Solutions and Lax Pairs based on Bilinear Bäcklund Transformations of Some Supersymmetric Equations

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SOLUTIONS AND LAX PAIRS BASED ON BILINEAR BÄCKLUND TRANSFORMATIONS OF SOME SUPERSYMMETRIC EQUATIONS

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The paper investigates solutions and Lax pairs through bilinear Bäcklund transformations for some supersymmetric equations. We derive variety of solutions from the known bilinear Bäcklund transformations. Besides, using the gauge invariance of (super) Hirota bilinear derivatives we may get deformed bilinear Bäcklund transformations and consequently deformed Lax pairs with fermionic parameters. Examples are the $\mathcal{N} = 1$ supersymmetric KdV equation and the $\mathcal{N} = 2$ supersymmetric KdV$_1$ equation.

Keywords: Supersymmetric equations; bilinear Bäcklund transformations; multiple-pole solutions; Lax pairs.

Mathematics Subject Classification: 35Q51, 35Q58, 35Q53

1. Introduction

Supersymmetry is a symmetry that makes particles of different spins and statistical properties link together, or it is a symmetry that makes fermions and bosons transform to each other, moreover, it is a new type of symmetry that makes space-time symmetry and internal symmetry combine together. In 1971, the term of supersymmetry [10] was first used in Gol'fand and Linkman's work. In recent decades, supersymmetry was widely applied in various fields, such as: relativity, non-relativity, nuclear physics, quantum field theory, superstring and so on. Mathematicians and theoretical physicists have aroused great interest in supersymmetry. Supersymmetry reflected in mathematics is based on the introduction of Grassmann variables. Meanwhile, in the field theory it is not only a Grassmann odd field but also an equation of motion.

For the integrability of supersymmetric (SUSY) systems, in 1984 Kupershmidt [17] introduced an integrable super KdV equation through the fermionic extension of the KdV equation. Later, Manin and Radul [26] obtained the SUSY KdV equation. From then on

many standard integrable equations and their integrable characteristics have been extended to their SUSY counterparts, see, for example, [20, 21, 24, 25, 28].

With regard to solutions, in 1978 Chaichian and Kulish [5] considered the inverse scattering problem and Bäcklund transformations (BTs) of the SUSY Liouville equation and the SUSY sine-Gordon equation. One of powerful solving tool is bilinear method. In 1993 McArthur and Yung [27] first introduced a super Hirota bilinear derivative and gave explicit bilinear forms of equations in the SUSY KdV hierarchy. In 2000 under the gauge-invariance principle [12] Carstea redefine the super Hirota derivatives and obtained bilinear forms for more SUSY equations, from which one can derived multi-super-soliton solutions [3, 4, 16]. In 2005 Liu and Hu [22] constructed the bilinear BT for the SUSY KdV equation from which they got a nonlinear superposition formula and a new Lax representation for the SUSY KdV equation. There are other interesting progress related bilinear method, such as quasi-periodic solutions [9] of SUSY equations and applications of super Bell polynomials [7, 8], and so on.

Using bilinear equations one can construct bilinear BTs [14]. Usually there is an arbitrary parameter in a bilinear BT and multi-soliton solutions can be derived step by step by taking different values for the parameter in each step. For the classical integrable systems, their bilinear BTs can be deformed. This is based on gauge invariance of Hirota bilinear derivatives. The deformation brings more freedom for choosing parameter values. As a result, many "new" solutions (which are different from solitons and in fact limit-solutions or multiple-pole solutions (cf. [31, 32])) can be derived [1, 6, 30].

In the paper, we will consider bilinear BTs and the related solutions and Lax pairs for some SUSY equations. Since the super Hirota bilinear derivatives also admit gauge invariance, one can deform bilinear BTs for equations. Surprisingly, different from what we mentioned previously for ordinary integrable systems, one can get variety of solutions but deformations of BTs are not necessary. However, the deformations do bring something new. A suitable deformation will bring a bilinear BT with a fermionic type parameter, which leads to a Lax pair with a fermionic type "spectral" parameter. We will also investigate bilinear BTs for the $\mathcal{N} = 2$ SUSY KdV$_1$ equation.

The paper is organized as follows. In Sec. 2, we introduce some notations and properties on the ordinary and super Hirota bilinear derivatives. This section also includes the deformed BT and related solutions for the KdV equation. Then, in Sec. 3, we give detail investigation on the bilinear BT of $\mathcal{N} = 1$ SUSY KdV equation and the related solutions, deformations and Lax pairs. Section 4 is for the $\mathcal{N} = 2$ SUSY KdV$_1$ equation.

2. Preliminary

2.1. *Notations and properties*

For two sufficiently differential ordinary functions $a = a(x, t)$ and $b = b(x, t)$, their Hirota bilinear derivative is defined as [15]:

$$D_x^m D_t^n a \cdot b = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n a(x, t) b(x', t')|_{x=x', t=t'}. \tag{2.1}$$

It can be proved that

$$\mathcal{F}(D_x, D_t)(e^{\zeta_1'} a) \cdot (e^{\zeta_2'} b) = e^{\zeta_1' + \zeta_2'} \mathcal{F}(D_x + (p_1' - p_2'), D_t + (q_1' - q_2'))a \cdot b, \tag{2.2}$$

where $\mathcal{F}(D_x, D_t)$ is a polynomial of the operators D_x and D_t, and ζ_i' are linear functions defined as

$$\zeta_i' = p_i'x + q_i't + \zeta_i^{(0)'}, \quad i = 1, 2. \tag{2.3}$$

When $\zeta_1' = \zeta_2'$, the equality (2.2) coincides with the gauge invariance of the Hirota bilinear equations [12].

For the Grassman-valued functions

$$f \doteq f(x, t, \theta), \quad g \doteq g(x, t, \theta),$$

where θ is a Grassmann variable denoting fermionic counterpart of the spatial variable x, one can define a super Hirota derivative as [3]:

$$S_x^N f \cdot g = \sum_{i=0}^{N} (-1)^{i|f| + \frac{i(i+1)}{2}} \begin{bmatrix} N \\ i \end{bmatrix} (\mathcal{D}^{N-i} f(x, t, \theta))(\mathcal{D}^i g(x, t, \theta)), \tag{2.4}$$

where $\mathcal{D}$ is the super covariant derivative

$$\mathcal{D} \doteq \mathcal{D}_\theta = \partial_\theta + \theta \partial_x,$$

$|f|$ is the Grassmann parity of the function f defined as

$$|f| = \begin{cases} 1, & f \text{ is odd (fermionic)}, \\ 0, & f \text{ is even (bosonic)}, \end{cases}$$

$\begin{bmatrix} N \\ i \end{bmatrix}$ is the superbinomial coefficient [26] defined as

$$\begin{bmatrix} N \\ i \end{bmatrix} = \begin{cases} 0, & (N, i) \equiv (0, 1) \mod 2, \\ C_{[N/2]}^{[i/2]} = \dfrac{[N/2]!}{([N/2] - [i/2])![i/2]!}, & \text{others}, \end{cases}$$

and $[k]$ is the integer part of the real number k, i.e., $([k] \le k \le [k] + 1)$. Since $\mathcal{D}^2 = \partial_x$ and consequently

$$S_x^{2N} f \cdot g = D_x^N f \cdot g,$$

one only needs to consider the first order super Hirota derivative which is as the following [3]

$$S_x f \cdot g = (\mathcal{D}f)g - (-1)^{|f|} f(\mathcal{D}g). \tag{2.5}$$

One can further finds that

$$S_x D_x^m D_t^n f \cdot g = (\mathcal{D}_\theta - \mathcal{D}_{\theta'})(\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x, t, \theta)g(t', x', \theta') \mid_{x=x', t=t', \theta=\theta'}, \tag{2.6}$$

where f, g are Grassmann even functions.

50

With regards to the properties of the operator S_x, if

$$\zeta_i = p_i x + q_i t + \theta \varrho_i + \zeta_i^{(0)} \tag{2.7}$$

with bosonic constants $(p_i, q_i, \xi_i^{(0)})$ and fermionic constant ϱ_i, it then follows that

$$S_x^{2N+1} e^{\zeta_1} \cdot e^{\zeta_2} = S_x D_x^N e^{\zeta_1} \cdot e^{\zeta_2} = [\varrho_1 - \varrho_2 + \theta(p_1 - p_2)](p_1 - p_2)^N e^{\zeta_1 + \zeta_2}, \tag{2.8a}$$

and

$$S_x D_x^m D_t^n e^{\zeta_1} f \cdot e^{\zeta_2} g$$
$$= e^{\zeta_1 + \zeta_2}[S_x + (\varrho_1 - \varrho_2) + \theta(p_1 - p_2)][D_x + (p_1 - p_2)]^m [D_t + (q_1 - q_2)]^n f \cdot g. \tag{2.8b}$$

Further we can find

$$\mathcal{F}(S_x, D_x, D_t)(e^{\zeta_1} f) \cdot (e^{\zeta_2} g) = e^{\zeta_1 + \zeta_2} \mathcal{F}(S_x + (\varrho_1 - \varrho_2) + \theta(p_1 - p_2), D_x$$
$$+ (p_1 - p_2), D_t + (q_1 - q_2)) f \cdot g, \tag{2.8c}$$

where $\mathcal{F}(S_x, D_x, D_t)$ is a polynomial of S_x, D_x, D_t with bosonic coefficients. Equation (2.8c) is an analogue of the formula (2.2), and exhibits the gauge invariance property of super Hirota derivative [3]. This formula will play an important role in getting the deformed bilinear BTs. The proof for the formula is given below,

$$S_x D_x^m D_t^n (e^{\zeta_1} f) \cdot (e^{\zeta_2} g)$$
$$= (\mathcal{D}_\theta - \mathcal{D}_{\theta'})\{e^{\zeta_1(\theta) + \zeta_2(\theta')}[D_x + (p_1 - p_2)]^m [D_t + (q_1 - q_2)]^n f(\theta) \cdot g(\theta')\}|_{\theta' = \theta}$$
$$= e^{\xi + \eta}(\mathcal{D}(\xi - \eta))[D_x + (p_1 - p_2)]^m [D_t + (q_1 - q_2)]^n f \cdot g$$
$$+ e^{\xi + \eta}(\mathcal{D}_\theta - \mathcal{D}_{\theta'})\{[D_x + (p_1 - p_2)]^m [D_t + (q_1 - q_2)]^n f(\theta) \cdot g(\theta')\}|_{\theta' = \theta}$$
$$= e^{\xi + \eta}(\mathcal{D}(\xi - \eta))[D_x + (p_1 - p_2)]^m [D_t + (q_1 - q_2)]^n f \cdot g$$
$$+ e^{\xi + \eta} S_x [D_x + (p_1 - p_2)]^m [D_t + (q_1 - q_2)]^n f \cdot g$$
$$= e^{\xi + \eta}[S_x + (\varrho_1 - \varrho_2) + \theta(p_1 - p_2)][D_x + (p_1 - p_2)]^m [D_t + (q_1 - q_2)]^n f \cdot g.$$

2.2. BTs of the KdV equation

The KdV equation always plays a paradigmatic role in classical integrable systems. To have a comparison with SUSY systems and also to understand the main ideas of the paper, let us recall the BT and related topics of the KdV equation. The KdV equation reads

$$u_t + 6u u_x + u_{xxx} = 0. \tag{2.9}$$

By the transformation

$$u = 2(\ln f)_{xx}, \tag{2.10}$$

it is bilinearized to [13],

$$(D_x D_t + D_x^4) f \cdot f = 0, \tag{2.11}$$

which generates the bilinear BT [14]

$$(D_x^2 - \lambda)f \cdot g = 0, \tag{2.12a}$$

$$(D_t + D_x^3 + 3\lambda D_x)f \cdot g = 0, \tag{2.12b}$$

where λ is a parameter and D is the well-known operator defined in (2.1). If f solves (2.11), so does g provided (f, g) satisfy the above BT, and

$$u = 2(\ln g)_{xx} \tag{2.13}$$

gives a new solution to the KdV equation. Besides, the BT (2.12) is related to Lax pair of the KdV equation, which is

$$\phi_{xx} = (\lambda - u)\phi, \tag{2.14a}$$

$$\phi_t = -4\phi_{xxx} - 3u_x\phi - 6u\phi_x, \tag{2.14b}$$

through [14]

$$\phi = \frac{g}{f}, \quad u = 2(\ln f)_{xx}. \tag{2.15}$$

Usually, with respect to generating multi-soliton solutions, the perturbation expansion technique (expanding f and g as (2.17) for the bilinear BT (2.12)) is not as effective as from the bilinear equation (2.11). In fact, at the first glance, when $f = 1$, g cannot be in the form of $g = 1 + \varepsilon g_1$ due to the term λfg in (2.12a) unless $\lambda = 0$. In practice, one may first take $f = 1$ and solve (2.12) with $\lambda = p_1'$ to get $g = g(p_1')$ which gives 1-soliton solution via (2.13). Then, in a new turn let $f = g(p_1')$ and solve (2.12) with $\mu = p_2' \neq p_1'$ for a new $g(p_1', p_2')$, which yields a 2-soliton solution via (2.13). We also note that in the second turn the restriction $p_2' \neq p_1'$ is necessary. Otherwise, g does not create new solutions. Repeating the procedure by taking f to be g which is obtained in the previous round and solving (2.12) with λ being a number which is different from those p_j''s previously used, the obtained g will provide a new multi-soliton solutions.

In order to use the perturbation expansion technique and also get more solutions than solitons, Chen *et al.* [6] deformed the bilinear BT (2.12) by replacing (f, g) by $(e^{\zeta_1'}f, e^{\zeta_2'}g)$ in (2.12), where ζ_i' are defined as (2.3). They got which by using (2.2) yielded

$$(D_x^2 + \lambda' D_x)f \cdot g = 0, \tag{2.16a}$$

$$(D_t + D_x^3)f \cdot g = 0, \tag{2.16b}$$

where the parametrization is

$$\lambda' = 2(p_1' - p_2'), \quad \lambda = (p_1' - p_2')^2, \quad q_1' - q_2' = 4(p_1' - p_2')^3, \quad \frac{\lambda'^2}{4} = \lambda.$$

Obviously, guaranteed by the gauge invariance of Hirota bilinear equations [12], if f solves (2.11) and (2.16) (together with g), then g satisfies (2.11). This means the deformed BT (2.16) does play a role of a bilinear BT for the KdV equation. Moreover, (2.16) admits more freedom than (2.12).

First, (2.16) admits perturbation solution of the following form,

$$f = 1 + f_1\varepsilon + f_2\varepsilon^2 + f_3\varepsilon^3 + \cdots, \tag{2.17a}$$

$$g = 1 + g_1\varepsilon + g_2\varepsilon^2 + g_3\varepsilon^3 + \cdots. \tag{2.17b}$$

Substituting them into (2.16) yields

$$(D_x^2 + \lambda' D_x)(1 \cdot g_1 + f_1 \cdot 1) = 0, \tag{2.18a}$$

$$(D_t + D_x^3)(1 \cdot g_1 + f_1 \cdot 1) = 0, \tag{2.18b}$$

$$(D_x^2 + \lambda' D_x)(1 \cdot g_2 + f_1 \cdot g_1 + f_2 \cdot 1) = 0, \tag{2.18c}$$

$$(D_t + D_x^3)(1 \cdot g_2 + f_1 \cdot g_1 + f_2 \cdot 1) = 0, \tag{2.18d}$$

$$\vdots$$

Taking $f = 1$, i.e., $f_i = 0 (i \geq 1)$, and $\lambda = -p_1'$, one can find that (2.16) admits a truncated series form for g:

$$g = 1 + \varepsilon g_1, \quad g_1 = e^{\sigma_1}, \quad \sigma_1 = p_1'x - (p_1')^3 t + \sigma_1^{(0)}, \tag{2.19}$$

which provides 1-soliton solution via (2.13), where p_1' and $\sigma_1^{(0)}$ are constants. If we start from $f = 1 + \varepsilon e^{\sigma_1}$ but still take $\lambda' = -p_1'$ which is as same as in the first round, we can find

$$g = 1 + \varepsilon g^{(1)} + \varepsilon^2 g^{(2)}, \quad g^{(1)} = (-2p_1'x + 6(p_1')^3 t + d_1)e^{\sigma_1}, \quad g^{(2)} = -e^{2\sigma_1}, \tag{2.20}$$

where d_1 is an arbitrary number. This gives a solution different from 2-soliton solution of the KdV equation. We note that λ' can also be zero which leads rational part in solutions. (For more examples one can see [6]).

Now Let us go to the Lax pair generated from (2.16) by taking $\Phi = \frac{g}{f}$, $u = 2(\ln f)_{xx}$. It reads

$$\Phi_{xx} + u\Phi - \lambda'\Phi_x = 0, \tag{2.21a}$$

$$\Phi_t - 3u\Phi_x + \Phi_{xxx} = 0, \tag{2.21b}$$

of which the compatibility condition yields the KdV equation (2.9). This Lax pair is related to the old one (2.14) by

$$\phi = e^\xi \Phi, \tag{2.22}$$

where ξ is some linear function of x, t.

To sum up for this subsection, we can see that the deformed BT (2.16) provides more freedom than the old one (2.12). Both BTs can lead to Lax pairs for the KdV equation and these Lax pairs are related to each other be a simple gauge transformation.

3. $\mathcal{N} = 1$ SUSY KdV Equation

3.1. *BT and Lax pair*

There are more than one supersymmetric extension for the KdV equation. A simple one contains one Grassmann variable, which is referred to as the $\mathcal{N} = 1$ SUSY KdV equation

and reads [26]

$$\Phi_t = -\Phi_{xxx} + 3(\Phi \mathcal{D}\Phi)_x, \tag{3.1}$$

where $\Phi \doteq \Phi(x,t,\theta)$ is a Grassmann odd variable depending on super spatial variables (x,θ) and temporal variable t and $\mathcal{D}$ is defined as in Sec. 2. This $\mathcal{N} = 1$ SUSY KdV has space supersymmetry invariance. It is a fermionic extension of the classical KdV equation. In other words, when all the anticommuting variables degenerate to zero, the equation goes back to the classical KdV equation.

By the transformation

$$\Phi = \Psi_x = -2\mathcal{D}(\ln f(x,t,\theta))_x, \tag{3.2}$$

the $\mathcal{N} = 1$ SUSY KdV equation (3.1) can be bilinearlized to [3, 27]

$$S_x(D_t + D_x^3)f(x,t,\theta) \cdot f(x,t,\theta) = 0, \tag{3.3}$$

which yields the bilinear BT [22],

$$S_x(D_x - \mu)f \cdot g = 0, \tag{3.4a}$$

$$[D_t + D_x^3 - 3\mu D_x(D_x - \mu)]f \cdot g = 0, \tag{3.4b}$$

where μ is an arbitrary bosonic constant. The BT means that if f is solution of (3.3), so is g. In other words, once we have a solution f for the bilinear equation (3.3), and solve g from (3.4), then

$$\Phi = -2\mathcal{D}(\ln g)_x \tag{3.5}$$

gives a solution to the $\mathcal{N} = 1$ SUSY KdV equation (3.1) as well. The Lax pair of the $\mathcal{N} = 1$ SUSY KdV equation derived from the above BT is [22]

$$\mathcal{D}\Sigma_x - \Psi_x\Sigma + \mu\mathcal{D}\Sigma = 0, \tag{3.6a}$$

$$\Sigma_t - 3\mu(\mathcal{D}\Psi_x)\Sigma + 3(\mu^2 - (\mathcal{D}\Psi_x)) + 3\mu\Sigma_{xx} + \Sigma_{xxx} = 0, \tag{3.6b}$$

with $\Sigma = g/f$ and (3.2). Here the bosonic constant μ plays a role of spectral parameter.

3.2. *Variety of solutions*

Surprisingly, the BT (3.4) admits solutions (f,g) in the form of (2.17). Compared with Sec. 2.2, this is unlike the "ordinary" BT (2.12) but similar to the deformed one (2.16).

In following for convenience we define some linear functions:

$$\xi_j = k_j x - k_j^3 t + \xi_j^{(0)}, \quad \eta_j = k_j x - 3k_j^3 t, \quad j = 1, 2 \tag{3.7}$$

where k_j, $\xi_j^{(0)}$ are even constants and ϱ_j $(j = 1, 2)$ are odd constants. Now we list several solution pairs derived from the bilinear BT (3.4).

(I)

$$f = 1, \quad \mu = -k_1, \tag{3.8a}$$

$$g = 1 + \varepsilon e^{\xi_1 + \theta \varrho_1}; \tag{3.8b}$$

(II)

$$f = 1 + \varepsilon e^{\xi_1 + \theta \varrho_1}, \quad \mu = -k_2, \tag{3.9a}$$

$$g = 1 + \varepsilon(e^{\xi_1 + \theta \varrho_1} + e^{\xi_2 + \theta \varrho_2}) + \varepsilon^2 A_{12} \left[1 + 2 \frac{\varrho_1 \varrho_2}{k_2 - k_1} \right] e^{\xi_1 + \xi_2 + \theta(\alpha_{12}\varrho_1 + \alpha_{21}\varrho_2)}, \tag{3.9b}$$

where

$$A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j} \right)^2, \quad \alpha_{ij} = \frac{k_i + k_j}{k_i - k_j};$$

(III)

$$f = 1 + \varepsilon e^{\xi_1 + \theta \varrho_1}, \quad \mu = -k_1, \tag{3.10a}$$

$$g = 1 - 2\varepsilon(\eta_1 + \theta \varrho_1)e^{\xi_1 + \theta \varrho_1} - \varepsilon^2 e^{2(\xi_1 + \theta \varrho_1)}, \tag{3.10b}$$

where η_j is defined in (3.7);

(IV)

$$f = 1, \quad \mu = 0, \tag{3.11a}$$

$$g = a_1 x + \theta c_1 + d_1, \tag{3.11b}$$

where a_1, c_1, d_1 are even constants;

Among the above solutions, through the transformation (3.5), (3.8b) and (3.9b) provide 1-soliton and 2-soliton solutions [3, 4, 22], while (3.10b) provide mixed type solutions.

Let us look at the relation between (3.10b) and (3.9b). We can rewrite (3.9b) as

$$g = 1 + \varepsilon \frac{k_1 + k_2}{k_2 - k_1}(e^{\xi_1 + \theta \varrho_1 \ln k_1} - e^{\xi_2 + \theta \varrho_1 \ln k_2}) - \varepsilon^2 e^{\xi_1 + \xi_2 + \theta[\alpha_{12}\varrho_1 \ln k_1 + \alpha_{21}\varrho_1 \ln k_2]}, \tag{3.12}$$

where we have replaced $e^{\xi_1^{(0)}}$ by $-\frac{k_1+k_2}{k_1-k_2}e^{\xi_1^{(0)}}$, $e^{\xi_2^{(0)}}$ by $\frac{k_1+k_2}{k_1-k_2}e^{\xi_1^{(0)}}$, ϱ_j by $\varrho_1 \ln k_j$, due to the arbitrariness of $\xi_j^{(0)}$ and ϱ_j. Then, taking limit $k_2 \to k_1$ and making use of Hospital's rule we get (3.10b). This is similar to the ordinary case [6].

3.3. Deformed BTs and Lax pairs

Although the BT (3.4) can bring variety of solutions, the deformation for the BT is still interesting.

Let us replace f and g in (3.4) by $e^{\zeta_1} f$ and $e^{\zeta_2} g$, respectively, where ζ_j are defined in (2.7). Then, by the formula (2.8c), one gets the following deformed version,

$$(S_x + \kappa)(D_x - \beta)f \cdot g = 0, \tag{3.13a}$$

$$(D_t + D_x^3 - 3\beta D_x^2 + 3\beta^2 D_x)f \cdot g = \tau f g, \tag{3.13b}$$

where

$$\beta = \mu - \alpha, \quad \kappa = \gamma + \alpha\theta, \quad \tau = q_2 - q_1 + \alpha^3 + 3\mu^2\alpha, \quad \alpha = p_1 - p_2, \quad \gamma = \varrho_1 - \varrho_2. \tag{3.14}$$

Note that in (3.13) there are four parameters among which α, β and τ are bosonic parameters while γ is a fermionic one. Such kind of BTs with mixed type parameters was reported in [7]. This deformed BT, together with

$$\Omega = \frac{g}{f}, \quad \Psi = -2\mathcal{D}(\ln f(x, t, \theta)),$$

contributes a linear pair

$$\mathcal{D}\Omega_x - \Psi_x\Omega + \beta\mathcal{D}\Omega - \kappa\Omega_x - \beta\kappa\Omega = 0, \tag{3.15a}$$

$$\Omega_t - 3\beta(\mathcal{D}\Psi_x)\Omega + 3[\beta^2 - (\mathcal{D}\Psi_x)]\Omega_x + 3\beta\Omega_{xx} + \Omega_{xxx} - \tau\Omega = 0, \tag{3.15b}$$

of which the compatibility condition $\mathcal{D}\Omega_{xt} = \mathcal{D}\Omega_{tx}$ yields the equation

$$\Psi_t + \Psi_{xxx} - 3\Psi_x\mathcal{D}\Psi_x = 0, \tag{3.16}$$

which is the above equation is the $\mathcal{N} = 1$ SUSY KdV equation (3.1) with $\Phi = \Psi_x$.

Taking the bosonic parameters $\alpha = \beta = \tau = 0$, (3.13) and (3.15) reduce to

$$D_x(S_x + \gamma)f \cdot g = 0, \tag{3.17a}$$

$$(D_t + D_x^3)f \cdot g = 0 \tag{3.17b}$$

and

$$\mathcal{D}\Omega_x - \Psi_x\Omega - \gamma\Omega_x = 0, \tag{3.18a}$$

$$\Omega_t - 3(\mathcal{D}\Psi_x)\Omega_x + \Omega_{xxx} = 0. \tag{3.18b}$$

Here (3.18) can serve as a Lax pair of the $\mathcal{N} = 1$ SUSY KdV equation but the "spectral" parameter γ is fermionic. Besides, comparing (3.4a) and (3.17a), one may find that when S_x and D_x switch places the bosonic parameter μ and fermionic parameter γ switch their places as well.

4. $\mathcal{N} = 2$ SUSY KdV

4.1. *BT and Lax pair*

There exist more than one supersymmetric extensions for the KdV equation. Let us look at the following equation

$$\Phi_t = -\Phi_{xxx} + 3(\Phi\mathcal{D}_1\mathcal{D}_2\Phi)_x + \frac{1}{2}(a-1)(\mathcal{D}_1\mathcal{D}_2\Phi^2)_x + 3a\Phi^2\phi_x, \tag{4.1}$$

56

which is called the $\mathcal{N} = 2$ SUSY KdV equation [18, 19], where $\Phi \doteq \Phi(x, t, \theta_1, \theta_2)$ is a superboson function depending on temporal variable t, spatial variable x and its fermionic counterparts θ_i $(i = 1, 2)$, and

$$\mathcal{D}_i = \partial_{\theta_i} + \theta_i \partial_x, \quad (i = 1, 2),$$

(4.1) is also referred to as the SKdV$_a$ equation corresponding to the value of real parameter a. It has been shown that the SKdV$_a$ equation can pass Painlevé test only when $a = -2, 0, 1, 4$ [2]. In this section, we focus on the SKdV$_1$ equation, i.e.,

$$\Phi_t = -\Phi_{xxx} + 3(\Phi\mathcal{D}_1\mathcal{D}_2\Phi)_x + 3\Phi^2\Phi_x. \tag{4.2}$$

Lax pairs of the equation have been given in [29, 33].

Through the transformation [33]

$$\Phi(x, t, \theta_1, \theta_2) = i\left(\ln\frac{f}{g}\right)_x - \theta_2(\mathcal{D}\ln fg)_x, \tag{4.3}$$

where $f = f(x, t, \theta_1)$ and $g = g(x, t, \theta_1)$ are bosonic functions, $i = \sqrt{-1}$, and here and below we make use of a simplified notation $\mathcal{D} = \mathcal{D}_1 = \partial_{\theta_1} + \theta_1\partial_x$, the SKdV$_1$ equation was bilinearized to

$$(D_t + D_x^3)f \cdot g = 0, \tag{4.4a}$$

$$S_x(D_t + D_x^3)f \cdot g = 70. \tag{4.4b}$$

The related bilinear BT reads [33]

$$(D_x - \mu)(g \cdot f' - f \cdot g') = 0, \tag{4.5a}$$

$$S_x(D_x - \mu)(g \cdot f' + f \cdot g') = 0, \tag{4.5b}$$

$$(D_t + D_x^3 - 3\mu D_x^2 + 3\mu^2 D_x)f \cdot f' = 0, \tag{4.5c}$$

$$(D_t + D_x^3 - 3\mu D_x^2 + 3\mu^2 D_x)g \cdot g' = 0, \tag{4.5d}$$

with a bosonic parameter μ. That is to say, if (f, g) solves (4.4) and (f, g), (f', g') satisfy the above BT, then (f', g') provides a solution to the SKdV$_1$ equation through

$$\Phi(x, t, \theta_1, \theta_2) = i\left(\ln\frac{f'}{g'}\right)_x - \theta_2(\mathcal{D}\ln f'g')_x. \tag{4.6}$$

Besides solutions, the BT (4.5) can lead to a Lax representation for the SKdV$_1$ equation, which reads [33]

$$U_x + \varphi_x V + \mu U = 0, \tag{4.7a}$$

$$\mathcal{D}V_x + \varphi_x \mathcal{D}U + \mu\mathcal{D}V + V\mathcal{D}\rho_x = 0, \tag{4.7b}$$

$$V_t + 3\mu\rho_{xx}V + 3\mu\varphi_{xx}U + V_{xxx} + 3\varphi_{xx}U_x + 3\rho_{xx}V_x + 3\mu V_{xx} + 3\mu^2 V_x = 0, \tag{4.7c}$$

$$U_t + 3\mu\rho_{xx}U + 3\mu\varphi_{xx}V + U_{xxx} + 3\varphi_{xx}U_x + 3\rho_{xx}U_x + 3\mu U_{xx} + 3\mu^2 U_x = 0, \tag{4.7d}$$

where

$$U = \frac{f'}{f} - \frac{g'}{g}, \quad V = \frac{f'}{f} + \frac{g'}{g}, \quad \varphi = \ln\frac{f}{g}, \quad \rho = \ln fg. \tag{4.8}$$

4.2. *Variety of solutions*

Similar to the $\mathcal{N} = 1$ SUSY KdV equation, here through the bilinear BT (4.5), one can get variety of solutions for the SKdV$_1$ equation. The following are some solutions, some of which have been derived in [33].

(I)

$$f = g = 1, \quad \mu = 0, \tag{4.9a}$$

$$f' = a_1 x + \theta_1 b_1 + c_2 - c_1, \quad g' = a_1 x - \theta_1 b_1 + c_1 + c_2, \tag{4.9b}$$

where a_1, b_1, c_1, c_2 are even constants;

(II)

$$f = 1, \quad g = 1, \quad \mu = -k_1, \tag{4.10a}$$

$$f' = 1 + \varepsilon e^{\xi_1 + \theta_1 \varrho_1}, \quad g' = 1 - \varepsilon e^{\xi_1 + \theta_1 \varrho_1}, \tag{4.10b}$$

where ξ_1 is defined in (3.7) and ϱ_1 is a odd constant;

(III)

$$f = 1 + \varepsilon e^{\xi_1 + \theta_1 \varrho_1}, \quad g = 1 - \varepsilon e^{\xi_1 + \theta_1 \varrho_1}, \quad \mu = -k_2, \tag{4.11a}$$

$$f' = 1 + \varepsilon (e^{\xi_1 + \theta_1 \varrho_1} + e^{\xi_2 + \theta_1 \varrho_2}) + \varepsilon^2 A_{12} e^{(\xi_1 + \xi_2) + \theta_1(\varrho_1 + \varrho_2)}, \tag{4.11b}$$

$$g' = 1 - \varepsilon (e^{\xi_1 + \theta_1 \varrho_1} + e^{\xi_2 + \theta_1 \varrho_2}) + \varepsilon^2 A_{12} e^{(\xi_1 + \xi_2) + \theta_1(\varrho_1 + \varrho_2)}, \tag{4.11c}$$

where

$$A_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)\left(\frac{k_2 - k_1 + 2\varrho_1 \varrho_2}{k_1 + k_2} + 2\theta_1 \frac{k_1 \varrho_2 - k_2 \varrho_1}{p_1 + p_2}\right),$$

ξ_i are defined in (3.7) and ϱ_i are odd constants;

(IV)

$$f = 1 + \varepsilon e^{\xi_1 + \theta_1 \varrho_1}, \quad g = 1 - \varepsilon e^{\xi_1 + \theta_1 \varrho_1}, \quad \mu = -k_1, \tag{4.12a}$$

$$f' = 1 - 2\varepsilon(\eta_1 + \theta_1 \varrho_1)e^{\xi_1 + \theta_1 \varrho_1} - \varepsilon^2 2 e^{2(\xi_1 + \theta_1 \varrho_1)}, \tag{4.12b}$$

$$g' = 1 + 2\varepsilon(\eta_1 + \theta_1 \varrho_1)e^{\xi_1 + \theta_1 \varrho_1} - \varepsilon^2 2 e^{2(\xi_1 + \theta_1 \varrho_1)}, \tag{4.12c}$$

where ξ_i, η_i are defined in (3.7) and ϱ_i are odd constants. Through the transformation (4.6) and (4.10b) gives an 1-soliton solution, (4.11b), (4.11c) gives a 2-soliton solution, which have been already known; while (4.9b) provides a rational solution and (4.12b), (4.12c) provides a mixed solution. Similar to the previous section, one can establish a limit relation between (4.11b), (4.11c) and (4.12b), (4.12c).

58

4.3. *Deformed BTs and Lax pairs*

Similar to Sec. 3.3, replacing (f, g) and (f', g') in (4.5) by $(e^{\zeta_1} f, e^{\zeta_1} g)$ and $(e^{\zeta_2} f', e^{\zeta_2} g')$, respectively, where ζ_i are given in (2.7) with $\theta = \theta_1$, we get

$$(D_x - \beta)(g \cdot f' - f \cdot g') = 0, \tag{4.13a}$$

$$(S_x + \gamma + \theta\,\alpha)(D_x - \beta)(g \cdot f' + f \cdot g') = 0, \tag{4.13b}$$

$$(D_t + D_x^3 - 3\beta D_x^2 + 3\beta^2 D_x)f \cdot f' = \tau f f', \tag{4.13c}$$

$$(D_t + D_x^3 - 3\beta D_x^2 + 3\beta^2 D_x)g \cdot g' = \tau g g', \tag{4.13d}$$

where $\beta, \gamma, \alpha, \tau$ are defined in (3.14). It can be reduced to

$$D_x(g \cdot f' - f \cdot g') = 0, \tag{4.14a}$$

$$(S_x + \gamma)D_x(g \cdot f' + f \cdot g') = 0, \tag{4.14b}$$

$$(D_t + D_x^3)f \cdot f' = 0, \tag{4.14c}$$

$$(D_t + D_x^3)g \cdot g' = 0, \tag{4.14d}$$

where the only one parameter γ is fermionic.

Through the transformation (4.8), the above deformed BT (4.14) contributes a deformed Lax pair

$$U_x + \varphi_x V = 0, \tag{4.15a}$$

$$\mathcal{D}V_x + \varphi_x \mathcal{D}U + V\mathcal{D}\rho_x - \gamma V_x - \gamma U \varphi_x = 0, \tag{4.15b}$$

$$V_t + V_{xxx} + 3\varphi_{xx}U_x + 3\rho_{xx}V_x = 0, \tag{4.15c}$$

$$U_t + U_{xxx} + 3\varphi_{xx}V_x + 3\rho_{xx}U_x = 0, \tag{4.15d}$$

with a fermionic "spectral" parameter γ, of which the compatibility condition (i.e., $U_{xt} = U_{tx}$ and $\mathcal{D}V_{xt} = \mathcal{D}V_{tx}$) yields the $\mathcal{N} = 2$ SKdV$_1$ equation (4.2).

5. Conclusion

In the paper, we have shown that the bilinear BTs can provide variety of solutions to the SUSY equations such as the $\mathcal{N} = 1$ supersymmetric KdV equation and the $\mathcal{N} = 2$ supersymmetric KdV$_1$ equation. By virtue of the gauge invariance of (super) Hirota bilinear derivatives, the bilinear BTs can be deformed and the deformations brought new BTs and Lax pairs only with fermionic parameters. It is well known that spectral parameters play important roles in the soliton theory. For example, for a given Lax integrable model, using its Lax pair and expanding some functions in the power series of the spectral parameter, it is possible to obtain infinitely many conserved quantities for the given model. However, in (3.18) and (4.15), the fermionic parameter γ takes the place of the bosonic spectral parameter. The fermionic parameter does not work in power expansion for seeking infinitely many conservation laws due to the property $\gamma^j \equiv 0$ when $j \geq 2$. However, this property may simplify the verification of the compatibility of Lax pair representations. For this moment we cannot explain the meaning of the fermionic parameter γ which appears in those deformed

Lax representations, but we do hope such it could bring something new or interesting. Besides, We believe similar investigation can be extended to other SUSY equations such as the SUSY modified KdV equation [23], the SUSY sine-Gordon equation [11] and so on.

Acknowledgments

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