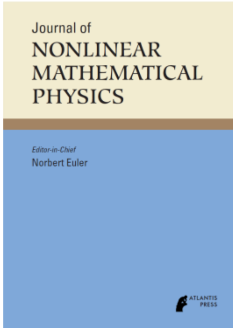

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Anca-Voichita Matioc

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STEADY INTERNAL WATER WAVES WITH A CRITICAL LAYER BOUNDED BY THE WAVE SURFACE

ANCA-VOICHITA MATIOC

Institute for Applied Mathematics
Leibniz University of Hanover, Welfengarten 1
30655 Hanover, Germany
matioca@ifam.uni-hannover.de

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In this paper we construct small amplitude periodic internal waves traveling at the boundary region between two rotational and homogeneous fluids with different densities. Within a period, the waves we obtain have the property that the gradient of the stream function associated to the fluid beneath the interface vanishes, on the wave surface, at exactly two points. Furthermore, there exists a critical layer which is bounded from above by the wave profile. Besides, we prove, without excluding the presence of stagnation points, that if the vorticity function associated to each fluid in part is real-analytic, bounded, and non-increasing, then capillary-gravity steady internal waves are *a priori* real-analytic. Our new method provides the real-analyticity of capillary and capillary-gravity waves with stagnation points traveling over a homogeneous rotational fluid under the same restrictions on the vorticity function.

Keywords: Internal waves; streamlines; vorticity; real-analytic.

Mathematics Subject Classification: 35Q35, 76B45, 76B55, 37N10

1. Introduction

In this paper we consider two-dimensional internal periodic waves traveling at the interface between two layers of immiscible fluids with different densities, under the rigid lid assumption. In our context, the fluids have constant vorticity and we construct internal traveling waves with a critical layer and stagnation points in both gravity and capillary-gravity regimes.

A critical layer is a region of fluid consisting entirely of closed streamlines, and stagnation points are fluid particles traveling horizontally with the same speed as the wave. Flows with stagnation points are known [5] to be relevant for the description of background states for tsunamis. Concerning waves traveling over a homogeneous fluid, it was observed in [3, 7] that the strong elliptic maximum principle rules out the existence of smooth irrotational waves with stagnation points or critical layers. On the other hand, it was shown in [31–34] that there exist extreme Stokes waves which are Lipschitz continuous and have stagnation points and a sharp corner at the crest. For the rotational case the picture is different. Existence of exact periodic traveling gravity waves with general vorticity has been established first in [6] by means of bifurcation and degree theory, and in [8] in the case of waves with bounded and discontinuous vorticity. By construction, the waves in [6, 8] do not possess stagnation points and critical layers. Linear gravity water waves with stagnation points which travel on currents with constant vorticity were studied in [15], and the exact picture of the waves has been obtained first in [35], and later on in [9] by employing complex methods. These waves possess at most a critical layer which is located within the fluid body.

One of the factors which can determine the presence of critical layers is the vertical stratification of the fluid. It was recently shown in [19] that continuously stratified gravity waves with constant vorticity may possess two critical layers and the qualitative picture of the streamlines may be different than that for homogeneous flows [9, 35]. Another factor is the presence of an affine vorticity distribution, situation analyzed in [13, 14], when the waves may possess arbitrarily many critical layers. For a survey on traveling waves with critical layers we refer to the article [16].

Internal waves are usually created by the presence of two different layers of water combined with a certain configuration of relief and current. They form where the water above and below the interface is either moving in opposite directions or in the same direction at different speeds. A well-known example are the internal waves which move from the Atlantic Ocean to the Mediterranean Sea, at the east of the Strait of Gibraltar. In this case the two layers correspond to different salinities, whereas the current is caused by the tide passing through the strait. The mathematical theory of waves at the interface between two layers of immiscible fluid of different densities has attracted a lot of interest and we refer to the survey article of Helfrich and Melville [20] which provides a good overview on steady internal solitary waves in such systems.

In this paper we construct periodic steady internal waves between two layers of homogeneous fluids with different densities in the gravity and capillary-gravity regimes. Particularly, we find solutions which possess exactly one critical layer which is bounded from above by the wave profile. Moreover, the gradient of the stream function associated to the fluid beneath the interface vanishes at exactly two points on the interface, meaning that, close to these points, fluid particles located below the interface move almost horizontally with velocity approaching the wave speed. The fluid above the interface does not sense the presence of these stagnation points. In contrast to the situation in [19], where waves with critical layers were constructed in an unstable regime, the solutions we find are stably stratified. While in [31, 33, 34] the wave surface is only Lipschitz continuous in a neighborhood of the stagnation point, our solutions are real-analytic. In the pure gravity case this follows by using a regularity result for free boundary value problems, cf. [26], whereas when capillary plays a role we provide a new idea based on parabolic theory. It is worthwhile to mention that our method may be applied to prove the *a priori* analyticity of water waves traveling over an homogeneous fluid, when capillary effects are incorporated, provided the vorticity function is bounded, analytic, and non-increasing even if stagnation points exist (situation complementary to that analyzed in [21, 23]).

We determine, by using higher order expansions and elliptic maximum principles, a precise picture of the streamlines in the fluids which could be used to describe the particle trajectories. Due to the analyticity of the interface, one could choose also linear theory, similarly as in [10, 15, 25, 29], to obtain an approximative picture of the exact particle paths.

The outline of the paper is as follows: in Sec. 2 we present the mathematical model, re-express the problem in terms of the stream functions and provide the main regularity result Theorem 2.1. In Sec. 3 we write the problem as a nonlocal equation for the wave profile, and use bifurcation theory to prove the existence statement Theorem 3.1 as well as higher order expansion for the bifurcation curves. In the last section, based on Theorem 2.1, we illustrate the precise picture of the streamlines for some of the waves obtained in Theorem 2.1.

2. The Governing Equations and a Regularity Result

In this section we present the governing equations for two-dimensional internal water waves traveling at the boundary region between two rotational fluids with different densities under the rigid lid condition at the top. The bottom of the ocean is assumed to be flat and we denote with $\mathbb{S}$ the unit circle, i.e. $\mathbb{S}$ stands for $\mathbb{R}/2\pi\mathbb{Z}$. The fluid domain $\Omega := \mathbb{S}\times(-1,1)$ contains two fluids layers separated by a sharp interface $y=\eta(t,x)$, the wave profile, which defines the two subsets

$$\Omega_\eta^b := \{(x,y) : x\in\mathbb{S} \text{ and } -1<y<\eta(t,x)\},\quad \Omega_\eta^t := \{(x,y) : x\in\mathbb{S} \text{ and } \eta(t,x)<y<1\}.$$

The domain Ω_η^b contains a Newtonian fluid with constant density ρ, velocity field (u,v), and pressure P, and we denote by $\overline{\rho}$, $(\overline{u},\overline{v})$, and $\overline{P}$ the density, velocity, and pressure of the fluid located at the top. The line $y=-1$ is the impermeable bottom of the ocean and $y=1$ is the rigid top of the two-fluid system. Assuming that both fluids are inviscid, the dynamics can be described by Euler's equations (see [27] for a justification of the inviscid flow). Being interested in traveling internal waves, we presuppose that there exists a positive constant c, the wave speed, such that $\eta(t,x)=\eta(x-ct)$,

$$(u,v,P)(t,x,y)=(u,v,P)(x-ct,y),\quad (\overline{u},\overline{v},\overline{P})(t,x,y)=(\overline{u},\overline{v},\overline{P})(x-ct,y),$$

and formulate the problem in a frame moving with the wave. The equations of motion are the steady state two-dimensional Euler equations

$$\begin{cases}(u-c)u_x+vu_y=-P_x/\rho & \text{in } \Omega_\eta^b,\\ (u-c)v_x+vv_y=-P_y/\rho-g & \text{in } \Omega_\eta^b,\\ u_x+v_y=0 & \text{in } \Omega_\eta^b,\end{cases}\quad \begin{cases}(\overline{u}-c)\overline{u}_x+\overline{v}\,\overline{u}_y=-\overline{P}_x/\overline{\rho} & \text{in } \Omega_\eta^t,\\ (\overline{u}-c)\overline{v}_x+\overline{v}\,\overline{v}_y=-\overline{P}_y/\overline{\rho}-g & \text{in } \Omega_\eta^t,\\ \overline{u}_x+\overline{v}_y=0 & \text{in } \Omega_\eta^t,\end{cases} \tag{2.1a}$$

subjected to the boundary conditions, see e.g. [2, 6],

$$\begin{cases}v=0 & \text{on } y=-1,\\ \overline{v}=0 & \text{on } y=1,\end{cases}\quad \text{and}\quad \begin{cases}P=\overline{P}-\sigma\eta''/(1+\eta'^2)^{3/2} & \text{on } y=\eta(x),\\ v=(u-c)\eta' & \text{on } y=\eta(x),\\ \overline{v}=(\overline{u}-c)\eta' & \text{on } y=\eta(x),\end{cases} \tag{2.1b}$$

with $\sigma\geq 0$ being the surface tension coefficient and g the constant of gravity. To be more precise, we are interested in finding solutions of problem (2.1) within the class

$$\eta\in C^{2+\alpha}(\mathbb{S}),\quad (u,v,P)\in(C^{1+\alpha}(\overline{\Omega_\eta^b}))^3,\quad (\overline{u},\overline{v},\overline{P})\in(C^{1+\alpha}(\overline{\Omega_\eta^t}))^3 \tag{2.2}$$

for some $\alpha \in (0,1)$. Similarly as in [6], we reformulate the problem (2.1) by introducing the stream functions $\psi : \Omega_\eta^b \to \mathbb{R}$ and $\overline{\psi} : \Omega_\eta^t \to \mathbb{R}$ defined by the relations

$$\psi(x,y) := m + \int_{-1}^{y} (u(x,s) - c)\, ds \quad \text{and} \quad \overline{\psi}(x,y) := \int_{\eta(x)}^{y} (\overline{u}(x,s) - c)\, ds.$$

Here m is a positive constant fixed such that $\psi = 0$ on $y = \eta(x)$. Indeed, taking into account that $\nabla\psi = (-v, u - c)$ and $\nabla\overline{\psi} = (-\overline{v}, \overline{u} - c)$ it follows, by using the chain rule and (2.1b), that ψ is constant on $y = \eta(x)$ and we can make this choice. Similarly, $\overline{\psi}$ is constant on the rigid lid $y = 1$, and we let $\overline{m} := \psi(0,1)$ be this constant. The above properties show that the streamlines coincide with the level curves of the stream functions. Both fluids being rotational, we introduce their vorticity by

$$\omega := u_y - v_x = \Delta\psi \quad \text{in } \Omega_\eta^b \quad \text{and} \quad \overline{\omega} := \overline{u}_y - \overline{v}_x = \Delta\overline{\psi} \quad \text{in } \Omega_\eta^t.$$

Next, we assume that

$$u - c < 0 \quad \text{in } \overline{\Omega_\eta^b} \quad \text{and} \quad \overline{u} - c < 0 \quad \text{in } \overline{\Omega_\eta^t}, \tag{2.3}$$

and exclude so the presence of stagnation points in the fluids. However, this assumption (2.3) will be dropped later on, after expressing the problem (2.1) in terms of the stream functions. This will allow us to find solutions of system (2.1) which do possess stagnation points.

Condition (2.3) guarantees [6, 30] that the vorticity is a single-valued function of the stream function, that is, there exist $\gamma \in C^\alpha([-m,0])$ and $\overline{\gamma} \in C^\alpha([0,-\overline{m}])$, the vorticity functions, with

$$\omega(x,y) = \gamma(-\psi(x,y)), \quad (x,y) \in \Omega_\eta^t \quad \text{and} \quad \overline{\omega}(x,y) = \overline{\gamma}(-\overline{\psi}(x,y)), \quad (x,y) \in \Omega_\eta^b.$$

Since, by Bernoulli's principle, the quantities

$$\frac{(u-c)^2 + v^2}{2} + \frac{P}{\rho} + gy - \int_0^{\psi} \gamma(-s)\, ds \quad \text{and} \quad \frac{(\overline{u}-c)^2 + \overline{v}^2}{2} + \frac{\overline{P}}{\overline{\rho}} + gy - \int_0^{\overline{\psi}} \overline{\gamma}(-s)\, ds$$

are constant in Ω_η^t and Ω_η^b, respectively, we find, by restricting to $y = \eta(x)$ that

$$\frac{|\nabla\psi|^2}{2} + gy + \frac{P}{\rho} = \text{const.} \quad \text{and} \quad \frac{|\nabla\overline{\psi}|^2}{2} + gy + \frac{\overline{P}}{\overline{\rho}} = \text{const.}$$

To conclude, we observe that ψ and $\overline{\psi}$ are solutions of the semilinear Dirichlet problems

$$\begin{cases} \Delta\psi = \gamma(-\psi) & \text{in } \Omega_\eta^b, \\ \psi = 0 & \text{on } y = \eta(x), \\ \psi = m & \text{on } y = -1, \end{cases} \quad \text{and} \quad \begin{cases} \Delta\overline{\psi} = \overline{\gamma}(-\overline{\psi}) & \text{in } \Omega_\eta^t, \\ \overline{\psi} = 0 & \text{on } y = \eta(x), \\ \overline{\psi} = \overline{m} & \text{on } y = 1, \end{cases} \tag{2.4a}$$

respectively, and they are coupled by the following equation

$$\overline{\rho}|\nabla\overline{\psi}|^2 - \rho|\nabla\psi|^2 + 2g(\overline{\rho} - \rho)y + \frac{2\sigma\eta''}{(1+\eta'^2)^{3/2}} = Q \quad \text{on } y = \eta(x) \tag{2.4b}$$

for some constant $Q \in \mathbb{R}$. If we integrate this last Eq. (2.4b) over the unit circle, we obtain, by taking into account that $\eta''/(1+\eta'^2)^{3/2} = (\eta'/(1+\eta'^2)^{1/2})'$, that

$$Q = \int_{\mathbb{S}} (\overline{\rho}|\nabla\overline{\psi}|^2(x,\eta(x)) - \rho|\nabla\psi|^2(x,\eta(x)) + 2g(\overline{\rho}-\rho)\eta(x))\,dx. \tag{2.4c}$$

The integral over $\mathbb{S}$ is normalized such that $\int_{\mathbb{S}} 1\,dx = 1$. It is not difficult to check that given two vorticity functions γ and $\overline{\gamma}$ and a classical solution $(\eta,\psi,\overline{\psi})$ of (2.4), then we can associate to this solution a unique solution of problem (2.1), even if the condition (2.3) is not satisfied. This formulation of problem (2.1) is much more convenient because we deal with two coupled semilinear elliptic problems. If γ and $\overline{\gamma}$ are such that we may solve (2.4a) for given η, which is definitely the case when the fluids have constant vorticity, then (2.4) reduces to the problem of determining η. As in [35], we remark that the solutions of (2.4) are solutions of (2.1) for any value $c \in (0,\infty)$ of the wave speed.

As a first result, we use in the pure gravity case $\sigma = 0$ a theorem on the regularity of free interfaces from [26], which was employed in [4, 21, 22, 30] to establish analyticity of traveling water waves without stagnation points on the profile in different wave regimes, to prove that internal traveling waves with real-analytic vorticity functions are *a priori* real-analytic. When we incorporate surface tension effects, we provide a new idea, which is based on parabolic theory, and show the real-analyticity without excluding the existence of stagnation points.

Theorem 2.1. *Let $\sigma \in [0,\infty)$ and assume that γ and $\overline{\gamma}$ are constant functions. Given a solution $(\eta,\psi,\overline{\psi})$ of* (2.4) *within the class* (2.2), *we assume, when $\sigma = 0$, that*

$$|\nabla\psi|^2 + |\nabla\overline{\psi}|^2 > 0 \quad \text{on } y = \eta(x). \tag{2.5}$$

Then, the wave profile η is real-analytic $\eta \in C^\omega(\mathbb{S})$.

Remark 2.1. It is clear from the proof of Theorem 2.1 that its assertion is true in the case $\sigma = 0$ for arbitrary analytic vorticity functions $\gamma, \overline{\gamma}$, provided (2.5) is fulfilled. When $\sigma > 0$, and $\gamma, \overline{\gamma} \in C^\omega(\mathbb{R})$ are bounded and satisfy additionally

$$\gamma' \le 0 \quad \text{and} \quad \overline{\gamma}' \le 0, \tag{2.6}$$

then the claim of Theorem 2.1 is still true. Relation (2.6) ensures the unique solvability of the Dirichlet problems (2.4a), see [24, Theorem 3.3], which permits us to express the problem (2.4) by the Eq. (2.7). Our method may be used to prove analyticity of the profile for traveling water waves with stagnation points in the capillary and capillary-gravity regime, provided the vorticity function is non-increasing, cf. [22, 23].

Proof of Theorem 2.1. Assuming $\sigma = 0$ first, we re-write the coupling Eq. (2.4b) in a different way. Since the boundary conditions on $y = \eta(x)$ of both problems (2.4a) imply that the tangential component of $\nabla\psi$ and $\nabla\overline{\psi}$ vanish at the wave profile, this implies

$$|\nabla\psi|^2 = (\partial_\nu\psi)^2 \quad \text{and} \quad |\nabla\overline{\psi}|^2 = (\partial_\nu\overline{\psi})^2 \quad \text{on } y = \eta(x).$$

Here, $\nu := (-\eta',1)/(1+\eta'^2)^{1/2}$ is the unit normal at $y = \eta(x)$. With this notation, Eq. (2.4b) is equivalent to the following relation

$$f(x,y,\partial_\nu\overline{\psi},\partial_\nu\psi) = \overline{\rho}(\partial_\nu\overline{\psi})^2 - \rho(\partial_\nu\psi)^2 + 2g(\overline{\rho}-\rho)y - Q = 0 \quad \text{on } y = \eta(x).$$

Clearly, f is real-analytic in all its arguments. Furthermore, our assumption (2.5) implies that

$$f_{\partial_\nu \overline{\psi}}\partial_\nu \overline{\psi} - f_{\partial_\nu \psi}\partial_\nu \psi = 2(\overline{\rho}(\partial_\nu \overline{\psi})^2 + \rho(\partial_\nu \psi)^2) \neq 0$$

on $y = \eta(x)$, which shows that the assumptions of [26, Theorem 3.2] (see also the Remark following it) are all satisfied and the claim follows at once.

The proof in the case when $\sigma \neq 0$ is different. Indeed, we can reduce the problem (2.4) (see e.g. Sec. 3) to an operator equation

$$\frac{\eta''}{(1+\eta'^2)^{3/2}} + \phi(\eta) = 0, \tag{2.7}$$

where setting

$$\mathcal{U} := \{\eta \in C^{2+\alpha}(\mathbb{S}) : |\eta| < 1\}, \tag{2.8}$$

we have that $\phi : \mathcal{U} \to C^{1+\alpha}(\mathbb{S})$ is a real-analytic operator with $\phi(C^\infty(\mathbb{S})) \subset C^\infty(\mathbb{S})$. Because we use parabolic theory later on in the proof, we introduce the small Hölder spaces $h^{k+\alpha}(\mathbb{S})$, $k \in \mathbb{N}$, as the closure of the smooth functions $C^\infty(\mathbb{S})$ in $C^{k+\alpha}(\mathbb{S})$. We define now $D := \mathcal{U} \cap h^{2+\alpha}(\mathbb{S})$, and observe that (2.7) may be written as

$$\eta' = \frac{\eta''}{(1+\eta'^2)^{3/2}} + \varphi(\eta), \tag{2.9}$$

with $\varphi \in C^\omega(D, h^{1+\alpha}(\mathbb{S}))$ given by $\varphi(\eta) := \eta' + \phi(\eta)$, $\eta \in D$. Indeed, one can easily see from (2.9) that $\eta \in C^\infty(\mathbb{S})$, which implies $\eta \in D$.

Let now $\xi : \mathbb{R} \to h^{2+\alpha}(\mathbb{S})$ be the function defined by $\xi(t,x) = \eta(t+x)$. If we remark that $\varphi(\eta(t+\cdot)) = \varphi(\eta)(t+\cdot)$ for all $t \in \mathbb{R}$, which is a direct consequence of the unique solvability of (2.4a) and of the fact that the variable x does not interfere into (2.4a), we find

$$\begin{aligned}\partial_t \xi(t,x) = \partial_x \eta(t+x) &= \frac{\partial_x^2 \xi}{(1+(\partial_x \xi)^2)^{1/2}}(t,x) + \varphi(\eta)(t+x) \\ &= \frac{\partial_x^2 \xi}{(1+(\partial_x \xi)^2)^{1/2}}(t,x) + \varphi(\eta(t+\cdot))(x) = \frac{\partial_x^2 \xi}{(1+(\partial_x \xi)^2)^{1/2}}(t,x) + \varphi(\xi(t))(x).\end{aligned}$$

Whence, ξ is a solution of the autonomous problem

$$\partial_t \xi = \Phi(\xi), \quad t > 0, \quad \xi(0) = \eta, \tag{2.10}$$

where $\Phi \in C^\omega(D, h^\alpha(\mathbb{S}))$ is the mapping

$$\Phi(\xi) = \frac{\partial_x^2 \xi}{(1+(\partial_x \xi)^2)^{1/2}} + \varphi(\xi), \quad \xi \in D.$$

Given $\xi \in D$, we have $\partial\Phi(\xi)[\zeta] = (1+(\partial_x \xi)^2)^{-1/2}\partial_x^2 \zeta + \mathbb{A}\zeta$ with $\mathbb{A} \in \mathcal{L}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))$. Using well known interpolation properties of the small Hölder spaces:

$$(h^{\sigma_0}(\mathbb{S}), h^{\sigma_1}(\mathbb{S}))_\theta = h^{(1-\theta)\sigma_0 + \theta\sigma_1}(\mathbb{S}),$$

if $\theta \in (0,1)$ and $(1-\theta)\sigma_0 + \theta\sigma_1 \in \mathbb{R}^+\setminus\mathbb{N}$, where $(\cdot,\cdot)_\theta$ denotes the continuous interpolation method of DaPrato and Grisvard [12] (see also [1, 28]), we obtain from [28, Corollary 3.1.9, Propositions 2.2.7 and 2.4.1] that the Fréchet derivative $\partial\Phi(\xi)$ is the generator of an analytic semigroup in $\mathcal{L}(h^\alpha(\mathbb{S}))$. In [28, Corollary 8.4.6] ensures that the unique solution of (2.10) to the initial data $\xi(0) = \eta$ is analytic, that is $\xi \in C^\omega((0,\infty), h^{2+\alpha}(\mathbb{S}))$, which implies the desired assertion. □

3. Bifurcation Analysis and the Main Result

In the remainder of this paper we restrict our analysis to the stable regime when the fluid in Ω_η^b is more dense than that located above, that is $\rho > \overline{\rho}$, and the fluids have constant vorticity

$$\omega = \gamma \in \mathbb{R} \quad \text{and} \quad \overline{\omega} = \overline{\gamma} \in \mathbb{R}. \tag{3.1}$$

Furthermore, the surface tension coefficient may take any value $\sigma \in [0,\infty)$.

As a first step, we introduce a constant λ into the problem which will allow us to describe the laminar flow solutions of (2.4), i.e. solutions with a flat wave profile, located at $y = 0$. Later on, we use this constant as a bifurcation parameter to find non-flat solutions of (2.4). To this end, when $\eta = 0$, we observe that the functions $(\psi, \overline{\psi}) =: (\psi_0, \overline{\psi}_0)$ solving (2.4a) are given by

$$\psi_0(x,y) := \frac{\omega y^2}{2} \quad \text{in } \Omega_0^b \quad \text{and} \quad \overline{\psi}_0(x,y) := \frac{\overline{\omega} y^2}{2} + \lambda y \quad \text{in } \Omega_0^t, \tag{3.2}$$

provided the constants m and $\overline{m}$ satisfy

$$m = \frac{\omega}{2} \quad \text{and} \quad \overline{m} = \lambda + \frac{\overline{\omega}}{2}. \tag{3.3}$$

Hence, given $\lambda \in \mathbb{R}$, the tuple $(\eta, \psi, \overline{\psi}) := (0, \psi_0, \overline{\psi}_0)$ is a solutions of (2.4) if the constants $m, \overline{m}$, and Q are given by (3.3) and (2.4c).

In order to determine non-flat solutions of (2.4) we re-write problem (2.4) as a nonlinear and nonlocal operator equation having (λ, η) as unknowns. Because the domains where the Dirichlet problems (2.4a) are posed depend upon η, we need to transform these problems and the Eq. (2.4b) on fixed reference manifolds. Therefore, we define the functions $\phi_\eta : \Omega_0^b \to \Omega_\eta^b$ and $\overline{\phi}_\eta : \Omega_0^t \to \Omega_\eta^t$ by the relations

$$\phi_\eta(x,y) := (x, y + (1+y)\eta(x)) \quad \text{in } \Omega_0^b \quad \text{and}$$

$$\overline{\phi}_\eta(x,y) := (x, y + (1-y)\eta(x)) \quad \text{in } \Omega_0^t,$$

and observe that if η belongs to $\mathcal{U}$ (see relation (2.8)), then ϕ_η and $\overline{\phi}_\eta$ are diffeomorphisms. We use these diffeomorphisms to transform the Laplace operator into a differential operator on the fixed domains Ω_0^b and Ω_0^t, respectively. More precisely, setting

$$\mathcal{A}(\eta)w := (\Delta(w \circ \phi_\eta^{-1})) \circ \phi_\eta, \quad w \in C^{2+\alpha}(\overline{\Omega_0^b}),$$

$$\overline{\mathcal{A}}(\eta)\overline{w} := (\Delta(\overline{w} \circ \overline{\phi}_\eta^{-1})) \circ \overline{\phi}_\eta, \quad \overline{w} \in C^{2+\alpha}(\overline{\Omega_0^t}),$$

we find the following expressions for the differential operators $\mathcal{A}(\eta)$ and $\overline{\mathcal{A}}(\eta)$:

$$\mathcal{A}(\eta)=\partial_{xx}^2-2\frac{(1+y)\eta'}{1+\eta}\partial_{xy}^2+\left(\frac{(1+y)^2\eta'^2}{(1+\eta)^2}+\frac{1}{(1+\eta)^2}\right)\partial_{yy}^2-(1+y)\frac{(1+\eta)\eta''-2\eta'^2}{(1+\eta)^2}\partial_y,$$

$$\overline{\mathcal{A}}(\eta)=\partial_{xx}^2-2\frac{(1-y)\eta'}{1-\eta}\partial_{xy}^2+\left(\frac{(1-y)^2\eta'^2}{(1-\eta)^2}+\frac{1}{(1-\eta)^2}\right)\partial_{yy}^2-(1-y)\frac{(1-\eta)\eta''+2\eta'^2}{(1-\eta)^2}\partial_y$$

for all $\eta\in\mathcal{U}$. From these explicit expressions we can easily see that the functions $\eta\mapsto\mathcal{A}(\eta)$ and $\eta\mapsto\overline{\mathcal{A}}(\eta)$ are both real-analytic. Furthermore, corresponding to the coupling condition (2.4b) we define the boundary operators $\mathcal{B}:\mathcal{U}\times C^{2+\alpha}(\overline{\Omega_0^b})\to C^{1+\alpha}(\mathbb{S})$ and $\overline{\mathcal{B}}:\mathcal{U}\times C^{2+\alpha}(\overline{\Omega_0^t})\to C^{1+\alpha}(\mathbb{S})$ by the relations

$$\mathcal{B}(\eta,w):=\operatorname{tr}(|\nabla(w\circ\phi_\eta^{-1})|^2\circ\phi_\eta)\quad\text{and}\quad\overline{\mathcal{B}}(\eta,w):=\operatorname{tr}(|\nabla(\overline{w}\circ\overline{\phi}_\eta^{-1})|^2\circ\overline{\phi}_\eta),$$

with tr being the trace operator with respect to the line $\mathbb{S}\times\{0\}\cong\mathbb{S}$. For these operators we find

$$\mathcal{B}(\eta,w):=\operatorname{tr}(\partial_x w)^2-\frac{2\eta'}{1+\eta}\operatorname{tr}\partial_x w\operatorname{tr}\partial_y w+\frac{1+\eta'^2}{(1+\eta)^2}\operatorname{tr}(\partial_y w)^2,\quad(\eta,w)\in\mathcal{U}\times C^{2+\alpha}(\overline{\Omega_0^b}),$$

$$\overline{\mathcal{B}}(\eta,\overline{w}):=\operatorname{tr}(\partial_x\overline{w})^2-\frac{2\eta'}{1-\eta}\operatorname{tr}\partial_x\overline{w}\operatorname{tr}\partial_y\overline{w}+\frac{1+\eta'^2}{(1-\eta)^2}\operatorname{tr}(\partial_y\overline{w})^2,\quad(\eta,\overline{w})\in\mathcal{U}\times C^{2+\alpha}(\overline{\Omega_0^t}),$$

which shows that $\mathcal{B}$ and $\overline{B}$ also depend analytically on their variables.

To conclude, we observe that if ψ and $\overline{\psi}$ are the solutions of the problems (2.4a) (for some $\eta\in\mathcal{U}$) when $\omega,\overline{\omega}$ are constant and the constants $m,\overline{m}$ are given by (3.3), then $w:=\psi\circ\phi_\eta=:\mathcal{T}(\eta)$ and $\overline{w}:=\overline{\psi}\circ\overline{\phi}_\eta=:\overline{\mathcal{T}}(\lambda,\eta)$ are the solutions of the Dirichlet problems

$$\begin{cases}\mathcal{A}(\eta)w=\omega & \text{in } \Omega_0^b,\\ w=0 & \text{on } y=0,\\ w=\omega/2 & \text{on } y=-1\end{cases}\quad\text{and}\quad\begin{cases}\overline{\mathcal{A}}(\eta)\overline{w}=\overline{\omega} & \text{in } \Omega_0^t,\\ \overline{w}=0 & \text{on } y=0,\\ \overline{w}=\lambda+\overline{\omega}/2 & \text{on } y=1,\end{cases}\tag{3.4}$$

respectively. Since, $\mathcal{A}$ and $\overline{\mathcal{A}}$ are real-analytic, and the right-hand side of the Eqs. (3.4) depends analytically on λ, we obtain that $\mathcal{T}$ and $\overline{\mathcal{T}}$ are also real-analytic. For the proof we refer to [17, Lemmas 2.2 and 2.3].

With this notation, finding the solutions $(\eta,\psi,\overline{\psi})$ of the coupled problem (2.4) when the fluids have constant vorticities and the constants $m,\overline{m},Q$ are given by relations (3.3) and (2.4c), reduces to determining the solutions (λ,η) of the nonlocal and nonlinear equation

$$\Psi(\lambda,\eta)=0,\tag{3.5}$$

where $\Psi:\mathbb{R}\times\mathcal{U}\to C^{\sigma'+\alpha}(\mathbb{S})$ is given by

$$\begin{aligned}\Psi(\lambda,\eta):=&\,\overline{\rho}\overline{\mathcal{B}}(\eta,\overline{\mathcal{T}}(\lambda,\eta))-\rho\mathcal{B}(\eta,\mathcal{T}(\eta))+2g(\overline{\rho}-\rho)\eta+\frac{2\sigma\eta''}{(1+\eta'^2)^{3/2}}\\&-\int_{\mathbb{S}}(\overline{\rho}\overline{\mathcal{B}}(\eta,\overline{\mathcal{T}}(\lambda,\eta))-\rho\mathcal{B}(\eta,\mathcal{T}(\eta))+2g(\overline{\rho}-\rho)\eta(x))\,dx,\quad(\lambda,\eta)\in\mathbb{R}\times\mathcal{U},\end{aligned}$$

with $\sigma'=0$ if $\sigma>0$ and $\sigma'=1$ for $\sigma=0$.

We note that the laminar flow solutions found at the beginning of the section are in correspondence with the trivial solutions $(\lambda, \eta) = (\lambda, 0)$ of (3.5). By applying the bifurcation theorem from simple eigenvalues due to Crandall and Rabinowitz [11] to Eq. (3.5) we show in Theorem 3.1 that infinitely many analytic branches consisting of non-flat solutions of (2.4) intersect this trivial set of solutions of (3.5). To this end, we restrict first the domain and range of Ψ. This is done by introducing the subspaces $C^{m+\alpha}_{k,0,ev}(\mathbb{S})$, $k, m \in \mathbb{N}$, of $C^{m+\alpha}(\mathbb{S})$ which contain only even, $2\pi/k$ periodic functions with integral mean zero. Choosing functions with integral mean zero corresponds to a choice of equal volumes of both liquids beneath the rigid lid, meaning that, at rest, the flat wave profile is located at $y = 0$. We use next the intrinsic properties of (2.4) to prove the following lemma.

Lemma 3.1. *Let* $\sigma \in [0, \infty)$ *and* $\mathcal{U}_{k,0,ev} := \mathcal{U} \cap C^{2+\alpha}_{k,0,ev}(\mathbb{S})$. *The operator* Ψ *is real-analytic*

$$\Psi \in C^{\omega}\left(\mathbb{R} \times \mathcal{U}_{k,0,ev}, C^{\sigma'+\alpha}_{k,0,ev}(\mathbb{S})\right),$$

and

$$\Psi(\lambda, \eta) = \overline{\rho}\overline{\mathcal{B}}(\eta, \overline{\mathcal{T}}(\lambda, \eta)) - \rho\mathcal{B}(\eta, \mathcal{T}(\eta)) + 2g(\overline{\rho} - \rho)\eta + \frac{2\sigma\eta''}{(1 + \eta'^2)^{3/2}}$$
$$- \int_{\mathbb{S}} (\overline{\rho}\overline{\mathcal{B}}(\eta, \overline{\mathcal{T}}(\lambda, \eta)) - \rho\mathcal{B}(\eta, \mathcal{T}(\eta)))\, dx$$

for all $(\lambda, \eta) \in \mathbb{R} \times \mathcal{U}_{k,0,ev}$.

Proof. The analyticity of Ψ is a consequence of the fact that the operators $\mathcal{T}, \overline{\mathcal{T}}, \mathcal{B}, \overline{\mathcal{B}}$ are real-analytic in their variables.

Let now $(\lambda, \eta) \in \mathbb{R} \times \mathcal{U}_{k,0,ev}$ be given. Since η is even, we may define the functions $w_1(x, y) := w(-x, y)$ for $(x, y) \in \Omega_0^b$ and $\overline{w}_1(x, y) := \overline{w}(-x, y)$ for $(x, y) \in \Omega_0^t$, where w and $\overline{w}$ denote the solutions of (3.4), respectively. Since the Dirichlet conditions are constant, it may be easily checked that w_1 and $\overline{w}_1$ are solutions of the problems (3.4), respectively, so that, by the weak elliptic maximum principle, $w = w_1$ and $\overline{w} = \overline{w}_1$. A similar argument shows that w and $\overline{w}$ are both $2\pi/k$ periodic in x and we obtain, by using the explicit relations for the boundary operators $\mathcal{B}$ and $\overline{\mathcal{B}}$, that $\Psi(\lambda, \eta)$ is $2\pi/k$ periodic and even. To finish the proof, we note, with our choice of Q in (2.4c), that $\Psi(\lambda, \eta)$ has integral mean zero if η has this property. □

An important step in our analysis is to determine the Fréchet derivative of Ψ at the trivial branch of laminar solutions.

Lemma 3.2. *Given* $\lambda \in \mathbb{R}$, *the Fréchet derivative* $\partial_\eta \Psi(\lambda, 0) \in \mathcal{L}(C^{2+\alpha}_{k,0,ev}(\mathbb{S}), C^{\sigma'+\alpha}_{k,0,ev}(\mathbb{S}))$ *is a Fourier multiplier*

$$\partial_\eta \Psi(\lambda, 0) \sum_{k=1}^{\infty} a_k \cos(kx) = \sum_{k=1}^{\infty} \mu_k(\lambda) a_k \cos(kx), \tag{3.6}$$

with symbol

$$\mu_k(\lambda) := 2\left[g(\overline{\rho} - \rho) - \sigma k^2 + \overline{\omega}\,\overline{\rho}\lambda + \overline{\rho}\frac{k}{\tanh(k)}\lambda^2\right], \quad k \geq 1. \tag{3.7}$$

Proof. Taking into account that $\mathcal{T}(0) = \psi_0$ and $\overline{\mathcal{T}}(\lambda, 0) = \overline{\psi}_0$ for all $\lambda \in \mathbb{R}$, we obtain from

$$\begin{aligned} \partial_\eta \mathcal{B}(0, \psi_0) &= 0, & \partial_w \mathcal{B}(0, \psi_0) &= 0, \\ \partial_\eta \overline{\mathcal{B}}(0, \overline{\psi}_0)[\eta] &= 2\lambda^2 \eta, & \partial_{\overline{w}} \overline{\mathcal{B}}(0, \overline{\psi}_0)[\overline{w}] &= 2\lambda \operatorname{tr} \partial_y \overline{w}, \end{aligned} \tag{3.8}$$

and the chain rule that

$$\partial_\eta \Psi(\lambda, 0)[\eta] = 2\overline{\rho}\lambda\left(\operatorname{tr}(\partial_y \partial_\eta \overline{\mathcal{T}}(\lambda, 0)[\eta]) - \int_{\mathbb{S}} \operatorname{tr}(\partial_y \partial_\eta \overline{\mathcal{T}}(\lambda, 0)[\eta])\, dx\right) \tag{3.9}$$

$$+ [2g(\overline{\rho} - \rho) + 2\lambda^2 \overline{\rho}]\eta + 2\sigma\eta'' \tag{3.10}$$

for all $\eta \in C^{2+\alpha}_{k,0,ev}(\mathbb{S})$. We determine now the Fréchet derivative $\overline{w} := \partial_\eta \overline{\mathcal{T}}(\lambda, 0)[\eta]$. Therefore, we differentiate the equations of the Dirichlet problem (3.4) solved by $\overline{\mathcal{T}}(\lambda, \eta)$ with respect to η at $\eta = 0$, and find that $\overline{w}$ is the solution of the problem

$$\Delta \overline{w} = -\partial \overline{\mathcal{A}}(0)[\eta]\overline{\psi}_0 \quad \text{in } \Omega_0^t, \quad \overline{w} = 0 \quad \text{on } \partial\Omega_0^t, \tag{3.11}$$

where, by the explicit expressions found at the beginning of the section

$$\partial \overline{\mathcal{A}}(0)[\eta]\overline{\psi}_0 = 2\overline{\omega}\eta - (1 - y)(\overline{\omega}y + \lambda)\eta''.$$

In order to determine relation (3.6) we use Fourier expansions for functions in $C^{2+\alpha}_{k,0,ev}(\mathbb{S})$ and make a similar ansatz for the solution $\overline{w} := \partial_\eta \overline{\mathcal{T}}(\lambda, 0)[\eta]$ of (3.11):

$$\eta = \sum_{k=1}^{\infty} a_k \cos(kx) \quad \text{and} \quad \overline{w} = \sum_{k=1}^{\infty} a_k \overline{w}_k(y) \cos(kx).$$

The right-hand side of the first equation of (3.11) may be then expanded as follows

$$-\partial \overline{\mathcal{A}}(0)[\eta]\psi_{0+} = \sum_{k=1}^{\infty} a_k b_k(y) \cos(kx) \quad \text{with } b_k(y) := -2\overline{\omega} - (1 - y)(\overline{\omega}y + \lambda)k^2,$$

and, plugging all these expansions into (3.11) and matching the coefficients corresponding to $\cos(kx)$, yields that $\overline{w}_k$ solves

$$\overline{w}_k'' - k^2 \overline{w}_k = b_k, \quad 0 < y < 1, \quad \text{and} \quad \overline{w}_k(0) = \overline{w}_k(1) = 0. \tag{3.12}$$

The solution of (3.12) is given by

$$\overline{w}_k(y) = \frac{\lambda}{\tanh(k)} \sinh(ky) - \lambda \cosh(ky) + \overline{\omega}(y - y^2) + \lambda(1 - y), \tag{3.13}$$

and together with (3.9) we obtain the desired relations (3.6) and (3.7). □

We recall now a theorem which provides a sufficient condition for an operator in order to be a Fourier multipliers between Hölder spaces of periodic functions.

Lemma 3.3 ([18, Theorem 3.4]). *Let r, s be two positive non-integer constants and let $(M_p)_{p\in\mathbb{Z}} \subset \mathbb{C}$ be a sequence satisfying the following conditions*

(i) $\sup_{p\in\mathbb{Z}\setminus\{0\}} |p|^{r-s}|M_p| < \infty$,
(ii) $\sup_{p\in\mathbb{Z}\setminus\{0\}} |p|^{r-s+1}|M_{p+1} - M_p| < \infty$,
(iii) $\sup_{p\in\mathbb{Z}\setminus\{0\}} |p|^{r-s+2}|M_{p+2} - 2M_{p+1} + M_p| < \infty$.

Then, the Fourier multiplier satisfies

$$\sum_{p\in\mathbb{Z}} a_p e^{ipx} \mapsto \sum_{p\in\mathbb{Z}} M_p a_p e^{ipx} \in \mathcal{L}(C^s(\mathbb{S}), C^r(\mathbb{S})).$$

Using Lemma 3.3 and the relations (3.6) and (3.7), it is not difficult to see that $\partial_\eta \Psi(\lambda, 0)$ is an isomorphism if λ is chosen such that $\mu_k(\lambda) \neq 0$ for all integers $k \geq 1$. To find these values of λ we solve the quadratic equation $\mu_k(\lambda) = 0$ and find the solutions

$$\Lambda_k^i := -\frac{\overline{\omega}}{2}\frac{\tanh(k)}{k} + (-1)^i\sqrt{\frac{g(\rho - \overline{\rho}) + \sigma k^2}{\overline{\rho}}\frac{\tanh(k)}{k} + \frac{\overline{\omega}^2}{4}\frac{\tanh^2(k)}{k^2}}, \quad i = 1, 2.$$

We consider the case $\sigma = 0$ first. Since $x \mapsto \tanh(x)/x$ is a strictly decreasing function mapping $[0, \infty)$ onto $(0, 1]$, and, for $B > 0$, $t \mapsto t + \sqrt{t^2 + Bt}$ is increasing (respectively, $t \mapsto t - \sqrt{t^2 + Bt}$ is decreasing) on $[0, 1]$, we conclude that (Λ_k^i) are strictly monotone sequences. If $\sigma \neq 0$, we choose $\sigma_0 > 0$ such that (Λ_k^i) are strictly monotone sequences for all $\sigma > \sigma_0$. By Lemma 3.3, we see that in both cases $\partial_\eta \Psi(\Lambda_k^i, 0), i = 1, 2$, is a Fredholm operator of index zero having a one-dimensional kernel.

This leads us to the existence result of this paper. Besides existence of analytic curves consisting of traveling internal waves, we determine the second order Taylor expansions for these curves in a neighborhood of the laminar flow solutions. These expansions give us sufficient information to provide the precise picture of the streamlines (the level curves of the stream functions) in the frame moving with wave speed c, see Theorem 4.1.

Theorem 3.1. *Assume that $\overline{\rho} < \rho$ and if $\sigma \neq 0$, then let $\sigma > \sigma_0$. Given $k \in \mathbb{N}$, $k \geq 1$, there exists $\varepsilon_k > 0$ and real-analytic curves*

$$(\lambda_k^i, \eta_k^i) : (-\varepsilon_k, \varepsilon_k) \to \mathbb{R} \times C^{2+\alpha}_{k,0,ev}(\mathbb{S}), \quad i = 1, 2,$$

consisting only of real-analytic solutions of (3.5) *of minimal period* $2\pi/k$ *and having exactly one crest and trough per period. These are the only solutions of* (3.5) *close to* $(\Lambda_k^i, 0)$, *and for* $s \to 0$ *we have*

$$\begin{aligned} \lambda_k^i(s) &= \Lambda_k^i + O(s^2), \\ \eta_k^i(s) &= -\cos(kx)s + \alpha_k^i \cos(2kx)s^2 + O(s^3) \quad \text{in } C^{2+\alpha}_{k,0,ev}(\mathbb{S}), \end{aligned} \qquad i = 1, 2, \tag{3.14}$$

with constants α_k^i *given by* (3.23) (*with* Λ *replaced by* Λ_k^i).

Proof. We verify first that the assumptions of the theorem on bifurcations from simple eigenvalues due to Crandall and Rabinowitz [11, Theorem 1.7] are satisfied. We already know that, when $\lambda = \Lambda_k^i$ for some $k \in \mathbb{N}$ and $i = 1, 2$, the Fréchet $\partial_\eta \Psi(\Lambda_k^i, 0)$ is a Fredholm operator with

$$\operatorname{Ker} \partial_\eta \Psi(\Lambda_k^i, 0) = \operatorname{span}\{\cos(kx)\} \quad \text{and} \quad \operatorname{Im} \partial_\eta \Psi(\Lambda_k^i, 0) \oplus \operatorname{span}\{\cos(kx)\} = C^{\sigma'+\alpha}_{k,0,ev}(\mathbb{S}).$$

Furthermore, differentiating (3.6) with respect to λ we obtain that

$$\partial^2_{\lambda\eta} \Psi(\Lambda_k^i, 0)[\cos(kx)] = \pm 4\overline{\rho} \sqrt{\frac{g(\rho - \overline{\rho}) + \sigma k^2}{\overline{\rho}} \frac{k}{\tanh(k)} + \frac{\overline{\omega}^2}{4}} \cos(kx),$$

which implies $\partial^2_{\lambda\eta} \Psi(\Lambda_k^i, 0)[\cos(kx)] \notin \operatorname{Im} \partial_\eta \Psi(\Lambda_k^i, 0)$. The existence of the analytic bifurcation curves follows now from the above mentioned theorem and Lemma 3.1.

We pick now a solution $(\lambda(s), \eta(s))$ of (3.5) located on one of the curves (λ_k^i, η_k^i), $i = 1, 2$, and denote by $\psi := \mathcal{T}(\eta(s)) \circ \phi^{-1}_{\eta(s)}$ and $\overline{\psi} := \overline{\mathcal{T}}(\lambda, \eta(s)) \circ \overline{\phi}^{-1}_{\eta(s)}$ the stream functions associated to it. In order to prove that $\eta(s)$ is real-analytic, we show that the assumption (2.5) is fulfilled provided ε_k is sufficiently small. Indeed, since $\mathcal{T}(0) = \psi_0$ and $\overline{\mathcal{T}}(\Lambda_k^i, 0) = \overline{\psi}_0$, with ψ_0 and $\overline{\psi}_0$ given by (3.2), we obtain from $|\nabla \psi_0|^2 + |\nabla \overline{\psi}_0|^2 = |\Lambda_k^i|^2 > 0$ on $y = 0$ that (2.5) is satisfied by the laminar flows $(\lambda_k^i(0), \eta_k^i(0))$, $i = 1, 2$. Choosing ε_k small enough, the real-analyticity of $\eta(s)$ follows now from Theorem 2.1, by making use of the continuity of the bifurcation curves and of the solutions operators $\mathcal{T}$ and $\overline{\mathcal{T}}$.

Further on, we prove the asymptotic expansions (3.14) and show that the internal traveling waves we obtain have exactly one crest and trough per period. To this end, we fix $k \in \mathbb{N}$, $k \geq 1$, and, to ease notation, we let in this final part of the proof $\Lambda := \Lambda_k^i$, for $i \in \{1, 2\}$, and denote by (λ, η) the corresponding branch of solutions (λ_k^i, η_k^i). Since the bifurcation curve (λ, η) is real-analytic, we obtain from [11, Theorem 1.7] that

$$\begin{aligned} \lambda(s) &= \Lambda + \lambda_s(0)s + O(s^2) \quad \text{and} \\ \eta(s) &= -s\cos(kx) + \tau(s) + O(s^2) \quad \text{in } C^{2+\alpha}_{k,0,ev}(\mathbb{S}), \end{aligned} \tag{3.15}$$

with $\tau(0) = \tau_s(0) = 0$ and where the index s denotes the derivative with respect to the variable s. Additionally, the function τ takes values in the closed complement X_0 of $\operatorname{span}\{\cos(kx)\}$ in $C^{2+\alpha}_{k,0,ev}(\mathbb{S})$. Proceeding similarly as in [35], we find first from (3.15) that

$$\eta' = sk\sin(kx) + O(s^2) \quad \text{and} \quad \eta'' = sk^2\cos(kx) + O(s^2) \quad \text{in } C(\mathbb{S}).$$

Therefore, $\eta''(0) > 0$, $\eta''(\pi/k) < 0$, and, since η' is odd, we also have $\eta'(0) = \eta'(\pi/k) = 0$. We resume that η' is positive on $(0, \pi/k)$ provided ε_k is small, meaning that the wave has its crest located at $x = \pi/k$ and the trough at $x = 0$.

The next step of the proof is to determine the derivatives $\lambda_s(0)$ and $\tau_{ss}(0)$. Differentiating the relation $\Psi(\lambda(s), \eta(s)) = 0$ twice with respect to s we find, at $s = 0$, that $\lambda_s(0)$ and $\tau_{ss}(0)$ are related by

$$-2\lambda_s(0)\partial^2_{\lambda,\eta}\Psi(\Lambda, 0)[\cos(kx)] + \partial^2_{\eta\eta}\Psi(\Lambda, 0)[\cos(kx)]^2 + \partial_\eta\Psi(\Lambda, 0)[\tau_{ss}(0)] = 0. \tag{3.16}$$

Clearly, we need to find the second order derivative

$$\partial^2_{\eta\eta}\Psi(\lambda,0)[\cos(kx)]^2 = \overline{\rho}\left(\overline{I} - \int_{\mathbb{S}} \overline{I}\,dx\right) - \rho\left(I - \int_{\mathbb{S}} I\,dx\right), \tag{3.17}$$

where, setting $\xi := \cos(kx)$, we made the following notation

$$\begin{aligned} I &:= \partial^2_{\eta\eta}\mathcal{B}(0,\psi_0)[\xi]^2 - 2\partial^2_{\eta w}\mathcal{B}(0,\psi_0)[\xi,\partial\mathcal{T}(0)[\xi]] \\ &\quad - \partial^2_{ww}\mathcal{B}(0,\psi_0)[\partial\mathcal{T}(0)[\xi]]^2 - \partial_w\mathcal{B}(0,\psi_0)[\partial^2\mathcal{T}(0)[\xi]^2], \\ \overline{I} &:= \partial^2_{\eta\eta}\overline{\mathcal{B}}(0,\overline{\psi}_0)[\xi]^2 + 2\partial^2_{\eta\overline{w}}\overline{\mathcal{B}}(0,\overline{\psi}_0)[\xi,\partial_\eta\overline{\mathcal{T}}(\Lambda,0)[\xi]] \\ &\quad + \partial^2_{\overline{w}\,\overline{w}}\mathcal{B}(0,\overline{\psi}_0)[\partial_\eta\overline{\mathcal{T}}(\Lambda,0)[\xi]]^2 + \partial_{\overline{w}}\overline{\mathcal{B}}(0,\overline{\psi}_0)[\partial^2_{\eta\eta}\overline{\mathcal{T}}(\Lambda,0)[\xi]^2]. \end{aligned} \tag{3.18}$$

In view of (3.2), we compute

$$\begin{aligned} \partial^2_{\eta\eta}\overline{\mathcal{B}}(0,\overline{\psi}_0)[\xi]^2 &= 2\Lambda^2(3\xi^2 + \xi'^2), \\ \partial^2_{\eta\overline{w}}\overline{\mathcal{B}}(0,\overline{\psi}_0)[\xi,\overline{v}] &= 4\Lambda\xi\,\mathrm{tr}\,\partial_y\overline{v} - 2\xi'\Lambda\,\mathrm{tr}\,\partial_x\overline{v}, \\ \partial^2_{\overline{w}\,\overline{w}}\mathcal{B}(\eta,w)[\overline{v}]^2 &= 2\,\mathrm{tr}((\partial_x\overline{v})^2 + (\partial_y\overline{v})^2), \\ \partial^2_{ww}\mathcal{B}(\eta,w)[v]^2 &= 2\,\mathrm{tr}((\partial_x v)^2 + (\partial_y v)^2), \\ \partial^2_{\eta\eta}\mathcal{B}(0,\psi_0) &= \partial^2_{\eta w}\mathcal{B}(0,\psi_0) = 0, \end{aligned}$$

and, recalling (3.8), we have $\partial_w\mathcal{B}(0,\psi_0) = 0$. Consequently, we need to determine only the derivatives $\partial\mathcal{T}(0)[\xi]$ and $\partial^2_{\eta\eta}\overline{\mathcal{T}}(\Lambda,0)[\xi]^2$ to obtain an explicit expression for the right-hand side of (3.17). Concerning $\partial\mathcal{T}(0)[\xi]$, we have to study a linear Dirichlet problem similar to (3.11), and one finds

$$\partial\mathcal{T}(0)[\cos(kx)] = \omega(y^2 + y)\cos(kx). \tag{3.19}$$

As for the second order derivative, we differentiate the Dirichlet problem (3.4) for $\overline{w}$ twice with respect to η and obtain, at $\eta = 0$, that $\overline{v} := \partial^2_{\eta\eta}\overline{\mathcal{T}}(\Lambda,0)[\xi]^2$ is the solution of the problem

$$\Delta\overline{v} = -2\partial\overline{\mathcal{A}}(0)[\xi]\partial_\eta\overline{\mathcal{T}}(\Lambda,0)[\xi] - \partial^2\overline{\mathcal{A}}(0)[\xi,\xi]\overline{\psi}_0 \quad \text{in } \Omega_0^t, \quad \overline{v} = 0 \quad \text{on } \partial\Omega_0^t, \tag{3.20}$$

where

$$\begin{aligned} \partial^2\overline{\mathcal{A}}(0)[\xi]^2\overline{\psi}_0 &= \overline{\omega}(2(1-y)^2\xi'^2 + 6\xi^2) - (1-y)(\overline{\omega}y + \Lambda)(4\xi'^2 + 2\xi\xi''), \\ \partial\overline{\mathcal{A}}(0)[\xi]\partial_\eta\overline{\mathcal{T}}(\Lambda,0)[\xi] &= (-2(1-y)\xi'\partial^2_{xy} + 2\xi\partial^2_{yy} - (1-y)\xi''\partial_y)(\partial_\eta\overline{\mathcal{T}}(\Lambda,0)[\xi]). \end{aligned}$$

By (3.13), we have $\partial_\eta\overline{\mathcal{T}}(\Lambda,0)[\cos(kx)] = w_k(y)\cos(kx)$, so that we can express the right-hand side of the first equation of (3.20) as follows

$$2\partial\overline{\mathcal{A}}(0)[\xi]\partial_\eta\overline{\mathcal{T}}(\Lambda,0)[\xi] + \partial^2\overline{\mathcal{A}}(0)[\xi]^2\overline{\psi}_0 = E_0(y) + E_{2k}(y)\cos(2kx),$$

whereby

$$E_0(y) := -\overline{\omega} + 2k^2\Lambda\left(\frac{\sinh(ky)}{\tanh(k)} - \cosh(ky)\right) - (1-y)k^3\Lambda\left(\frac{\cosh(ky)}{\tanh(k)} - \sinh(ky)\right),$$

$$E_{2k}(y) := -\overline{\omega} + 2k^2\overline{\omega}(1-y)^2 + 2k^2\Lambda\left(\frac{\sinh(ky)}{\tanh(k)} - \cosh(ky)\right) + 3(1-y)k^3\Lambda\left(\frac{\cosh(ky)}{\tanh(k)} - \sinh(ky)\right).$$

Due to the linearity of (3.20), we write $\partial^2_{\eta\eta}\overline{\mathcal{T}}(\Lambda,0)[\cos(kx)]^2 = c_k(y) + d_k(y)\cos(2kx)$, with c_k denoting the solution of (3.12) when $(k, b_k) = (0, -E_0)$ and d_k being the solution of (3.12) with (k, b_k) are replaced by $(2k, -E_{2k})$. Since all Fréchet derivatives of $\mathcal{T}$ and $\overline{\mathcal{T}}$ with respect to η have zero boundary values, of relevance for our purpose are only the first derivatives

$$c_k'(0) := -k^2\Lambda - \frac{\overline{\omega}}{2},$$

$$d_k'(0) := -k^2\Lambda - \frac{k\Lambda}{\tanh(k)} + \frac{2k^2\Lambda}{\tanh(k)\tanh(2k)} - \overline{\omega} + \frac{k\overline{\omega}}{\tanh(2k)}.$$

Summarizing, we have shown that

$$I = \omega^2 + \omega^2\cos(2kx),$$

$$\overline{I} = \left(\frac{k\Lambda}{\tanh(k)} + \overline{\omega}\right)^2 + \frac{2k\Lambda^2}{\tanh(k)} + \overline{\omega}^2 + \Lambda\overline{\omega} - \Lambda^2k^2 + \left[\left(\frac{k\Lambda}{\tanh(k)} + \overline{\omega}\right)^2 + \frac{4k^2\Lambda^2}{\tanh(k)\tanh(2k)} + \frac{2k\overline{\omega}\Lambda}{\tanh(2k)} - 3k^2\Lambda^2\right]\cos(2kx).$$

Recalling (3.17), we find that $\partial^2_{\eta\eta}\Psi(\Lambda,0)[\cos(kx)]^2 = A_k\cos(2kx)$, whereby

$$A_k := \left[\left(\frac{k\Lambda}{\tanh(k)} + \overline{\omega}\right)^2 + \frac{4k^2\Lambda^2}{\tanh(k)\tanh(2k)} + \frac{2k\overline{\omega}\Lambda}{\tanh(2k)} - 3k^2\Lambda^2\right] - \rho\omega^2. \quad (3.21)$$

To finish, we observe that $\partial^2_{\eta\eta}\Psi(\Lambda,0)[\cos(kx)]^2$ belongs to $\operatorname{Im}\partial_\eta\Psi(\Lambda,0)$, meaning that $\cos(kx)$ is orthogonal on $\partial^2_{\eta\eta}\Psi(\Lambda,0)[\cos(kx)]^2$ in $L_2(\mathbb{S})$. Whence, if we multiply the relation (3.16) by $\cos(kx)$ and integrate it then over the unit circle, we get that $\lambda_s(0) = 0$. Moreover, since the restriction $\partial_\eta\Psi(\Lambda,0) : X_0 \to \operatorname{Im}\partial_\eta\Psi(\Lambda,0)$ is an isomorphism and $\tau_{ss}(0) \in X_0$, (3.16) yields

$$\tau_{ss}(0) = -(\partial_\eta\Psi(\Lambda,0))^{-1}\partial^2_{\eta\eta}\Psi(\Lambda,0)[\cos(kx)]^2. \quad (3.22)$$

Setting

$$\alpha_k := -\frac{A_k}{2\mu_{2k}(\Lambda)}, \quad (3.23)$$

we conclude from (3.6) that $\tau_{ss}(0) = 2\alpha_k\cos(2kx)$, and together with (3.15) we find the desired expansion for η. This completes the proof. □