Properties of the zeros of the sum of three polynomials

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LETTER TO THE EDITOR

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A system of algebraic equations satisfied by the zeros of the sum of three polynomials are reported.

1. Introduction

In a previous paper we highlighted certain properties of the zeros of the sum of two polynomials [6]. In the last section ("Outlook") of that paper we wrote that the "extension of these findings to the zeros of the sums of more than two polynomials will be reported in subsequent papers if these results will be deemed sufficiently novel and neat to justify their publication." In this paper we indeed report the extension of some of those findings to the sum of *three* polynomials.

Let $p_{N_j}^{(j)}(z)$, $j = 1, 2, 3$, be 3 arbitrary (for definiteness, *monic*) polynomials of the independent (complex) variable z, of degree N_j with, for definiteness and simplicity, $N_1 \leq N_2 < N_3$; and let us define as follows the, also *monic*, polynomial $\psi_N(z; t_1, t_2)$ of degree $N \equiv N_3$ in z and depending on the 2 parameters t_1 and t_2:

$$\psi_N(z; t_1, t_2) = p_N^{(3)}(z) + t_2\, p_{N_2}^{(2)}(z) + t_1\, p_{N_1}^{(1)}(z)\,. \tag{1.1}$$

Here and hereafter N is an *arbitrary positive integer* (say, *larger than unity*, to avoid that some of the formulas written below become too degenerate and trivial).

Hereafter we denote as $z_n \equiv z_n(t_1, t_2)$ the N (generally complex) zeros of this polynomial:

$$\psi_N(z; t_1, t_2) = \prod_{n=1}^{N} [z - z_n(t_1, t_2)]\,. \tag{1.2}$$

For simplicity, we hereafter assume these N zeros to be *all different among themselves*; the formulas displayed below remain generally valid when this assumption is violated, but possibly only after taking an appropriate limit.

In this paper we report a system of algebraic equations satisfied by these N zeros $z_n(t_1, t_2)$. A *solvable* system of N nonlinear second-order partial differential equations (PDEs) satisfied by these

N functions of two variables is also identified; and it is shown that it belongs to a known class of "C-integrable" PDEs in multidimensions [2,7]. In the following Section 2 these findings are displayed, and they are then proven in Section 3. A terse Section 4 mentions possible extensions of these findings.

2. Results

Hereafter we denote as $W(f;g)$ the wronskian of the two (monic) polynomials $f(z)$ and $g(z)$,

$$W(f;g) \equiv f'(z)\, g(z) - g'(z)\, f(z)\,. \tag{2.1}$$

Here and hereafter appended primes denote differentiations with respect to the variable z. Of course $W\left(p_{N_j}^{(j)}; p_{N_k}^{(k)}\right)$ is a polynomial of degree $N_j + N_k - 1$ if $N_j \neq N_k$ (and then $W\left(p_{N_j}^{(j)}; p_{N_k}^{(k)}\right)/(N_j - N_k)$ is *monic*); while if instead $N_j = N_k$, then $W\left(p_{N_j}^{(j)}; p_{N_k}^{(k)}\right)$ is a (generally *not* monic) polynomial of degree $N_j + N_k - 2$.

Proposition 2.1. *The N zeros $z_n \equiv z_n(t_1, t_2)$ of the polynomial $\psi_N(z; t_1, t_2)$, see (1.1) and (1.2), satisfy the following system of N nonlinear algebraic equations:*

$$\sum_{j=1,2,3 \bmod(3)} \left\{ 3 \left[p_{N_j}^{(j)''}(z_n)\, w_{j+1,j+2}(z_n) \right] \cdot \right.$$

$$\cdot \sum_{\ell=1,\ \ell \neq n, m \neq n, \ell \neq m}^{N} \left[\left(\frac{1}{z_n - z_\ell} \right) \left(\frac{1}{z_n - z_m} \right) \right]$$

$$-2\, p_{N_j}^{(j)'''}(z_n)\, w_{j+1,j+2}(z_n) \sum_{\ell=1,\ \ell \neq n}^{N} \left(\frac{1}{z_n - z_\ell} \right)$$

$$\left. + \left[p_{N_j}^{(j)'''}(z_n)\, w_{j+1,j+2}'(z_n) \right] \right\} = 0\,, \quad n = 1, \ldots, N\,, \tag{2.2a}$$

where

$$w_{j,k}(z) \equiv W\left(p_{N_j}^{(j)}(z),\ p_{N_k}^{(k)}(z) \right)\,, \quad j,k = 1,2,3 \ \bmod(3)\,. \tag{2.2b}$$

Note that, for notational simplicity, we omitted to display explicitly in (2.2a) the dependence of the *N* zeros $z_k \equiv z_k(t_1, t_2)$, $k = 1, \ldots, N$, on the two parameters t_1 and t_2. And of course, above and hereafter, the notation $f'(z_n)$ denotes the *z*-derivative of the function $f(z)$ evaluated at $z = z_n \equiv z_n(t_1, t_2)$.

Proposition 2.2. *The N zeros $z_n \equiv z_n(t_1, t_2)$ of the polynomial $\psi_N(z; t_1, t_2)$, see (1.1) and (1.2), satisfy the following system of N nonlinear second-order hyperbolic PDEs,*

$$z_{n,t_1 t_2} = \sum_{\ell=1,\ \ell \neq n}^{N} \left(\frac{z_{n,t_1}\, z_{\ell,t_2} + z_{n,t_2}\, z_{\ell,t_1}}{z_n - z_\ell} \right)\,, \quad n = 1, \ldots, N\,, \tag{2.3}$$

provided the boundary conditions $z_n(t,0)$ and $z_n(0,t)$ are assigned so that $z_n(t,0)$ are the N zeros of the polynomial $\psi_N(z; t, 0) = p_N^{(3)}(z) + t\, p_{N_2}^{(2)}(z)$ and $z_n(0,t)$ are the N zeros of the polynomial

$$\psi_N(z;0,t) = p_N^{(3)}(z) + t\, p_{N_1}^{(1)}(z):$$

$$\psi_N(z;t,0) = p_N^{(3)}(z) + t\, p_{N_2}^{(2)}(z) = \prod_{n=1}^{N}[z - z_n(t,0)] \ , \tag{2.4a}$$

$$\psi_N(z;0,t) = p_N^{(3)}(z) + t\, p_{N_1}^{(1)}(z) = \prod_{n=1}^{N}[z - z_n(0,t)] \ . \tag{2.4b}$$

Note that again, for notational simplicity, we omitted to display explicitly in (2.3) the dependence of the N zeros $z_k \equiv z_k(t_1,t_2)$, $k = 1,\ldots,N$, on the two parameters t_1 and t_2. And of course, above and hereafter, the subscripted variables t_1 and t_2 denote partial differentiations with respect to these variables, $z_{n,t_1} \equiv z_{n,t_1}(t_1,t_2) \equiv \partial z_n(t_1,t_2)/\partial t_1$, $z_{n,t_2} \equiv z_{n,t_2}(t_1,t_2) \equiv \partial z_n(t_1,t_2)/\partial t_2$, $z_{n,t_1 t_2} \equiv z_{n,t_1 t_2}(t_1,t_2) \equiv \partial^2 z_n(t_1,t_2)/\partial t_1 \partial t_2$.

Via the change of variables

$$z_n(t_1,t_2) = \varphi_n(x,t) \ , \quad t_1 = x - c\,t \ , \ t_2 = x + c\,t \ ; \quad n = 1,\ldots,N \ , \tag{2.5}$$

the system of PDEs (2.3) becomes

$$\varphi_{n,tt} - c^2\, \varphi_{n,xx} = \sum_{\ell=1,\ \ell \neq n}^{N} \left(\frac{\varphi_{n,t}\, \varphi_{\ell,t} - c^2\, \varphi_{n,x}\, \varphi_{\ell,x}}{\varphi_n - \varphi_\ell} \right) \ , \quad n = 1,\ldots,N \ , \tag{2.6}$$

which has the interesting feature of being invariant under (two-dimensional) Lorentz transformations. It is moreover clear that this system of nonlinear PDEs is a neat extension of the standard "goldfish" model [4,1,3,5], to which the system (2.6) indeed reduces for $c = 0$ (and likewise the system (2.3) for $t_1 = t_2 = t$, implying $N_1 = N_2$, $p_{N_1}^{(1)}(z) = p_{N_2}^{(2)}(z)$; see (2.4)). Clearly this system, (2.6), is a special case of the systems of C-integrable PDEs introduced in [2] and investigated (including the display of solutions in the guise of animations) in [7].

3. Proofs

To prove Proposition 2.1 we start from the well-known observation that, up to its normalization and an appropriate identification of the 2 parameters t_1 and t_2 (both specified above, see (1.1)), the polynomial $\psi_N(z;t_1,t_2)$ can be identified as the solution of the third-order ODE

$$\det \begin{pmatrix} \psi_N'''(z;t_1,t_2) & p_N^{(3)\prime\prime\prime}(z) & p_{N_2}^{(2)\prime\prime\prime}(z) & p_{N_1}^{(1)\prime\prime\prime}(z) \\ \psi_N''(z;t_1,t_2) & p_N^{(3)\prime\prime}(z) & p_{N_2}^{(2)\prime\prime}(z) & p_{N_1}^{(1)\prime\prime}(z) \\ \psi_N'(z;t_1,t_2) & p_N^{(3)\prime}(z) & p_{N_2}^{(2)\prime}(z) & p_{N_1}^{(1)\prime}(z) \\ \psi_N(z;t_1,t_2) & p_N^{(3)}(z) & p_{N_2}^{(2)}(z) & p_{N_1}^{(1)}(z) \end{pmatrix} = 0 \ , \tag{3.1a}$$

or equivalently

$$a_3(z)\, \psi_N'''(z;t_1,t_2) - a_2(z)\, \psi_N''(z;t_1,t_2)$$
$$+ a_1(z)\, \psi_N'(z;t_1,t_2) - a_0(z)\, \psi_N(z;t_1,t_2) = 0 \tag{3.1b}$$

where (see (2.1))

$$a_3(z) = \sum_{j=1,2,3 \ \mathrm{mod}(3)} \left[p_{N_j}^{(j)\prime\prime}(z)\, W\left(p_{N_{j+1}}^{(j+1)}(z),\, p_{N_{j+2}}^{(j+2)}(z) \right) \right] \ , \tag{3.2a}$$

$$a_2(z) = \sum_{j=1,2,3 \text{ mod}(3)} \left[p_{N_j}^{(j)'''}(z) \, W\left(p_{N_{j+1}}^{(j+1)}(z), \, p_{N_{j+2}}^{(j+2)}(z) \right) \right] , \qquad (3.2b)$$

$$a_1(z) = \sum_{j=1,2,3 \text{ mod}(3)} \left[p_{N_j}^{(j)'''}(z) \, W'\left(p_{N_{j+1}}^{(j+1)}(z), \, p_{N_{j+2}}^{(j+2)}(z) \right) \right] , \qquad (3.2c)$$

$$a_0(z) = \sum_{j=1,2,3 \text{ mod}(3)} \left[p_{N_j}^{(j)'}(z) \, W\left(p_{N_{j+1}}^{(j+1)''}(z), \, p_{N_{j+2}}^{(j+2)''}(z) \right) \right] . \qquad (3.2d)$$

At $z = z_n \equiv z_n(t_1, t_2)$ (where $\psi_N(z; t_1, t_2)$ vanishes, see (1.2)) formula (3.1b) clearly entails that

$$\begin{aligned} & a_3(z_n) \, \psi_N'''(z_n; t_1, t_2) - a_2(z_n) \, \psi_N''(z_n; t_1, t_2) \\ & + a_1(z_n) \, \psi_N'(z_n; t_1, t_2) = 0 , \quad n = 1, \ldots, N . \end{aligned} \qquad (3.3)$$

Here and below, for notational simplicity, we often omit to display explicitly the dependence of the zeros $z_n \equiv z_n(t_1, t_2)$ on the two parameters t_1 and t_2; as we already did in the preceding section.

On the other hand logarithmic z-differentiation of (1.2) implies

$$\psi_N'(z; t_1, t_2) = \psi_N(z; t_1, t_2) \sum_{\ell=1}^{N} \left(\frac{1}{z - z_\ell} \right) , \qquad (3.4a)$$

a second z-differentiation yields (via (3.4a) and a neat cancellation of the terms featuring double poles)

$$\psi_N''(z; t_1, t_2) = \psi_N(z; t_1, t_2) \sum_{\ell,m=1, \, \ell \neq m}^{N} \left[\left(\frac{1}{z - z_\ell} \right) \left(\frac{1}{z - z_m} \right) \right] , \qquad (3.4b)$$

and a third z-differentiation yields (again via (3.4a) and a neat cancellation of the terms featuring double poles)

$$\psi_N'''(z; t_1, t_2) = \psi_N(z; t_1, t_2) \cdot$$
$$\cdot \sum_{\ell,m,k=1, \, \ell \neq m, m \neq k, k \neq \ell}^{N} \left[\left(\frac{1}{z - z_\ell} \right) \left(\frac{1}{z - z_m} \right) \left(\frac{1}{z - z_k} \right) \right] . \qquad (3.4c)$$

Hence

$$\frac{\psi_N''(z; t_1, t_2)}{\psi_N'(z; t_1, t_2)} = \sum_{\ell,m=1, \, \ell \neq m}^{N} \left[\left(\frac{1}{z - z_\ell} \right) \left(\frac{1}{z - z_m} \right) \right] \left\{ \sum_{n=1}^{N} \left(\frac{1}{z - z_n} \right) \right\}^{-1} , \qquad (3.5a)$$

$$\frac{\psi_N'''(z; t_1, t_2)}{\psi_N'(z; t_1, t_2)} = \sum_{\ell,m,k=1, \, \ell \neq m, m \neq k, k \neq \ell}^{N} \left[\left(\frac{1}{z - z_\ell} \right) \left(\frac{1}{z - z_m} \right) \left(\frac{1}{z - z_k} \right) \right] \cdot$$
$$\cdot \left\{ \sum_{n=1}^{N} \left(\frac{1}{z - z_n} \right) \right\}^{-1} . \qquad (3.5b)$$

And clearly for $z = z_n \equiv z_n(t_1, t_2)$ these formulas yield (for $n = 1, \ldots, N$)

$$\frac{\psi_N''(z_n; t_1, t_2)}{\psi_N'(z_n; t_1, t_2)} = 2 \sum_{\ell=1,\ \ell \neq n}^{N} \left(\frac{1}{z_n - z_\ell} \right) , \tag{3.6a}$$

$$\frac{\psi_N'''(z_n; t_1, t_2)}{\psi_N'(z_n; t_1, t_2)} = 3 \sum_{\ell=1,\ \ell \neq n, m \neq n, \ell \neq m}^{N} \left[\left(\frac{1}{z_n - z_\ell} \right) \left(\frac{1}{z_n - z_m} \right) \right] . \tag{3.6b}$$

Hence the insertion of these formulas in (3.3) (divided by $\psi_N'(z_n; t_1, t_2)$) yields the equation

$$3\, a_3(z_n) \sum_{\ell=1,\ \ell \neq n, m \neq n, \ell \neq m}^{N} \left[\left(\frac{1}{z_n - z_\ell} \right) \left(\frac{1}{z_n - z_m} \right) \right]$$

$$- 2\, a_2(z_n) \sum_{\ell=1,\ \ell \neq n}^{N} \left(\frac{1}{z_n - z_\ell} \right) + a_1(z_n) = 0 , \quad n = 1, \ldots, N , \tag{3.7}$$

and via the first 3 formulas (3.2) one then gets (2.2). Proposition 2.1 is thereby proven. □

To prove Proposition 2.2 we start from the obvious observation that the polynomial $\psi_N(z; t_1, t_2)$, see (1.1), satisfies the PDE

$$\psi_{N, t_1 t_2}(z; t_1, t_2) = 0 . \tag{3.8}$$

Let us recall that here and below subscripted variables denote partial differentiations, $\psi_{N, t_1 t_2} \equiv \partial^2 \psi / \partial t_1 \partial t_2$.

We then note that (logarithmic, partial) t_1-differentiation of (1.2) yields

$$\psi_{N, t_1}(z; t_1, t_2) = -\psi_N(z; t_1, t_2) \sum_{n=1}^{N} \left(\frac{z_{n, t_1}}{z_n - z_\ell} \right) , \tag{3.9a}$$

and a (partial) t_2-differentiation of this formula yields (after a neat cancellation of the double-pole terms, and an appropriate renaming of dummy indices), the formula

$$\psi_{N, t_1 t_2}(z; t_1, t_2) = \psi_N(z; t_1, t_2) \sum_{n=1}^{N} \left\{ (z - z_n)^{-1} \cdot \right.$$

$$\left. \cdot \left[-z_{n, t_1 t_2} + \sum_{\ell=1,\ \ell \neq n}^{N} \left(\frac{z_{n, t_1}\, z_{\ell, t_2} + z_{n, t_2}\, z_{\ell, t_1}}{z_n - z_\ell} \right) \right] \right\} . \tag{3.9b}$$

Note that again, for notational simplicity, in the right-hand side of this formula we omitted to display the dependence of the zeros $z_n \equiv z_n(t_1, t_2)$ on the two parameters t_1 and t_2. And clearly this formula, via (3.8), implies (2.3). While the formulas (2.4) are an easy consequence of (1.1). Proposition 2.2 is thereby proven. □

4. Outlook

A natural extension of these results is to treat a linear combination of an *arbitrary* number of polynomials. Other interesting extensions may obtain by replacing polynomials with trigonometric polynomials or other combinations of seed functions, and/or replacing the parameters t_k with

functions of (a smaller number of) other parameters. And of course other, potentially interesting, findings may emerge from the investigation of *isochronous* variants of the solvable systems (2.3) and (2.6)—obtained via the, by now well-known, "trick" generating such systems, see for instance Section 2.1 (entitled "The trick") and Chapter 7 (entitled "Isochronous PDEs") of [5]—and from further investigations of the application of this development to the more general class of solvable systems considered in [2] and [7].

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