. In particular, a direct proof of its hereditary property is given.

*Keywords:* Burgers equation; noncommutativity; recursion operators; Bäcklund transformations; hereditariness.

### 1. Introduction

This article is devoted to the derivation and study of a recursion operator of the noncommutative (nc) Burgers equation  $s_t = s_{xx} + 2ss_x$ . Until very recently no recursion operator was known in the noncommutative setting, and even doubts about its existence were formulated in [15]. Nevertheless a hierarchy was constructed by Kupershmidt in [11], based on the Cole–Hopf transformation linking the nc heat equation and the nc Burgers equation (see also [12] for pioneering research). Inspired by this, here a candidate is derived by transferring the well-known recursion operator of the nc heat equation via the Cole–Hopf transformation. The recursion operator derived here was set down in [3]. It has been obtained independently and in another manner by Gürses, Karasu and Turhan [10]. Their approach is based on the Lax pair formulation of the nc Burgers equation and a method developed in [9].

A further motivation to use Bäcklund transformations to derive new recursion operators from known ones stems from a general structure theorem of Fokas and Fuchssteiner [5]. There it is shown that important structural properties are preserved under Bäcklund

transformations. However the argumentation in [5] only applies when the Bäcklund transformation  $B(u, s) = 0$  can be locally solved with respect to each of the variables  $u$  and  $s$ . To avoid restrictions of this kind, we prefer, in the present article, to prove the properties required of a recursion operator without referring to the Cole–Hopf transformation. Besides avoiding unnecessary extra assumptions, this has the advantage of leading to an argument which is essentially algebraic in nature. It can be easily extended to functions with values in suitable associative algebras with derivation (see [14] for substantial results on this level of generality).

In the present context, the most involved and interesting part is the proof of hereditariness, a concept introduced by Fuchssteiner in [6]. Once an operator is known to be hereditary, the proof of several further important properties of the whole hierarchy generated by the operator (strong symmetry, pairwise involutivity of the flows of the hierarchy) reduce to the verification of the appropriate property for the first member in the hierarchy or of identities involving the operator (see [6, 7]). On the other hand, verification that an operator is hereditary can require computations of considerable complexity even in the commutative setting (see [1, 8]). In the nc setting the first proof of the hereditary property was given in [18] for the recursion operator of the nc KdV hierarchy. For a study of the solutions to the nc KdV hierarchy based on its recursion operator, we refer to [2, 4] and for Bäcklund transformations see [16, 17].

The material is organized as follows. In Sec. 2 the nc Burgers recursion operator is derived via the Cole–Hopf link. In Sec. 3 we state its main structural properties and give the first part of the proof. Moreover it is observed that the resulting hierarchy coincides with that constructed in [11]. Section 4 is devoted to a direct proof that the nc Burgers recursion operator is hereditary, thereby completing the proof started in Sec. 3.

## 2. The nc Burgers Recursion Operator

In the present section a candidate for a recursion operator of the nc Burgers equation is derived via the Cole–Hopf transformation. We regard the nc heat equation

$$u_t = K(u) \tag{1}$$

and the nc Burgers equation

$$s_t = G(s) \tag{2}$$

as ordinary differential equations on infinite-dimensional spaces  $\mathcal{U}$  and  $\mathcal{S}$  of  $x$ -dependent functions  $u = u(x)$  and  $s = s(x)$  with values in a (possibly noncommutative) Banach algebra, where the vector fields on the right-hand sides are  $K(u) = u_{xx}$  and  $G(s) = s_{xx} + 2ss_x$ . Throughout the usual convention will be adopted to identify vector fields, i.e. sections of the tangent bundles  $T\mathcal{U}$  and  $T\mathcal{S}$ , with self-mappings of  $\mathcal{U}$  and  $\mathcal{S}$ .

A classical result states that the Cole–Hopf transform  $s = u^{-1}u_x$  maps solutions of (1) with values in the invertible elements to solutions of (2). To avoid additional assumptions, it is often convenient to view the Cole–Hopf transformation as the subset of  $\mathcal{U} \times \mathcal{S}$  defined by

$$B(u, s) = 0, \tag{3}$$

where  $B(u, s) = us - u_x$ . To give a meaning to  $B(u, s)$ ,  $\mathcal{U}$  and  $\mathcal{S}$  are assumed to be copies of the same space of algebra-valued functions, so that the product  $us$  is defined. Near points where (3) can be solved for  $s = s(u)$ , the differential of  $s(u)$  is  $T = -B_s^{-1}B_u$ , by the implicit function theorem. Note that  $T$  has to be read as a linear mapping sending the tangent space  $T_u\mathcal{U}$  to  $T_{s(u)}\mathcal{S}$ . Symmetrically, the differential becomes  $\tilde{T} = -B_u^{-1}B_s$  where (3) can be solved for  $u = u(s)$ . At pairs  $(u, s)$  where one can solve with respect to each of the two variables, we obviously have  $\tilde{T} = T^{-1}$ .

The aim is to derive the recursion operator of the nc Burgers equation by transferring the well-known recursion operator  $\Phi = D$  of the nc heat equation via the Cole–Hopf transformation. Here the recursion operator  $\Phi$  has to be regarded as a  $u$ -dependent family of linear self-mappings on the fibers of  $T\mathcal{U}$ , whence

$$\Psi = T\Phi T^{-1}.$$

This approach, which at first only is legitimate for pairs  $(u, s)$  where one can solve with respect to each of the two variables, leads to the right formula for the recursion operator.

We start with the calculation of  $T$ . Since the directional derivatives of  $B$  are easily calculated to be

$$\begin{aligned} B_u[V] &= \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} ((u + \epsilon V)s - (u + \epsilon V)_x) = Vs - V_x, \\ B_s[W] &= \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} (u(s + \epsilon W) - u_x) = uW, \end{aligned}$$

for  $V \in T_u\mathcal{U}$ ,  $W \in T_s\mathcal{S}$ , we have  $B_u = R_s - D$ , i.e.  $B_u = -(D - R_s)$ , and  $B_s = L_u$ , where  $L_u$  and  $R_u$  denote left- and right-multiplication by  $u$ . Thus,

$$T = L_{u^{-1}}(D - R_s). \quad (4)$$

To compute  $\Psi$  we require the following lemma.

**Lemma 1.** *If  $B(u, s) = 0$ , then  $(D - R_s)L_u = L_u(D + C_s)$ , where  $C_s$  denotes the commutator  $C_s = L_s - R_s$ .*

**Proof.** From the product rule  $DL_u = L_uD + L_{u_x}$ , and because left- and right-multiplication commute,

$$(D - R_s)L_u = L_u(D - R_s) + L_{u_x}.$$

Since  $B(u, s) = 0$ , one has  $L_{u_x} = L_{us} = L_uL_s$ . Thus  $(D - R_s)L_u = L_u(D - R_s + L_s) = L_u(D + C_s)$ .  $\square$

From Lemma 1, we get  $L_{u^{-1}}(D - R_s) = (D + C_s)L_{u^{-1}}$ ,  $(D - R_s)^{-1}L_u = L_u(D + C_s)^{-1}$ , and thus

$$\begin{aligned} \Psi(s) &= TDT^{-1} \\ &= L_{u^{-1}}(D - R_s)D(D - R_s)^{-1}L_u \\ &= (D + C_s)L_{u^{-1}}DL_u(D + C_s)^{-1} \\ &= (D + C_s)(L_{u^{-1}}(L_uD + L_{u_x}))(D + C_s)^{-1} \\ &= (D + C_s)(D + L_{u^{-1}u_x})(D + C_s)^{-1}. \end{aligned}$$

Since  $u^{-1}u_x = s$ , we obtain the  $u$ -independent expression

$$\Psi(s) = (D + C_s)(D + L_s)(D + C_s)^{-1}. \quad (5)$$

In [3] the recursion operator of the nc Burgers equation was introduced in the form (5). To rewrite (5) in a slightly different way, we use the identity

$$[D + C_s, D + L_s] = R_{s_x} \quad (6)$$

which follows from  $[D + C_s, D + L_s] = [(D + L_s) - R_s, D + L_s] = -[R_s, D + L_s] = -[R_s, D] = R_{s_x}$ , since left- and right-multiplication commute. Hence

$$\Psi(s) = ((D + L_s)(D + C_s) + R_{s_x})(D + C_s)^{-1} = D + L_s + R_{s_x}(D + C_s)^{-1}.$$

The form

$$\Psi(s) = D + L_s + R_{s_x}(D + C_s)^{-1} \quad (7)$$

of the recursion operator was derived independently by Gürses, Karasu, and Turhan [10]. Their approach, which relies on a Lax pair representation for the nc Burgers hierarchy, is completely different from our derivation, which is only based on the Cole–Hopf link.

At first glance, the condition that the operator  $D + C_s$  is invertible might seem very restrictive. The following observation shows that in fact for  $u \in \mathcal{U}$ ,  $s \in \mathcal{S}$  related via the Cole–Hopf link, invertibility of  $D + C_s$  simply amounts to invertibility of  $D$ .

**Proposition 2.** *For  $u \in \mathcal{U}$  invertible,  $s \in \mathcal{S}$  with  $B(u, s) = 0$  it follows that*

$$D + C_s = L_{u^{-1}} R_u D R_{u^{-1}} L_u.$$

**Proof.** The product rule gives  $R_u D R_{u^{-1}} = (DR_u - R_{u_x})R_{u^{-1}} = D - R_{u^{-1}u_x} = D - R_s$ , and similarly  $L_{u^{-1}} D L_u = D + L_s$ . Since right- and left-multiplication commute,

$$L_{u^{-1}} R_u D R_{u^{-1}} L_u = L_{u^{-1}}(D - R_s)L_u = D + L_s - R_s = D + C_s. \quad \square$$

Note that, although the derivation of the nc Burgers recursion operator (5) is quite natural, up to now we did not say anything about its properties. This is done in the next section.

### 3. Structural Properties of the nc Burgers Recursion Operator

In this section the main properties of the recursion operator (7) are stated and proved under the assumption that  $\Psi$  is hereditary. The proof of hereditarity is given in the next section. The section is concluded with a verification that the hierarchy generated by the recursion operator (7) is the same as the one obtained in another manner by Kuperschmidt [11].

Recall that a vector field  $V(s)$  is called a symmetry of an evolution equation  $s_t = H(s)$  if  $V$  and  $H$  are in involution, i.e. if  $[V, H]$  vanishes identically. An operator-valued function  $\Upsilon(s)$  is called a strong symmetry of  $s_t = H(s)$  if for every symmetry  $V$  of  $s_t = H(s)$  the vector field  $\Upsilon V$  is again a symmetry. It is proved in [6] that  $\Upsilon$  is a strong symmetry of  $s_t = H(s)$  if  $\Upsilon'[H]V = H'[\Upsilon V] - \Upsilon H'[V]$  holds for any vector field  $V$ .

Using the operator (7), we introduce the nc Burgers hierarchy

$$s_{t_n} = \Psi(s)^{n-1} s_x, \quad n \geq 1, \quad (8)$$

the lowest members of which read

$$\begin{aligned} s_{t_1} &= s_x, \\ s_{t_2} &= s_{xx} + 2ss_x, \\ s_{t_3} &= s_{xxx} + 3ss_{xx} + 3s_x^2 + 3s^2 s_x. \end{aligned}$$

The main result in this section is the following.

**Theorem 3.** *The nc Burgers recursion operator  $\Psi(s)$  given in (7) is a strong symmetry for all equations of the nc Burgers hierarchy (8).*

As a special case, Theorem 3 implies that  $\Psi(s)$  is a recursion operator of the nc Burgers equation in the sense of [13], meaning that  $\Psi(s)$  is a strong symmetry of  $s_t = s_{xx} + 2ss_x$ .

The next proposition shows that  $\Psi$  is a strong symmetry for  $s_t = s_x$ . According to Theorem 5 in Sec. 4,  $\Psi$  is hereditary. In [6] it is proved that, if  $\Upsilon$  is a strong symmetry for  $s_t = H(s)$ , then  $\Upsilon$  is a strong symmetry for the hierarchy generated by  $\Upsilon$ , provided that  $\Upsilon$  is hereditary. Hence, the proof of Theorem 3 will be complete once the following proposition and Theorem 5 are proved.

**Proposition 4.** *The nc Burgers recursion operator  $\Psi(s)$  given in (7) is a strong symmetry for  $s_t = s_x$ .*

As a preliminary to the proof of Proposition 4, we note that, for the particular vector field  $G_0(s) = s_x$ , we have  $G'_0(s) = D$ . Thus, an operator-valued function  $\Upsilon$  is a strong symmetry for  $G_0$  if and only if

$$\Upsilon'(s)[s_x] = [D, \Upsilon(s)]. \quad (9)$$

We now need the variational derivative of the noncommutative Burgers operator (7), namely

$$\Psi'(s)[V] = L_V + R_{Vx}(D + C_s)^{-1} - R_{s_x}(D + C_s)^{-1}C_V(D + C_s)^{-1}. \quad (10)$$

It is straightforward to check that  $C'_s[V] = C_V$ , and the variational derivative of the  $s$ -independent operator  $D$  vanishes. Applying the derivation rule for inverses,<sup>a</sup> we obtain

$$((D + C_s)^{-1})'[V] = -(D + C_s)^{-1}C_V(D + C_s)^{-1}.$$

On combination with  $L'_s[V] = L_V$ ,  $R'_{s_x}[V] = R_{Vx}$ , and the product rule, this delivers relation (10).

**Proof of Proposition 4.** As for the right-hand side of (9), we compute

$$\begin{aligned} [D, \Psi(s)] &= [D, L_s] + [D, R_{s_x}(D + C_s)^{-1}] \\ &= L_{s_x} + R_{s_{xx}}(D + C_s)^{-1} + R_{s_x}[D, (D + C_s)^{-1}] \\ &= L_{s_x} + R_{s_{xx}}(D + C_s)^{-1} - R_{s_x}(D + C_s)^{-1}C_{s_x}(D + C_s)^{-1}, \end{aligned} \quad (11)$$

<sup>a</sup>Recall that  $(\Upsilon^{-1}(s))'[V] = \Upsilon(s)^{-1}(-\Upsilon'(s)[V])\Upsilon(s)^{-1}$  holds for an operator-valued function  $\Upsilon(s)$ .

where we used  $[D, (D + C_s)^{-1}] = -(D + C_s)^{-1}C_{sx}(D + C_s)^{-1}$ , which follows from  $[D, D + C_s] = [D, C_s] = C_{sx}$ . Comparison of (11) with (10) shows (9).  $\square$

In the following our work is connected to that of Kupershmidt [11]. The construction of the nc Burgers hierarchy in [11] starts off from the nc heat hierarchy  $u_{t_n} = D^n u = \Phi(u)^{n-1}u_x$ . Setting  $s_n := u^{-1}(D^n u)$ , the recursion relation  $s_{n+1} = (D + L_s)s_n$  for  $n \geq 1$  is derived and used to show that  $s = s_1$  satisfies

$$s_{t_n} = -s_n s + s_{n+1} = (D + C_s)s_n.$$

This produces the nc Burgers hierarchy

$$\begin{aligned} s_{t_n} &= (D + C_s)(D + L_s)^{n-1}s \\ &= ((D + C_s)(D + L_s)(D + C_s)^{-1})^{n-1}(D + C_s)s \\ &= \Psi(s)^{n-1}s_x. \end{aligned}$$

Hence the nc Burgers hierarchy (8) coincides with the hierarchy constructed in [11].

#### 4. The Hereditary Property of the nc Burgers Recursion Operator: A Direct Verification

In this section, it is proved that the nc Burgers recursion operator is hereditary. Hereditarity is a fundamental concept that was introduced in [6] (see also [7]) in the context of integrable systems. Recall that an operator-valued function  $\Upsilon$  is called hereditary if for every  $s \in \mathcal{S}$  where  $\Upsilon$  is defined, the bilinear form

$$\Upsilon\Upsilon'[V]W - \Upsilon'[\Upsilon V]W, \quad (12)$$

is symmetric in  $V, W \in T_s\mathcal{S}$ . The following is the main result of the section.

**Theorem 5.** *The nc Burgers recursion operator given in (7) is hereditary.*

A structure theorem of Fokas and Fuchssteiner [5] states that the hereditary property is preserved under Bäcklund transformations. It implies that the nc Burgers recursion operator is hereditary at certain points of  $\mathcal{S}$  satisfying extra conditions related to the local solvability of (3) (note that the recursion operator  $D$  of the heat equation is trivially hereditary). In what follows, a direct proof without reference to the Cole–Hopf transformation is presented. Besides the advantage of avoiding extrinsic extra conditions, this gives an essentially algebraic argument which carries over to any algebraic setting where the operations involved allow an appropriate interpretation.

The deformed differential operator

$$\Lambda(s) = D + C_s$$

will be central in our calculations. We start by rewriting the recursion operator  $\Psi(s)$  and its directional derivative  $\Psi'(s)[V]$  in terms of  $\Lambda(s)$ . Using the elementary fact that  $\Lambda(s)s = s_x$ , we get

$$\Psi(s) = \Lambda(s) + R_s + R_{\Lambda(s)s}\Lambda(s)^{-1}, \quad (13)$$

$$\Psi'(s)[V] = L_V + ((R_{\Lambda(s)V} - R_{[s,V]}) - R_{\Lambda(s)s}\Lambda(s)^{-1}C_V)\Lambda(s)^{-1}. \quad (14)$$

It is remarked that for the operator  $\Lambda(s)$  the usual product rule and partial integration rule hold so that

$$\Lambda(s)L_V = L_V\Lambda(s) + L_{\Lambda(s)V}, \quad (15)$$

$$\Lambda(s)^{-1}L_{\Lambda(s)V}\Lambda(s)^{-1} = L_V\Lambda(s)^{-1} - \Lambda(s)^{-1}L_V. \quad (16)$$

Note that in the above identities, the role of the multiplication from the left can be replaced by multiplication from the right (and also by commutator or anticommutator).

In the sequel we also deal with bilinear operators  $B : (V, W) \rightarrow B(V, W)$  which map  $V, W \in T_s\mathcal{S}$  to a vector in  $T_s\mathcal{S}$ . For such operators we introduce the following equivalence relation: two operators  $B_1, B_2$  are called equivalent,  $B_1 \simeq B_2$ , if  $B = B_1 - B_2$  is symmetric, that is  $B(V, W) = B(W, V)$  for all  $V, W$ .

For  $s \in \mathcal{S}$ ,  $V \in T_s\mathcal{S}$  fixed,  $W \mapsto B(V, W)$  defines a linear operator  $B_V$  on  $T_s\mathcal{S}$ . Obviously  $B_V$  depends linearly on  $V$ . This way bilinear operators are identified with operators depending linearly on  $V$ . This allows a certain simplification in our formulas as we can suppress the variable  $W$ . Furthermore we write  $(B_1)_V \simeq (B_2)_V$  if  $B_1 \simeq B_2$ .

**Example 6.** The bilinear form  $B(V, W) = \{V, W\}$  is identified with the operator  $B_V = L_V + R_V$ . Since  $\{V, W\}$  is symmetric, we can write

$$L_V \simeq -R_V.$$

Note that this also implies  $\Upsilon(s)L_V \simeq -\Upsilon(s)R_V$  for every operator-valued function  $\Upsilon(s)$ . Moreover, the form  $V(\Upsilon(s)W) + W(\Upsilon(s)V)$  is also symmetric and

$$L_V\Upsilon(s) \simeq -R_{\Upsilon(s)V}. \quad (17)$$

The following lemma is a straightforward generalization of Example 6.

**Lemma 7.** *If  $\Upsilon(s), \Xi(s)$  are operator-valued functions then*

$$L_{\Xi(s)V}\Upsilon(s) \simeq -R_{\Upsilon(s)V}\Xi(s).$$

**Proof.** This follows directly from the fact that the bilinear form  $(V, W) \mapsto (L_{\Xi(s)V}\Upsilon(s) + R_{\Upsilon(s)V}\Xi(s))W = (\Xi(s)V)(\Upsilon(s)W) + (\Xi(s)W)(\Upsilon(s)V)$  is symmetric.  $\square$

In particular, Lemma 7 gives the following identities

$$L_{(\Lambda(s)^{-1}V)(s)} \simeq -R_{sV}\Lambda(s)^{-1}, \quad (18)$$

$$L_{(\Lambda(s)^{-1}V)(\Lambda(s)s)} \simeq -R_{(\Lambda(s)s)V}\Lambda(s)^{-1}, \quad (19)$$

$$L_{(\Lambda(s)^{-1}V)s}\Lambda(s)^{-1} \simeq -R_{s(\Lambda(s)^{-1}V)}\Lambda(s)^{-1}. \quad (20)$$

To obtain (18) and (19) one can take e.g.  $\Xi(s) = \Lambda(s)^{-1}$ , and  $\Upsilon(s) = L_s$ ,  $\Upsilon(s) = L_{\Lambda(s)s}$ , respectively. The choice  $\Xi(s) = R_s\Lambda(s)^{-1}$ ,  $\Upsilon(s) = \Lambda(s)^{-1}$  leads to (20).

Finally, we need the following identity.

**Lemma 8.** *It holds*

$$\Lambda(s)^{-1}C_V\Lambda(s)^{-1} \simeq -R_{\Lambda(s)^{-1}V}\Lambda(s)^{-1}.$$

**Proof.** This follows from  $C_V \Lambda(s)^{-1} + \Lambda(s) R_{\Lambda(s)^{-1} V} \Lambda(s)^{-1} = C_V \Lambda(s)^{-1} + (R_{\Lambda(s)^{-1} V} \Lambda(s) + R_V) \Lambda(s)^{-1} = L_V \Lambda(s)^{-1} + R_{\Lambda(s)^{-1} V} \simeq 0$ , see Example 6.  $\square$

Now, we are in the position to give a direct proof of the hereditary property of  $\Psi(s)$ .

**Proof of Theorem 5.** We decompose the directional derivative of the nc Burgers recursion operator (14) as  $\Upsilon_V := \Psi'[V] = \sum_{j=1}^4 \Upsilon_{V,j}$  according to

$$\Upsilon_{V,1} = L_V, \quad \Upsilon_{V,2} = R_{\Lambda V} \Lambda^{-1}, \quad \Upsilon_{V,3} = -R_{[s,V]} \Lambda^{-1}, \quad \Upsilon_{V,4} = -R_{\Lambda s} \Lambda^{-1} C_V \Lambda^{-1}.$$

In what follows we calculate stepwise the bilinear forms

$$S_j(V, W) = \Psi \Upsilon_{V,j} W - \Upsilon_{\Psi V, j} W,$$

the aim being to show that  $S = \sum_{j=1}^4 S_j$  is symmetric. As above, it is convenient to suppress the dependence of the argument  $s$  in the operator-functions.

**Step 1:** In the following, we use the identities  $L_V L_W = L_{VW}$ ,  $R_V R_W = R_{WV}$ , and the fact that left- and right-multiplication commute. Thus,

$$\begin{aligned} S_1(V, \cdot) &= \Psi L_V - L_{\Psi V} \\ &= (\Lambda + R_s + R_{\Lambda s} \Lambda^{-1}) L_V - L_{\Lambda V + V s + (\Lambda^{-1} V)(\Lambda s)} \\ &= (\Lambda L_V - L_{\Lambda V}) + (R_s L_V - L_{V s}) + R_{\Lambda s} \Lambda^{-1} L_V - L_{(\Lambda^{-1} V)(\Lambda s)} \\ &\stackrel{(15)}{=} L_V \Lambda - (L_V R_s - L_V L_s) + R_{\Lambda s} \Lambda^{-1} L_V - L_{(\Lambda^{-1} V)(\Lambda s)} \\ &= \boxed{L_V \Lambda - L_V C_s} + R_{\Lambda s} \Lambda^{-1} L_V - \boxed{L_{(\Lambda^{-1} V)(\Lambda s)}}. \end{aligned} \tag{21}$$

Moreover,

$$\begin{aligned} S_2(V, \cdot) &= \Psi R_{\Lambda V} \Lambda^{-1} - R_{\Lambda \Psi V} \Lambda^{-1} \\ &= (\Lambda + R_s + R_{\Lambda s} \Lambda^{-1}) R_{\Lambda V} \Lambda^{-1} - R_{\Lambda(\Lambda V + V s + (\Lambda^{-1} V)(\Lambda s))} \Lambda^{-1} \\ &= (\Lambda R_{\Lambda V} - R_{\Lambda^2 V}) \Lambda^{-1} + (R_{(\Lambda V)s} - R_{\Lambda(Vs)}) \Lambda^{-1} \\ &\quad + R_{\Lambda s} \Lambda^{-1} R_{\Lambda V} \Lambda^{-1} - R_{\Lambda((\Lambda^{-1} V)(\Lambda s))} \Lambda^{-1} \\ &\stackrel{(15)}{=} R_{\Lambda V} - R_{V(\Lambda s)} \Lambda^{-1} + R_{\Lambda s} (\Lambda^{-1} R_{\Lambda V} \Lambda^{-1}) - R_{\Lambda((\Lambda^{-1} V)(\Lambda s))} \Lambda^{-1} \\ &\stackrel{(16)}{=} R_{\Lambda V} - R_{V(\Lambda s)} \Lambda^{-1} + R_{\Lambda s} (R_V \Lambda^{-1} - \Lambda^{-1} R_V) - R_{\Lambda((\Lambda^{-1} V)(\Lambda s))} \Lambda^{-1} \\ &= R_{\Lambda V} - R_{V(\Lambda s)} \Lambda^{-1} + (R_{V(\Lambda s)} \Lambda^{-1} - R_{\Lambda s} \Lambda^{-1} R_V) - R_{\Lambda((\Lambda^{-1} V)(\Lambda s))} \Lambda^{-1} \\ &= \boxed{R_{\Lambda V}} - R_{\Lambda s} \Lambda^{-1} R_V - \boxed{R_{\Lambda((\Lambda^{-1} V)(\Lambda s))} \Lambda^{-1}}. \end{aligned} \tag{22}$$

The identity  $[s, V]s = [s, Vs]$  allows us to show that the underlined terms in the following calculation cancel.

$$\begin{aligned}
 S_3(V, \cdot) &= -\Psi R_{[s,V]} \Lambda^{-1} + R_{[s,\Psi V]} \Lambda^{-1} \\
 &= -(\Lambda + R_s + R_{\Lambda s} \Lambda^{-1}) R_{[s,V]} \Lambda^{-1} + R_{[s,\Lambda V + Vs + (\Lambda^{-1}V)(\Lambda s)]} \Lambda^{-1} \\
 &= -\Lambda R_{[s,V]} \Lambda^{-1} - \underline{R_{[s,V]} s \Lambda^{-1}} - R_{\Lambda s} \Lambda^{-1} R_{[s,V]} \Lambda^{-1} \\
 &\quad + R_{[s,\Lambda V]} \Lambda^{-1} + \underline{R_{[s,Vs]} \Lambda^{-1}} + R_{[s,(\Lambda^{-1}V)(\Lambda s)]} \Lambda^{-1} \\
 &= -(\Lambda R_{[s,V]} - R_{[s,\Lambda V]}) \Lambda^{-1} - R_{\Lambda s} \Lambda^{-1} R_{[s,V]} \Lambda^{-1} + R_{[s,(\Lambda^{-1}V)(\Lambda s)]} \Lambda^{-1} \\
 &\stackrel{(15)}{=} -(R_{[s,V]} \Lambda + R_{\Lambda[s,V]} - R_{[s,\Lambda V]}) \Lambda^{-1} - R_{\Lambda s} \Lambda^{-1} R_{[s,V]} \Lambda^{-1} \\
 &\quad + R_{[s,(\Lambda^{-1}V)(\Lambda s)]} \Lambda^{-1} \\
 &= \boxed{-R_{[s,V]}} - \boxed{R_{[\Lambda s,V]} \Lambda^{-1}} - R_{\Lambda s} \Lambda^{-1} R_{[s,V]} \Lambda^{-1} + R_{[s,(\Lambda^{-1}V)(\Lambda s)]} \Lambda^{-1}. \quad (23)
 \end{aligned}$$

On application of (17) with  $\Upsilon = \Lambda$  and  $\Upsilon = C_s$ , respectively, we observe that the boxed terms in (21), (22) and (23) cancel out. Furthermore, for the double-boxed expressions, we get from (19)

$$\begin{aligned}
 L_{(\Lambda^{-1}V)(\Lambda s)} + R_{\Lambda(\Lambda^{-1}V)(\Lambda s)} \Lambda^{-1} + R_{[\Lambda s,V]} \Lambda^{-1} \\
 \simeq -R_{(\Lambda s)V} \Lambda^{-1} + R_{V(\Lambda s) + (\Lambda^{-1}V)(\Lambda^2 s)} \Lambda^{-1} + R_{(\Lambda s)V - V(\Lambda s)} \Lambda^{-1} \\
 = R_{(\Lambda^{-1}V)(\Lambda^2 s)} \Lambda^{-1}.
 \end{aligned}$$

Summing up we get

$$\begin{aligned}
 (S_1 + S_2 + S_3)(V, \cdot) &\simeq \boxed{R_{\Lambda s} \Lambda^{-1} C_V - R_{(\Lambda^{-1}V)(\Lambda^2 s)} \Lambda^{-1}} \\
 &\quad - R_{\Lambda s} \Lambda^{-1} R_{[s,V]} \Lambda^{-1} + R_{[s,(\Lambda^{-1}V)(\Lambda s)]} \Lambda^{-1}. \quad (24)
 \end{aligned}$$

**Step 2:** Next we calculate and simplify the remaining term, and indicate in a box those terms which cancel with the boxed terms in (24). Note that in the calculations below, terms corresponding to underlined expressions (at least partially) cancel out.

$$\begin{aligned}
 S_4(V, \cdot) &= -\Psi R_{\Lambda s} \Lambda^{-1} C_V \Lambda^{-1} + R_{\Lambda s} \Lambda^{-1} C_{\Psi V} \Lambda^{-1} \\
 &= -(\Lambda + R_s + R_{\Lambda s} \Lambda^{-1}) R_{\Lambda s} \Lambda^{-1} C_V \Lambda^{-1} + R_{\Lambda s} \Lambda^{-1} C_{\Lambda V + Vs + (\Lambda^{-1}V)(\Lambda s)} \Lambda^{-1} \\
 &= -(\Lambda R_{\Lambda s}) \Lambda^{-1} C_V \Lambda^{-1} + R_{\Lambda s} (\Lambda^{-1} C_{\Lambda V} \Lambda^{-1}) - (R_s + R_{\Lambda s} \Lambda^{-1}) R_{\Lambda s} \Lambda^{-1} C_V \Lambda^{-1} \\
 &\quad + R_{\Lambda s} \Lambda^{-1} C_{Vs + (\Lambda^{-1}V)(\Lambda s)} \Lambda^{-1} \\
 &\stackrel{(15),(16)}{=} -\underline{R_{\Lambda s} \Lambda + R_{\Lambda^2 s}} \Lambda^{-1} C_V \Lambda^{-1} + R_{\Lambda s} (\underline{C_V \Lambda^{-1}} - \Lambda^{-1} C_V) \\
 &\quad - (R_s + R_{\Lambda s} \Lambda^{-1}) R_{\Lambda s} \Lambda^{-1} C_V \Lambda^{-1} + R_{\Lambda s} \Lambda^{-1} C_{Vs + (\Lambda^{-1}V)(\Lambda s)} \Lambda^{-1}
 \end{aligned}$$

$$\begin{aligned}
&= -(R_{\Lambda^2 s} + (R_s + R_{\Lambda s} \Lambda^{-1}) R_{\Lambda s})(\Lambda^{-1} C_V \Lambda^{-1}) \\
&\quad - R_{\Lambda s} \Lambda^{-1} C_V + R_{\Lambda s} \Lambda^{-1} C_{V s + (\Lambda^{-1} V)(\Lambda s)} \Lambda^{-1} \\
&\stackrel{\text{L8}}{\simeq} (R_{\Lambda^2 s} + R_{(\Lambda s)s} + R_{\Lambda s} \Lambda^{-1} R_{\Lambda s}) R_{\Lambda^{-1} V} \Lambda^{-1} \\
&\quad - R_{\Lambda s} \Lambda^{-1} C_V + R_{\Lambda s} \Lambda^{-1} C_{V s + (\Lambda^{-1} V)(\Lambda s)} \Lambda^{-1} \\
&= R_{(\Lambda^{-1} V)(\Lambda^2 s)} \Lambda^{-1} + (R_{(\Lambda^{-1} V)(\Lambda s)s} + R_{\Lambda s} \Lambda^{-1} \underline{R_{(\Lambda^{-1} V)(\Lambda s)}}) \Lambda^{-1} \\
&\quad - R_{\Lambda s} \Lambda^{-1} C_V + R_{\Lambda s} \Lambda^{-1} (C_{V s} + \underline{C_{(\Lambda^{-1} V)(\Lambda s)}}) \Lambda^{-1} \\
&= \boxed{R_{(\Lambda^{-1} V)(\Lambda^2 s)} \Lambda^{-1} - R_{\Lambda s} \Lambda^{-1} C_V} + R_{(\Lambda^{-1} V)(\Lambda s)s} \Lambda^{-1} \\
&\quad + R_{\Lambda s} \Lambda^{-1} C_{V s} \Lambda^{-1} + R_{\Lambda s} \Lambda^{-1} L_{(\Lambda^{-1} V)(\Lambda s)} \Lambda^{-1}. \tag{25}
\end{aligned}$$

It remains to compare the terms in (24) and (25) which are not boxed. In this connection, we first observe that it is sufficient to verify

$$-R_{\Lambda s} \Lambda^{-1} R_{sV} \Lambda^{-1} + R_{s(\Lambda^{-1} V)(\Lambda s)} \Lambda^{-1} \simeq -R_{\Lambda s} \Lambda^{-1} (L_{V s} + L_{(\Lambda^{-1} V)(\Lambda s)}) \Lambda^{-1}.$$

Indeed,

$$\begin{aligned}
R_{\Lambda s} \Lambda^{-1} (L_{V s} + L_{(\Lambda^{-1} V)(\Lambda s)}) \Lambda^{-1} &= R_{\Lambda s} (\Lambda^{-1} L_{\Lambda((\Lambda^{-1} V)(s))} \Lambda^{-1}) \\
&\stackrel{(16)}{=} R_{\Lambda s} (L_{(\Lambda^{-1} V)(s)} \Lambda^{-1} - \Lambda^{-1} L_{(\Lambda^{-1} V)(s)}) \\
&= R_{\Lambda s} (L_{(\Lambda^{-1} V)(s)} \Lambda^{-1}) - R_{\Lambda s} \Lambda^{-1} (L_{(\Lambda^{-1} V)(s)}) \\
&\stackrel{(20),(18)}{=} -R_{\Lambda s} (R_{s(\Lambda^{-1} V)} \Lambda^{-1}) + R_{\Lambda s} \Lambda^{-1} (R_{sV} \Lambda^{-1}) \\
&= R_{s(\Lambda^{-1} V)(\Lambda s)} \Lambda^{-1} - R_{\Lambda s} \Lambda^{-1} R_{sV} \Lambda^{-1}.
\end{aligned}$$

Thus the non-boxed terms in (24) and (25) also cancel.

As a consequence of Step 1 and 2, we have shown  $S(V, \cdot) \simeq 0$ . This completes the proof of Theorem 5, and hence also the proof of Theorem 3.  $\square$

## References

- [1] S. Carillo and B. Fuchssteiner, The abundant symmetry structure of hierarchies of nonlinear equations obtained by reciprocal links, *J. Math. Phys.* **30** (1989) 1606–1613.
- [2] S. Carillo and C. Schiebold, Noncommutative KdV and mKdV hierarchies via recursion methods, *J. Math. Phys.* **50** (2009) 073510.
- [3] S. Carillo and C. Schiebold, A non-commutative operator-hierarchy of Burgers equations and Bäcklund transformations, in *Applied and Industrial Mathematics in Italy III: Selected Contributions from the 9th SIMAI Conf.*, 15–19 September 2008, eds. E. De Bernardis, R. Spigler and V. Valente, Series on Advances in Mathematics for Applied Sciences, Vol. 82 (World Scientific, 2010), pp. 175–185.
- [4] S. Carillo and C. Schiebold, Matrix Korteweg–de Vries and modified Korteweg–de Vries hierarchies: Noncommutative soliton solutions, *J. Math. Phys.* **52** (2011) 053507.
- [5] A. S. Fokas and B. Fuchssteiner, Bäcklund transformation for hereditary symmetries, *Nonlinear Anal.* **5**(4) (1981) 423–432.

- [6] B. Fuchssteiner, Application of hereditary symmetries to nonlinear evolution equations, *Nonlinear Anal.* **3**(6) (1979) 849–862.
- [7] B. Fuchssteiner, Linear aspects in the theory of solitons and nonlinear integrable equations, *J. Phys. Soc. Japan* **60** (1991) 1473–1496.
- [8] B. Fuchssteiner, W. Oevel and W. Wiwianka, Computer-algebra methods for investigating hereditary operators of higher order soliton equations, *Comput. Phys. Comm.* **44** (1987) 47–55.
- [9] M. Gürses, A. Karasu and V. V. Sokolov, On construction of recursion operators from Lax representation, *J. Math. Phys.* **40** (1999) 6473–6490.
- [10] M. Gürses, A. Karasu and R. Turhan, On non-commutative integrable Burgers equations, *J. Nonlinear Math. Phys.* **17** (2010) 1–6.
- [11] B. A. Kupershmidt, On a group of automorphisms of the noncommutative Burgers hierarchy, *J. Nonlinear Math. Phys.* **12**(4) (2005) 539–549.
- [12] D. Levi, O. Ragnisco and M. Bruschi, Continuous and discrete matrix Burgers' hierarchies, *Il Nuovo Cimento B* **74** (1983) 33–51.
- [13] P. J. Olver, Evolution equations possessing infinitely many symmetries, *J. Math. Phys.* **18** (1977) 1212–1215.
- [14] P. J. Olver and V. V. Sokolov, Integrable evolution equations on nonassociative algebras, *Comm. Math. Phys.* **193** (1998) 245–268.
- [15] P. J. Olver and J. P. Wang, Classification of integrable one-component systems on associative algebras, *Proc. London Math. Soc.* **81** (2000) 566–586.
- [16] C. Rogers and W. K. Schief, *Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory*, Cambridge Texts in Applied Mathematics (Cambridge University Press, Cambridge, 2002).
- [17] C. Rogers and W. F. Shadwick, *Bäcklund Transformations and Their Applications*, Mathematics in Science and Engineering, Vol. 161 (Academic Press, New York, 1982).
- [18] C. Schiebold, Structural properties of the noncommutative KdV recursion operator, *J. Math. Phys.* **52** (2011) 113504.