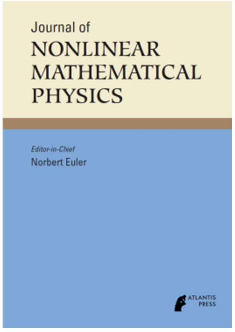

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Hybrid Ermakov-Painlevé IV Systems

Colin Rogers
Australian Research Council Centre of Excellence for Mathematics & Statistics of Complex Systems, School of Mathematics and Statistics, The University of New South Wales, Sydney, NSW2052, Australia.
c.rogers@unsw.edu.au

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Ermakov-Painlevé IV coupled systems are introduced and associated Ermakov-type invariants isolated. These invariants are used to obtain systematic reduction of the system in terms of the canonical Painlevé IV equation. The procedure is applied to a Ermakov-Painlevé IV symmetry reduction of a coupled derivative resonant nonlinear Schrödinger triad incorporating de Broglie-Bohm potential terms.

1. Introduction

The six classical Painlevé equations arise in a wide range of physical applications and have a well-established and fundamental role in modern soliton theory (see e.g. Conte [13] and Clarkson [12] together with literature cited therein). On the other hand, Ermakov-type systems and their generalisations likewise have extensive applications, notably in nonlinear optics [15, 20, 21, 23, 48, 49, 62] where, inportantly, they have been used to model the evolution of the size and shape of the light spot and wave front in elliptical Gaussian beams [15, 23].

Nonlinear coupled systems of Ermakov-Ray-Reid type have their origin in independent classical studies of Ermakov [18] and Steen [61] and were originally introduced by Ray and Reid in [38, 39]. The systems adopt the form

$$\begin{aligned} \ddot{\phi} + \omega(t)\phi &= \frac{1}{\phi^2\psi}\,\Phi(\psi/\phi)\ , \\ \ddot{\psi} + \omega(t)\psi &= \frac{1}{\psi^2\phi}\,\Psi(\phi/\psi) \end{aligned} \tag{1.1}$$

and admit a distinctive integral of motion, namely the invariant

$$\mathscr{I} = \frac{1}{2}(\phi\dot{\psi} - \psi\dot{\phi})^2 + \int^{z=\psi/\phi} \Phi(z)dz + \int^{w=\phi/\psi} \Psi(w)dw\ , \tag{1.2}$$

where, in the above, the dot indicates a derivative with respect to the independent variable t. Subsequently 2+1-dimensional versions of (1.1) were introduced in [47] while extensions to arbitrary order and dimension which preserve the characteristic invariant were presented in [59]. Multi-component Ermakov-Ray-Reid systems were introduced in a hydrodynamics context in [42] via a symmetry reduction of a 2+1-dimensional multi-layer fluid model. Moreover, sequences of two-component Ermakov-Ray-Reid systems were shown therein to be linked via Darboux transformations. Discretisation aspects of particular Ermakov-type equations have been investigated in [55,57].

Stability and periodic properties of a Ermakov-Ray-Reid system arising out of such a two-layer hydrodynamic model have been investigated by Athorne [4] while novel algebraic aspects underlying the multi-component Ermakov-type systems introduced in [42] were uncovered in [5]. In recent work [45], a Hamiltonian Ermakov-Ray-Reid system has been obtained in the context of a classical 2+1-dimensional rotating shallow water system with an underlying rigid circular paraboloidal bottom topography. The Ermakov variables in that case have the physical interpretation that they describe the time evolution of the semi-axes of the elliptical moving shoreline on the paraboloidal basin. The route to the Ermakov-Ray-Reid system, interestingly involves application of extensions of Ball-type moment of inertia theorems [7, 8] which, in turn, may be shown to have a Lie group origin [30, 40].

Ermakov-Ray-Reid systems have also been uncovered in magnetogasdynamics [43, 53, 60] and remarkably, have been shown to admit pulsrodon-type phenomena analogous to those observed in an elliptic warm core eddy context (Rogers [41], Rubino and Brandt [56]). Lyapunov stability aspects of pulsrodons and their duals has been discussed by Holm [24]. The magneto-gasdynamics pulsrodon describes a rotating, pulsating elliptic cylindrical plasma column bounded by a vacuum state [53]. Ermakov-Ray-Reid structure has also recently been isolated in [53] in connection with gas cloud evolution of a type originally investigated by Ovsiannikov [34] and Dyson [17] and subsequently by Gaffet (see [19] and work cited therein). Connection with integrability has been made in the 2+1-dimensional case by construction of a Lax pair [24].

In recent work by Desyatnik *et al* [16], the contribution of orbital angular momentum to the suppression of the collapse of spiralling elliptic solitons in Kerr media has been investigated via a variational approach. An analogous system modelled by a 2+1-dimensional NLS equation incorporating an harmonic trap was subsequently derived by Abdullaev *et al* [1] by means of a variational approximation in the context of elliptic cloud evolution in a Bose-Einstein condensate. In [48] it has been shown that the remarkable repeated occurrence of Hamiltonian Ermakov-Ray-Reid structure in nonlinear optics extends to both the spiralling elliptic soliton system of [16] and to its generalization in the Bose-Einstein setting in [1].

The preceding attests to the established importance of Ermakov-Ray-Reid systems in a wide range of physical applications. Like solitonic systems and their associated classical Painlevé equations which admit nonlinear superposition principles (permutability theorems) associated with invariance under Bäcklund transformations [22, 50, 52] it is known that Ermakov-Ray-Reid systems also possess underlying nonlinear superposition laws albeit of another kind [38, 39]. Moreover, just as solitonic and associated Painlevé equations generically admit linear representations, Ermakov-Ray-Reid systems also have been shown to have an underlying linear structure [6]. However, despite these key common properties, the two areas of soliton and of Ermakov-Ray-Reid theory have generally proceeded independently. The only known hybrid solitonic-Ermakov system seems to be that set down in [59] where, alignment of the integrable Ernst system of general relativity, with a 2+1-dimensional Ermakov system was shown to lead to a composition of an integrable 2+1-dimensional sinh Gordon system (see [25–27, 51, 58]) and of a Ermakov-Ray-Reid system. However, recently a hybrid Ermakov-Painlevé II system was derived as a reduction of a coupled N+1-dimensional Manakov-type NLS system in [44]. Ermakov invariants admitted by the hybrid system were key to its systematic reduction in terms of a single component Ermakov-Painlevé II equation which, in turn, may be linked to the integrable Painlevé II equation. In terms of physical applications, the latter equation has been shown to arise in the steady, two-ion reduction with valency relation $\nu_1 + \nu_2 = 0$ of the classical Nernst-Planck ion system descriptive of the migration of charged particles through material barriers [9, 11, 33, 36, 37]. The hybrid Ermakov-Painlevé II equation of [44] arises directly in a pair of three-ion cases with valency ratios $\nu_1 : \nu_2 : \nu_3 = 1 : -2 : -1$ as previously isolated by Conte *et al* in [14]. It adopts the form

$$\Omega_{xx} + \frac{x}{2}\,\Omega + \varepsilon\Omega^3 = -\frac{1}{4\Omega^3}\,(\gamma - \frac{\varepsilon}{2})^2 \ , \qquad (\varepsilon = \pm 1) \tag{1.3}$$

where Ω is connected to the electric field E by a relation of the type $E \sim \Omega_x/\Omega$. The nonlinear equation (1.3) is, in turn, connected to the classical Painlevé II equation

$$\omega_{xx} = 2\omega^3 + x\omega + \gamma \ , \tag{1.4}$$

via the relation

$$\omega = \frac{\varepsilon}{2\Omega^2}\,[\,\gamma - \frac{\varepsilon}{2} - 2\Omega\Omega_x\,] \tag{1.5}$$

(c.f. Gromak [22]).

In [44], a two-component 'resonant' nonlinear Schrödinger system

$$\begin{gathered} i\,\Phi_z + \nabla^2\Phi - s\,[\,\nabla^2|\Phi|/|\Phi|\,]\,\Phi + \nu\,[\,|\Phi|^2 + |\Psi|^2\,]\,\Phi = 0 \ , \\ i\,\Psi_z + \nabla^2\Psi - s\,[\,\nabla^2|\Psi|/|\Psi|\,]\,\Psi + \nu\,[\,|\Phi|^2 + |\Psi|^2\,]\,\Psi = 0 \\ (\,\nabla^2 := \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_N^2\,) \end{gathered} \tag{1.6}$$

was investigated which incorporates de-Broglie-Bohm-type potential terms in $\nabla^2|\Phi|/|\Phi|$ and $\nabla^2|\Psi|/|\Psi|$. Wave packet representations were introduced into (1.6) which resulted in a symmetry reduction to a novel coupled Ermakov Painlevé II system for the amplitudes $|\Phi|$, $|\Psi|$ in the form

$$\begin{aligned} |\Phi|_{\xi\xi} + [\,c_{\mathrm{I}} + c_{\mathrm{II}}\xi\,]\,|\Phi| + c_{\mathrm{III}}\,(\,|\Phi|^2 + |\Psi|^2\,)\,|\Phi| &= \frac{1}{1-s}\left(\frac{\mathscr{I}}{N}\right)^2 \frac{1}{|\Phi|^3} \ , \\ |\Psi|_{\xi\xi} + [\,c_{\mathrm{I}} + c_{\mathrm{II}}\xi\,]\,|\Psi| + c_{\mathrm{III}}\,(\,|\Phi|^2 + |\Psi|^2\,)\,|\Psi| &= \frac{1}{1-s}\left(\frac{\mathscr{K}}{N}\right)^2 \frac{1}{|\Psi|^3} \end{aligned} \tag{1.7}$$

where $\xi = \lambda z + \mu z^2 + x_1 + \cdots + x_N$ and $\mathscr{I}$, $\mathscr{K}$ are appropriate invariants.

The pair (1.7) constitutes a particular Ermakov-Ray-Reid system and admits the characteristic invariant $\mathscr{E}$ where

$$(\,|\Phi|_\xi|\Psi| - |\Phi||\Psi|_\xi\,)^2 + \frac{1}{1-s}\,[\,\left(\frac{\mathscr{K}}{N}\right)^2\left|\frac{\Phi}{\Psi}\right|^2 + \left(\frac{\mathscr{I}}{N}\right)^2\left|\frac{\Psi}{\Phi}\right|^2\,] = \mathscr{E} \tag{1.8}$$

and this, together with a (non-local) Hamiltonian may be used to obtain reduction of the system (1.7) in terms of the canonical single component Ermakov-Painlevé II equation in $\Omega = \sqrt{|\Phi|^2 + |\Psi|^2}$, namely

$$\Omega_{\xi\xi} + [\ c_{\rm I} + c_{\rm II}\ \xi\ ]\ \Omega + c_{\rm III}\Omega^3 = \frac{\mathscr{E}}{\Omega^3}\ . \tag{1.9}$$

If we now set $c_{\rm II}^2 = 2$ and $c_{\rm III} = 2\varepsilon$ together with

$$\mathscr{E} = -\frac{1}{4}\ (\gamma - \frac{\varepsilon}{2})^2 \leq 0 \tag{1.10}$$

then ω as given by (1.5) is governed by the integrable Painlevé II equation. The integrals of motion involving $\mathscr{I}$, $\mathscr{K}$ may be exploited to systematically construct the original wave packet representations. It is interesting to observe that the Ermakov invariant (1.8) together with the constraint (1.10) requires that the parameter $s > 1$. This corresponds to the 'resonant' situation when the de Broglie-Bohm potentials cannot be removed by the procedure described in [29, 35]. The 1+1-dimensional resonant NLS reduction with $s > 1$ is of particular physical interest since then novel solitonic fission and fusion phenomena are admitted (see e.g. [36] and literature cited therein).

The object of the present paper which introduces the notion of hybrid Ermakov-Painlevé IV systems is to exploit underlying Ermakov invariants therein to obtain reduction in terms of a single canonical Painlevé IV equation. The procedure is motivated by the occurrence of such a Ermakov-Painlevé IV system as a symmetry reduction of an N-component derivative nonlinear Schrödinger system. In terms of physical applications, two-component derivative NLS systems were originally derived by Morris and Dodd [32] in a study of the modulational stability of circularly polarized long Alfvén waves. Alfvén solitonic phenomena in such coupled derivative NLS systems in plasma physics have subsequently been investigated by Xu *et al* in [63] while Darboux transformations for two-component derivative NLS systems have been constructed by Ling and Liu [31]. Resonant derivative NLS equations have been recently derived in the context of Chern-Simons theory by Lee and Pashaev in [28].

2. Ermakov-Painlevé IV Reduction of a Derivative Nonlinear Schrödinger System

Here, we consider an N-component coupled 'resonant' derivative nonlinear Schrödinger system

$$\begin{aligned}
&i\ \Phi_{1t} - \nu i\ [\ (\ |\Phi_1|^2 + |\Phi_2|^2 + \cdots + |\Phi_N|^2\ )\ \Phi_1\ ]_x - \varepsilon\Phi_{1xx} = s\ (\ |\Phi_1|_{xx}/|\Phi_1|\ )\ \Phi_1\ ,\\
&i\ \Phi_{2t} - \nu i\ [\ (\ |\Phi_1|^2 + |\Phi_2|^2 + \cdots + |\Phi_N|^2\ )\ \Phi_2\ ]_x - \varepsilon\Phi_{2xx} = s\ (\ |\Phi_2|_{xx}/|\Phi_2|\ )\ \Phi_2\ ,\\
&\qquad\qquad\cdots\cdots\cdots\cdots\cdots\cdots\\
&i\ \Phi_{Nt} - \nu i\ [\ (\ |\Phi_1|^2 + |\Phi_2|^2 + \cdots + |\Phi_N|^2\ )\ \Phi_N\ ]_x - \varepsilon\Phi_{Nxx} = s\ (\ |\Phi_N|_{xx}/|\Phi_N|\ )\ \Phi_N
\end{aligned} \tag{2.1}$$

which incorporates de Broglie-Bohm type potential terms. It is noted that, recently, Lee and Pashaev [28] derived a single component resonant derivative nonlinear Schrödinger equation via a reduction of a Chern-Simons system.

Wave packet representations are now introduced into the system (2.1) according to

$$\Phi_i = (\ 2\ t\ )^{-1/4}\sigma_i(\zeta)e^{i\alpha(\zeta)}\ ,\quad i = 1, \cdots N \tag{2.2}$$

where $\zeta = x/(2t)^{1/2}$. In the single component case, in the absence of a de Broglie-Bohm potential, such an ansatz is known to lead to an integrable Painlevé IV reduction in a gauge-equivalent NLS equation (Ablowitz, Ramani and Segur [2, 3]).

Insertion of the representations (2.2) into the derivative NLS-type system (2.1) produces a nonlinear coupled system in the σ_i and α, namely

$$
\begin{aligned}
&(\ \varepsilon+s\ )\ddot{\sigma}_1-\varepsilon\dot{\alpha}^2\sigma_1-\nu\dot{\alpha}\sigma_1[\ \sigma_1^2+\sigma_2^2+\cdots+\sigma_N^2\ ]=\dot{\alpha}\sigma_1\zeta\ ,\\
&\varepsilon[2\dot{\alpha}\dot{\sigma}_1+\ddot{\alpha}\sigma_1]+2\nu\sigma_1[\ \sigma_1\dot{\sigma}_1+\sigma_2\dot{\sigma}_2+\cdots+\sigma_N\dot{\sigma}_N\ ]+\nu\dot{\sigma}_1[\ \sigma_1^2+\sigma_2^2+\cdots+\sigma_N^2\ ]\\
&\qquad\qquad=-\zeta\dot{\sigma}_1-\frac{\sigma_1}{2}\ ,\\
&(\ \varepsilon+s\ )\ddot{\sigma}_2-\varepsilon\dot{\alpha}^2\sigma_2-\nu\dot{\alpha}\sigma_2[\ \sigma_1^2+\sigma_2^2+\cdots+\sigma_N^2\ ]=\dot{\alpha}\sigma_2\zeta\ ,\\
&\varepsilon[2\dot{\alpha}\dot{\sigma}_2+\ddot{\alpha}\sigma_2]+2\nu\sigma_2[\ \sigma_1\dot{\sigma}_1+\sigma_2\dot{\sigma}_2+\cdots+\sigma_N\dot{\sigma}_N\ ]+\nu\dot{\sigma}_2[\ \sigma_1^2+\sigma_2^2+\cdots+\sigma_N^2\ ]\\
&\qquad\qquad=-\zeta\dot{\sigma}_2-\frac{\sigma_2}{2}\ ,\\
&\qquad\qquad\cdots\cdots\cdots\cdots\cdots\cdots\\
&(\ \varepsilon+s\ )\ddot{\sigma}_N-\varepsilon\dot{\alpha}^2\sigma_N-\nu\dot{\alpha}\sigma_N[\ \sigma_1^2+\sigma_2^2+\cdots+\sigma_N^2\ ]=\dot{\alpha}\sigma_N\zeta\ ,\\
&\varepsilon[2\dot{\alpha}\dot{\sigma}_N+\ddot{\alpha}\sigma_N]+2\nu\sigma_N[\ \sigma_1\dot{\sigma}_1+\sigma_2\dot{\sigma}_2+\cdots+\sigma_N\dot{\sigma}_N\ ]+\nu\dot{\sigma}_N[\ \sigma_1^2+\sigma_2^2+\cdots+\sigma_N^2\ ]\\
&\qquad\qquad=-\zeta\dot{\sigma}_N-\frac{\sigma_N}{2}
\end{aligned}
\tag{2.3}
$$

where, in the present context, the dot indicates a derivative with respect to ζ. The relations $(2.3)_2$, $(2.3)_4$, ... $(2.3)_{2N}$ together show that

$$
\begin{aligned}
&\varepsilon[\ 2\dot{\alpha}(\ \sigma_1\dot{\sigma}_1+\sigma_2\dot{\sigma}_2+\cdots+\sigma_N\dot{\sigma}_N\ )+\ddot{\alpha}(\ \sigma_1^2+\sigma_2^2+\cdots+\sigma_N^2\ )\ ]\\
&\quad+3\nu(\ \sigma_1^2+\sigma_2^2+\cdots+\sigma_N^2\ )(\ \sigma_1\dot{\sigma}_1+\sigma_2\dot{\sigma}_2+\cdots+\sigma_N\dot{\sigma}_N\ )\\
&\qquad=-\zeta(\ \sigma_1\dot{\sigma}_1+\sigma_2\dot{\sigma}_2+\cdots+\sigma_N\dot{\sigma}_N\ )-\frac{1}{2}\ (\ \sigma_1^2+\sigma_2^2+\cdots+\sigma_N^2\ )
\end{aligned}
\tag{2.4}
$$

whence, one obtains the integral of motion

$$\varepsilon\dot{\alpha}\Sigma_N+\frac{3\nu}{4}\ \Sigma_N^2+\Sigma_N\frac{\zeta}{2}=\mathscr{I} \tag{2.5}$$

so that

$$\dot{\alpha}=\frac{1}{\varepsilon\Sigma}\ [\ \mathscr{I}-\frac{3\nu}{4}\ \Sigma^2-\frac{\zeta}{2}\ \Sigma\ ] \tag{2.6}$$

where $\Sigma_N=\sigma_1^2+\sigma_2^2+\cdots+\sigma_N^2$.

Substitution of $\dot{\alpha}$ as given by (2.6) into (2.3) now produces a ‘Ermakov-Painlevé IV’ system, here termed of Type I, namely

$$
\begin{aligned}
&4\varepsilon(\,\varepsilon+s\,)\sigma_{1\zeta\zeta}+[\,\frac{3\nu^2}{4}\,(\,\sigma_1^2+\sigma_2^2+\cdots+\sigma_N^2\,)^2+2\zeta\nu(\,\sigma_1^2+\sigma_2^2+\cdots+\sigma_N^2\,)+\zeta^2-\gamma\,]\sigma_1\\
&\qquad\qquad=\frac{4\mathscr{I}^2\sigma_1}{(\,\sigma_1^2+\sigma_2^2+\cdots+\sigma_N^2\,)^2}\ ,\\
&4\varepsilon(\,\varepsilon+s\,)\sigma_{2\zeta\zeta}+[\,\frac{3\nu^2}{4}\,(\,\sigma_1^2+\sigma_2^2+\cdots+\sigma_N^2\,)^2+2\zeta\nu(\,\sigma_1^2+\sigma_2^2+\cdots+\sigma_N^2\,)+\zeta^2-\gamma\,]\sigma_2\\
&\qquad\qquad=\frac{4\mathscr{I}^2\sigma_2}{(\,\sigma_1^2+\sigma_2^2+\cdots+\sigma_N^2\,)^2}\ ,\\
&\qquad\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&4\varepsilon(\,\varepsilon+s\,)\sigma_{N\zeta\zeta}+[\,\frac{3\nu^2}{4}\,(\,\sigma_1^2+\sigma_2^2+\cdots+\sigma_N^2\,)^2+2\zeta\nu(\,\sigma_1^2+\sigma_2^2+\cdots+\sigma_N^2\,)+\zeta^2-\gamma\,]\sigma_N\\
&\qquad\qquad=\frac{4\mathscr{I}^2\sigma_N}{(\,\sigma_1^2+\sigma_2^2+\cdots+\sigma_N^2\,)^2}\ ,
\end{aligned}
\tag{2.7}
$$

where $\gamma=-2\nu\mathscr{I}$. This system admits the constant Wronskian invariants

$$\sigma_{i\zeta}\sigma_j-\sigma_i\sigma_{j\zeta}=\mathscr{W}_{ij}\ ,\quad i\neq j\ ,\quad i,j=1,\cdots N \tag{2.8}$$

as well as the (non-local) Hamiltonian

$$4\varepsilon(\,\varepsilon+s\,)(\,\sigma_{1\zeta}^2+\sigma_{2\zeta}^2+\cdots+\sigma_{N\zeta}^2\,)+\frac{\nu^2}{4}\,\Sigma_N^3+2\nu\int\zeta\Sigma_N d\Sigma_N+\int(\,\zeta^2-\gamma\,)d\Sigma_N+\frac{4\mathscr{I}^2}{\Sigma_N}=2\mathscr{H}\ . \tag{2.9}$$

On use of the identity

$$
\begin{aligned}
&(\sigma_1^2+\sigma_2^2+\cdots+\sigma_N^2)(\sigma_{1\zeta}^2+\sigma_{2\zeta}^2+\cdots+\sigma_{N\zeta}^2)\\
&\quad-[(\sigma_{1\zeta}\sigma_2-\sigma_1\sigma_{2\zeta})^2+(\sigma_{1\zeta}\sigma_3-\sigma_1\sigma_{3\zeta})^2+\cdots(\sigma_{N-1\zeta}\sigma_N-\sigma_{N-1}\sigma_{N\zeta})^2]\\
&\qquad\equiv(\sigma_1\sigma_{1\zeta}+\sigma_2\sigma_{2\zeta}+\cdots+\sigma_N\sigma_{N\zeta})^2
\end{aligned}
\tag{2.10}
$$

it is seen that

$$
\begin{aligned}
&4\varepsilon(\,\varepsilon+s\,)[\,(\sigma_{1\zeta}\sigma_2-\sigma_1\sigma_{2\zeta})^2+(\sigma_{1\zeta}\sigma_3-\sigma_1\sigma_{3\zeta}\,)^2+\cdots+(\sigma_{N-1\zeta}\sigma_N-\sigma_{N-1}\sigma_{N\zeta})^2+\frac{1}{4}\,\Sigma_{N\zeta}^2\,]\\
&\qquad=\Sigma_N[\,2\mathscr{H}-\frac{\nu^2}{4}\,\Sigma_N^3-2\nu\int\zeta\Sigma_N d\Sigma_N-\int(\zeta^2-\gamma)d\Sigma_N-\frac{4\mathscr{I}^2}{\Sigma_N}\,]
\end{aligned}
\tag{2.11}
$$

whence, in view of the Wronskian invariant relations (2.8),

$$\begin{aligned}&\Sigma_N\ [\ 2\mathscr{H} - \frac{\nu^2}{4}\ \Sigma_N^3 - 2\nu \int \zeta\Sigma_N d\Sigma_N - \int (\zeta^2-\gamma)d\Sigma_N\ ] - 4\varepsilon(\ \varepsilon+s\ )\Sigma\mathscr{W}_{ij}^2 - 4\mathscr{I}^2 \\ &\quad = \varepsilon(\ \varepsilon+s\ )\left(\frac{d\Sigma_N}{d\zeta}\right)^2\ .\end{aligned} \tag{2.12}$$

Accordingly,

$$\Sigma_{N\zeta\zeta} = \frac{1}{2}\ \Sigma_{N\zeta}^2/\Sigma_N - \frac{1}{4\varepsilon(\ \varepsilon+s\ )}\ [\ \frac{3\nu^2}{2}\ \Sigma_N^3 + 4\nu\zeta\Sigma_N^2 + 2(\ \zeta^2-\gamma\ )\Sigma_N\ ] + \frac{2}{\Sigma_N}\ [\ \Sigma\mathscr{W}_{ij}^2 + \frac{\mathscr{I}^2}{\varepsilon(\varepsilon+s)}\ ]\ , \tag{2.13}$$

so that, with the scalings

$$\Sigma_N^* = \mu^{-1}\Sigma_N\ \ ,\ \ d\zeta^* = \lambda d\zeta \tag{2.14}$$

where

$$\begin{aligned}\lambda^2 = -\frac{1}{4\varepsilon(\varepsilon+s)}\ \ ,\ \ \varepsilon(\varepsilon+s)<0 \\ \lambda\mu\nu = 1\ \ ,\end{aligned} \tag{2.15}$$

reduction is obtained to the standard Painlevé IV equation

$$\Sigma^*_{N\zeta^*\zeta^*} = \frac{1}{2}\ \Sigma_{N\zeta}^{*2}/\Sigma_N^* + \frac{3}{2}\ \Sigma_N^{*3} + 4\zeta^*\Sigma_N^{*2} + 2(\ \zeta^{*2}-\delta^*\ )\Sigma_N^* + \frac{\beta^*}{\Sigma_N} \tag{2.16}$$

where

$$\beta^* = 2\nu^2[\ \Sigma\mathscr{W}_{ij}^2 + \frac{\mathscr{I}}{\varepsilon(\varepsilon+s)}\ ]\ \ ,\ \ \delta^* = \gamma/\lambda^2\ \ . \tag{2.17}$$

Importantly, in the present context, any solution Σ_N^* of this canonical Painlevé IV equation delivers a solution $\Omega = \Sigma_N^{*1/2}$ of the Ermakov-Painlevé IV-type equation

$$\Omega_{\zeta^*\zeta^*} - [\ \frac{3}{4}\ \Omega^4 + 2\zeta^*\Omega^2 + \zeta^{*2} - \delta^*\ ]\ \Omega = \frac{\mathscr{W}}{\Omega^3} \tag{2.18}$$

where

$$\mathscr{W} = \nu^2[\ \Sigma\mathscr{W}_{ij}^2 + \frac{\mathscr{I}}{\varepsilon(\varepsilon+s)}\ ]\ \ . \tag{2.19}$$

It is observed that, for a specified solution Σ_N^* of the Painlevé IV equation (2.17), substitution into the system (2.7) produces reduction to N copies of a now explicit second order linear equation. However, the presence of Painlevé IV transcendents in the potential of the latter makes it, in general, intractable as it stands. Accordingly, here an alternative approach is adopted wherein admitted Ermakov invariants are exploited to obtain explicit solution involving a single quadrature. Thus, a sequence of Ermakov invariants may now be constructed. In this connection, $(2.7)_1$ may be written as, on setting $\mu = 1$,

$$\sigma_{1\zeta\zeta} + \frac{1}{4\varepsilon(\ \varepsilon + s\ )}\ [\ \frac{3\nu^2}{4}\ \Omega^4 + 2\zeta\nu\Omega + \zeta^2 - \gamma\ ]\ \sigma = \frac{\mathscr{I}^2\sigma_1}{\varepsilon(\varepsilon + s)\Omega^4}\ , \tag{2.20}$$

while (2.13) yields

$$\Omega_{\zeta\zeta} + \frac{1}{4\varepsilon(\ \varepsilon + s\ )}\ [\ \frac{3\nu^2}{4}\ \Omega^4 + 2\zeta\nu\Omega + \zeta^2 - \gamma\ ]\ \Omega = \frac{1}{\Omega^3}\ [\ \Sigma\mathscr{W}_{ij}^2 + \frac{\mathscr{I}^2}{\varepsilon(\varepsilon + s)}\ ]\ . \tag{2.21}$$

The pair (2.20), (2.21) constitute a Ermakov-Ray-Reid system with invariant

$$\frac{1}{2}\ (\ \sigma_1\Omega_\zeta - \Omega\sigma_{1\zeta}\ )^2 + \frac{1}{2}\ [\ \Sigma\mathscr{W}_{ij}^2\ ]\left(\frac{\sigma_1}{\Omega}\right)^2 = \mathscr{I}_1\ . \tag{2.22}$$

In a similar manner, the pairs $(2.7)_2$, (2.21), $(2.7)_3$, (2.21) ..... $(2.7)_N$, (2.21) constitute Ermakov-Ray-Reid systems which, in turn, admit the invariants

$$\frac{1}{2}\ (\ \sigma_2\Omega_\zeta - \Omega\sigma_{2\zeta}\ )^2 + \frac{1}{2}\ [\ \Sigma\mathscr{W}_{ij}^2\ ]\left(\frac{\sigma_2}{\Omega}\right)^2 = \mathscr{I}_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \tag{2.23}$$

$$\frac{1}{2}\ (\ \sigma_N\Omega_\zeta - \Omega\sigma_{2\zeta}\ )^2 + \frac{1}{2}\ [\ \Sigma\mathscr{W}_{ij}^2\ ]\left(\frac{\sigma_N}{\Omega}\right)^2 = \mathscr{I}_N$$

It is seen that $\mathscr{I}_i > 0,\ i = 1,\cdots N$.

On introduction of the new independent variable $\bar{\zeta}$ according to

$$d\bar{\zeta} = \Omega^{-2}d\zeta = \Sigma_N^{-1}d\zeta \tag{2.24}$$

and of new dependent variables

$$U_i = \frac{\sigma_i}{\Omega}\ ,\quad i = 1,\cdots,N\ , \tag{2.25}$$

the Ermakov invariant relations (2.22), (2.23) are seen to yield

$$U_i = \sqrt{\frac{2\mathscr{I}_i}{\mathscr{W}^*}}\ \sin(\pm\sqrt{\mathscr{W}^*}\ \bar{\zeta} + \mathbb{K}_i\ ) \tag{2.26}$$

where $\mathscr{W}^* = \Sigma\mathscr{W}_{ij}^2$, $\Sigma U_i^2 = 1$ and the $\mathbb{K}_i$ are arbitrary constants of integration.

Thus, in summary, a solution set $\{\sigma_i, i = 1,\cdots,N\}$ of the Ermakov-Painlevé IV system (2.7) may be generated via a seed solution Σ_N of the integrable Painlevé IV equation (2.13) according to the relations

$$\sigma_i = \sqrt{\left(\frac{2\mathscr{I}_i}{\mathscr{W}^*}\Sigma_N\right)}\ \sin(\pm\sqrt{\mathscr{W}^*}\ \bar{\zeta} + \mathbb{K}_i) \tag{2.27}$$

where

$$\bar{\zeta} = \int \Sigma_N^{-1} d\zeta \tag{2.28}$$

Addition of the Ermakov invariant relations (2.22)−(2.23) and application of the identity (2.10) together establish a connection between the invariants, namely

$$\Sigma\mathscr{W}_{ij}^2 = \mathscr{I}_1 + \mathscr{I}_2 + \cdots + \mathscr{I}_N \ . \tag{2.29}$$

It is interesting to observe that reduction in the present context to the canonical Painlevé IV equation requires that $\varepsilon(\varepsilon + s) < 0$ so that, in the standard case with $\varepsilon = -1$ then $s > 1$. This corresponds to the situation when resonant solitonic behaviour may be observed in the derivative nonlinear Schrödinger equation incorporating de Broglie Bohm quantum potential terms [28].

3. A Type II Ermakov-Painlevé IV Triad

Here, a second kind of hybrid Ermakov-Painlevé IV system is introduced, namely the triad

$$\begin{aligned}
u_{xx} - [\ \frac{3}{4}\ (\ u^2+v^2+w^2\ )^2 + 2x\ (\ u^2+v^2+w^2\ ) + x^2 - \delta\ ]\ u &= \frac{\beta}{2u^3}\ , \\
v_{xx} - [\ \frac{3}{4}\ (\ u^2+v^2+w^2\ )^2 + 2x\ (\ u^2+v^2+w^2\ ) + x^2 - \delta\ ]\ v &= \frac{\beta}{2v^3}\ , \\
w_{xx} - [\ \frac{3}{4}\ (\ u^2+v^2+w^2\ )^2 + 2x\ (\ u^2+v^2+w^2\ ) + x^2 - \delta\ ]\ w &= \frac{\beta}{2w^3}\ .
\end{aligned} \tag{3.1}$$

It is again noted that the single component Ermakov-Painlevé IV equation

$$\Omega_{xx} - [\ \frac{3}{4}\ \Omega^4 + 2x\Omega^2 + x^2 - \delta\ ]\ \Omega = \frac{\beta}{2\Omega^3}\ , \tag{3.2}$$

on insertion of $\omega = \Omega^2$, leads to the canonical Painlevé IV equation (see e.g. [12, 13] and literature cited therein), namely

$$\omega_{xx} = \frac{\omega_x^2}{2\omega} + \frac{3}{2}\ \omega^3 + 4x\omega^2 + 2\ (\ x^2 - \delta\ )\ \omega + \frac{\beta}{\omega}\ . \tag{3.3}$$

It is noted that a coupled nonlinear system of the type (3.1) is readily derived as a symmetry reduction of an overarching appropriately x-modulated coupled NLS triad. In terms of applications, in Bassom *et al* [10], the avatar (3.2) of Painlevé IV was investigated with parameter $\beta = 0$ in connection with solutions $\Omega(\alpha : x)$ constrained by the boundary condition $\Omega(\infty) = 0$. Bound state solutions were obtained via an integral equation formulation which, importantly, decay exponentially as $x \to \infty$.

Here, the Ermakov-Painlevé IV system (3.1) is seen to admit a triad of characteristic Ermakov invariants, namely

$$\begin{aligned} (\ u_x v - v_x u\ )^2 + \frac{\beta}{2}\ [\ \left(\frac{u}{v}\right)^2 + \left(\frac{v}{u}\right)^2\ ] &= \mathscr{E}_{\mathrm{I}}\ , \\ (\ v_x w - w_x v\ )^2 + \frac{\beta}{2}\ [\ \left(\frac{v}{w}\right)^2 + \left(\frac{w}{v}\right)^2\ ] &= \mathscr{E}_{\mathrm{II}}\ , \\ (\ w_x u - u_x w\ )^2 + \frac{\beta}{2}\ [\ \left(\frac{w}{u}\right)^2 + \left(\frac{u}{w}\right)^2\ ] &= \mathscr{E}_{\mathrm{III}}\ ,\ . \end{aligned} \tag{3.4}$$

Moreover, the system (3.1) admits the (non-local) Hamiltonian

$$u_x^2 + v_x^2 + w_x^2 - \frac{1}{4}\,\Sigma^3 - 2\int x\Sigma d\Sigma - \int (\ x^2 - \gamma\ )\ d\Sigma + \frac{\beta}{2}\ [\ \frac{1}{u^2} + \frac{1}{v^2} + \frac{1}{w^2}\ ] = 2\mathscr{H} \tag{3.5}$$

where $\Sigma := u^2 + v^2 + w^2$.

The identity

$$\begin{aligned} &(\ u^2 + v^2 + w^2\ )(\ u_x^2 + v_x^2 + w_x^2\ ) - [\ (\ u_x v - v_x u\ )^2 + (\ v_x w - w_x v\ )^2 + (\ w_x u - u_x w\ )^2\ ] \\ &\qquad \equiv (\ u u_x + v v_x + w w_x\ )^2 \end{aligned} \tag{3.6}$$

is now employed, whence the Ermakov invariant relations (3.4) together with the Hamiltonian (3.5) show that

$$\begin{aligned} &\Sigma\ [\ 2\mathscr{H} + \frac{1}{4}\,\Sigma^3 + 2\int x\Sigma d\Sigma + \int (\ x^2 - \gamma\ )\ d\Sigma - \frac{\beta}{2}\ (\ \frac{1}{u^2} + \frac{1}{v^2} + \frac{1}{w^2}\ ) \\ &\quad - [\ \mathscr{E}_{\mathrm{I}} + \mathscr{E}_{\mathrm{II}} + \mathscr{E}_{\mathrm{III}} - \frac{\beta}{2}\ (\ \left(\frac{u}{v}\right)^2 + \left(\frac{v}{u}\right)^2 + \left(\frac{v}{w}\right)^2 + \left(\frac{w}{v}\right)^2 + \left(\frac{w}{u}\right)^2 + \left(\frac{u}{w}\right)^2\ )\ ] \\ &\quad = \frac{1}{4}\left(\frac{d\Sigma}{dx}\right)^2 \end{aligned} \tag{3.7}$$

that is,

$$\Sigma\ [\ 2\mathscr{H} + \frac{1}{4}\,\Sigma^3 + 2\int x\Sigma d\Sigma + \int (\ x^2 - \gamma\ )\ d\Sigma\ ] - \frac{\mathscr{E}}{2} = \frac{1}{4}\left(\frac{d\Sigma}{dx}\right)^2 \tag{3.8}$$

where

$$\mathscr{E} = 2\ (\ \mathscr{E}_{\mathrm{I}} + \mathscr{E}_{\mathrm{II}} + \mathscr{E}_{\mathrm{III}}\ ) + 3\beta\ \ . \tag{3.9}$$

Equation (3.8), in turn, leads to a Painlevé IV equation

$$\Sigma_{xx} = \frac{\Sigma_x^2}{2\Sigma} + \frac{3}{2}\,\Sigma^3 + 4x\Sigma^2 + 2\ (\ x^2 - \gamma\ )\Sigma + \frac{\mathscr{E}}{\Sigma} \tag{3.10}$$

in Σ which, in terms of $\Lambda = \Sigma^{1/2}$, produces the associated Ermakov-Painlevé IV equation

$$\Lambda_{xx} - [\ \frac{3}{4}\,\Lambda^4 + 2x\Lambda^2 + x^2 - \gamma\ ]\ \Lambda = \frac{\mathscr{E}}{2\Lambda^3}\ \ . \tag{3.11}$$

Importantly, the latter together with each of the constituent nonlinear equations in the triad (3.1) taken in turn, constitute Ermakov-Ray-Reid systems. These pairs admit an additional triad of

Ermakov invariant relations, namely

$$(u_x\Lambda - \Lambda_x u)^2 + \frac{1}{2} [\mathscr{E} \left(\frac{u}{\Lambda}\right)^2 + \beta \left(\frac{\Lambda}{u}\right)^2] = \mathbb{R}_{\mathrm{I}} \ , \tag{3.12}$$

$$(v_x\Lambda - \Lambda_x v)^2 + \frac{1}{2} [\mathscr{E} \left(\frac{v}{\Lambda}\right)^2 + \beta \left(\frac{\Lambda}{v}\right)^2] = \mathbb{R}_{\mathrm{II}} \ , \tag{3.13}$$

$$(w_x\Lambda - \Lambda_x w)^2 + \frac{1}{2} [\mathscr{E} \left(\frac{w}{\Lambda}\right)^2 + \beta \left(\frac{\Lambda}{w}\right)^2] = \mathbb{R}_{\mathrm{III}} \ . \tag{3.14}$$

On introduction of the new independent variable $\bar{x}$ according to

$$d\bar{x} = \Lambda^{-2} dx \ , \tag{3.15}$$

and new dependent variables U, V, W via

$$U = \left(\frac{u}{\Lambda}\right)^2 \ , \ V = \left(\frac{v}{\Lambda}\right)^2 \ , \ W = \left(\frac{w}{\Lambda}\right)^2 \tag{3.16}$$

it is seen that (3.12)−(3.14) yield

$$d\, U^{1/2}/d\bar{x} = \pm\sqrt{\mathbb{R}_{\mathrm{I}} - \frac{1}{2} [\mathscr{E}U + \beta U^{-1}]} \tag{3.17}$$

$$d\, V^{1/2}/d\bar{x} = \pm\sqrt{\mathbb{R}_{\mathrm{II}} - \frac{1}{2} [\mathscr{E}V + \beta V^{-1}]} \tag{3.18}$$

together with

$$d\, W^{1/2}/d\bar{x} = \pm\sqrt{\mathbb{R}_{\mathrm{III}} - \frac{1}{2} [\mathscr{E}W + \beta W^{-1}]} \ . \tag{3.19}$$

However, it is observed that once U, V have been calculated via (3.17) and (3.18) respectively then W may be determined immediately by the relation

$$U + V + W = 1 \ . \tag{3.20}$$

The equations (3.17) and (3.18) show that, if $\mathscr{E} > 0$, then

$$U = \frac{1}{\mathscr{E}} [\mathbb{R}_{\mathrm{I}} + \sqrt{\mathbb{R}_{\mathrm{I}}^2 - \beta\mathscr{E}} \ \sin(\pm\sqrt{\frac{\mathscr{E}}{2}}\bar{x} + \mathbb{K}_{\mathrm{I}})] \tag{3.21}$$

and

$$V = \frac{1}{\mathscr{E}} [\mathbb{R}_{\mathrm{II}} + \sqrt{\mathbb{R}_{\mathrm{II}}^2 - \beta\mathscr{E}} \ \sin(\pm\sqrt{\frac{\mathscr{E}}{2}}\bar{x} + \mathbb{K}_{\mathrm{II}})] \tag{3.22}$$

where $\mathbb{K}_{\mathrm{I}}$ and $\mathbb{K}_{\mathrm{II}}$ are constants of integration and it is required that $\mathbb{R}_{\mathrm{I}}^2 > \beta\mathscr{E}$ and $\mathbb{R}_{\mathrm{II}}^2 > \beta\mathscr{E}$. The residual quantity W is then given via the relation (3.20). Moreover, addition of the integrals of motion (3.12)−(3.14) together with the identity (3.6) and the Ermakov invariant relations (3.4) is seen to impose the constraint

$$\mathscr{E}_{\rm I}+\mathscr{E}_{\rm II}+\mathscr{E}_{\rm III}+\frac{\mathscr{E}}{2}+\frac{3\beta}{2}=\mathbb{R}_{\rm I}+\mathbb{R}_{\rm II}+\mathbb{R}_{\rm III} \tag{3.23}$$

whence, on use of the relation (3.9),

$$\mathscr{E}=\mathbb{R}_{\rm I}+\mathbb{R}_{\rm II}+\mathbb{R}_{\rm III} \ . \tag{3.24}$$

In summary, solution sets $\{u,v,w\}$ of the Ermakov-Painlevé IV system (3.1) are generated in terms of seed solutions Σ of the canonical integrable Painlevé IV equation (3.10) via the relations

$$u=\pm\sqrt{\frac{1}{\mathscr{E}}\,[\,\mathbb{R}_{\rm I}+\sqrt{\mathbb{R}_{\rm I}^2-\beta\mathscr{E}}\sin(\,\pm\sqrt{\frac{\mathscr{E}}{2}}\bar{x}+\mathbb{K}_{\rm I}\,)\,]\,\Sigma}\ , \tag{3.25}$$

$$v=\pm\sqrt{\frac{1}{\mathscr{E}}\,[\,\mathbb{R}_{\rm II}+\sqrt{\mathbb{R}_{\rm II}^2-\beta\mathscr{E}}\sin(\,\pm\sqrt{\frac{\mathscr{E}}{2}}\bar{x}+\mathbb{K}_{\rm II}\,)\,]\,\Sigma} \tag{3.26}$$

$$(\mathscr{E}>o\,)$$

while

$$w=\pm\sqrt{\Sigma-u^2-v^2} \tag{3.27}$$

and $\bar{x}$ is obtained in terms of x by integration of the relation (3.15), namely

$$d\bar{x}=\Sigma^{-1}dx \ . \tag{3.28}$$

It is remarked that seed solutions Σ of Painlevé IV may be generated, in particular, via the application of established Bäcklund transformations (Gromak [22]).

In summary, it has been established that novel hybrid Ermakov-Painlevé IV systems of two types may be introduced, both of which are amenable to algorithmic solution in terms of the canonical Painlevé IV equation via the systematic application of Ermakov invariants.

References

[1] J. Abdullaev, A.S. Desyatnikov and E.A. Ostravoskaya, Suppression of collapse for matter waves with orbital angular momentum, *J. Opt.* **13** (2011) 064023.

[2] M.J. Ablowitz, A. Ramani and H. Segur, A connection between nonlinear evolution equations and ordinary differential equations of p-type, *J. Math. Phys.* **21** (1980) 1006–1015.

[3] M.J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, (SIAM Studies in Applied Mathematics, Philadelphia, 1981).

[4] C. Athorne, Stability and periodicity in coupled Pinney equations, *J. Diff. Equations* **100** (1992) 82–94.

[5] C. Athorne, Projective lifts and generalised Ermakov and Bernoulli systems, *J. Math. Anal. Appl.* **233** (1999) 552–563.

[6] C. Athorne, C. Rogers, U. Ramgulam and A. Osbaldestin, On linearisation of the Ermakov system, *Phys. Lett. A* **143** (1990) 207–212.

[7] F.K. Ball, Some general theorems concerning the finite motion of a shallow liquid lying on a paraboloid, *J. Fluid Mech.* **17** (1963) 240–256.

[8] F.K. Ball, The effect of rotation on the simpler modes of motion in an elliptic paraboloid, *J. Fluid Mech.* **22** (1965) 529–545.

[9] L. Bass, J.J.C. Nimmo, C. Rogers and W.K. Schief, Electrical structures of interfaces: a Painlevé II model, *Proc. Roy. Soc. Lond. A* **466** (2010) 2117–2136.

[10] A.P. Bassom, P.A. Clarkson, A. C. Hicks and J.B. Mcleod, Integral relations and exact solutions for the fourth Painlevé equation, *Proc. Roy. Soc. Lond. A* **437** (1992) 1–24.

[11] A.J. Bracken, L. Bass and C. Rogers, Bäcklund flux quantization in a model of electrodiffusion based on Painlevé II, *J. Phys. A: Math. & Theor.* **45** (2012) 105204.

[12] P.A. Clarkson, Painlevé equations - Nonlinear special functions, *J. Comput. Appl. Math.* **153** (2003) 127–140.

[13] R. Conte (Ed.), *The Painlevé Property: One Century Later* (Springer, New York, 1999).

[14] R. Conte, C. Rogers and W.K. Schief, Painlevé structure of a multi-ion electrodiffusion system, *J. Phys. A: Math. & Theor.* **40** (2007) F1031–F1040.

[15] F. Cornolti, M. Lucchesi and B. Zambon, Elliptic gaussian beam self-focussing in nonlinear media, *Opt. Commun.* **75** (1990) 129–135.

[16] A.S. Desyatnikov, D. Buccoliero, M.R. Dennis and Y.S. Kivshar, Suppression of collapse for spiralling elliptic solitons, *Phys. Rev. Lett.* **104** (2010) 053902-1–053902-4.

[17] F.J. Dyson, Dynamics of a spinning gas cloud, *J. Math. Mech.* **18** (1968) 91–101.

[18] V.P. Ermakov, Second-order differential equations: conditions of complete integrability, Univ. Izy. Kiev **20** (1880) 1–25.

[19] B. Gaffet, Spinning gas clouds with precession: a new formulation, *J. Phys. A: Math. Theor.* **43** (2010) 165207.

[20] A.M. Goncharenko, Y.A. Logvin, A.M. Samson and P.S. Shapovalov, Rotating ellipsoidal gaussian beams in nonlinear media, *Opt. Commun.* **81** (1991) 225–230.

[21] A.M. Goncharenko, V.G. Kukushkin, Yu. A. Logvin and A.M. Samson, Self-focussing of two orthogonally polarised light beams in a nonlinear medium, *Opt. Quantum Electron.* **25** (1999) 97–104.

[22] V.I. Gromak, Bäcklund transformations of Painlevé equations and their applications, in *The Painlevé Property. One Century Later* (Ed. R. Conte) CRM Series in Mathematical Physics (Springer, New York, pp 687–734, 1999).

[23] C.R. Guiliano, J.H. Marburger and A. Yariv, Enhancement of self-focussing threshold in sapphire with elliptical beams, *Appl. Phys. Lett.* **21** (1972) 58–60.

[24] D.D. Holm, Elliptical vortices and integrable Hamiltonian dynamics of the rotating shallow water equations, *J. Fluid Mech.* **227** (1991) 393–406.

[25] B. Konopelchenko and C. Rogers, On 2+1-dimensional nonlinear systems of Loewner-type, *Phys. Lett. A* **158** (1991) 391–397.

[26] B. Konopelchenko and C. Rogers, On generalised Loewner systems: novel integrable equations in 2+1-dimensions, *J. Math. Phys.* **34** (1993) 214–242.

[27] B. Konopelchenko, W.K. Schief and C. Rogers, The 2+1-dimensional sine Gordon equation: its auto Bäcklund transformation, *Phys. Lett. A* **172** (1992) 39–48.

[28] J.H. Lee and O.K. Pashaev, Chiral resonant solitons in a Broer-Kaup type new hydrodynamic system, *arXiv*: 1012, 1241 VI [nlin.SI] (2010).

[29] J.H. Lee, O.K. Pashaev, C. Rogers and W.K. Schief, The resonant nonlinear Schrödinger equation in cold plasma physics. Application of Bäcklund-Darboux transformations and superposition principles, *J. Plasma Physics* **73** (2007) 257–272.

[30] D. Levi, M.C. Nucci, C. Rogers and P. Winternitz, Group theoretical analysis of a rotating shallow liquid in a rigid container, *J. Phys. A: Math. & Gen.* **22** (1989) 4743–4767.

[31] L. Ling and Q.P. Liu, Darboux transformations for a two-component derivative nonlinear Schrödinger equation, *J. Phys. A: Math. Theor.* **43** (2010) 434023.

[32] H.C. Morris and R.K. Dodd, The two-component derivative nonlinear Schrödinger equation, *Physica Scripta* **20** 505–508.

[33] W. Nernst, Zur Kinetik der in Lösung befindlichen Körper. Erste Abhandlung. Theorie der Diffusion, *Z. Phys. Chem.* **2** (1888) 613–637.

[34] L.V. Ovsiannnikov, New solutions of equations of hydrodynamics, *Dokl. akad. Nauk* **111** (1956) 47–49.
[35] O.K. Pashaev and J.H. Lee, Resonance solitons as black holes in Madelung fluid, *Mod. Phys. Lett. A* **17** (2002) 1601–1619.
[36] O.K. Pashaev, J.H. Lee and C. Rogers, Soliton resonances in a generalized nonlinear Schrödinger equation, *J. Phys. A. Math. & Theor.* **41** (2008) 452001–452009.
[37] M. Planck, Über die Erregung von Electricität und Wärme in Electrolyten, *Ann. Phys. Chem.* **39** (1890) 161–186.
[38] J.R. Ray, Nonlinear superposition law for generalised Ermakov systems, *Phys. Lett. A* **78** (1980) 4–6.
[39] J.L. Reid and J.R. Ray, Ermakov systems, nonlinear superposition and solution of nonlinear equations of motion, *J. Math. Phys.* **21** (1980) 1583–1587.
[40] C. Rogers, Elliptic warm-core theory: the pulsrodon, *Phys. Lett. A* **138** (1989) 267–273.
[41] C. Rogers, Generation of invariance theorems for nonlinear boundary-value problems in shallow water theory: application of MACSYMA, Numerical and Applied Mathematics, IMACS Meeting Proceedings, Paris (1989) 69–74.
[42] C. Rogers and W.K. Schief, Multi-component Ermakov systems: structure and linearization, *J. Math. Anal. Appl.* **198** (1996) 194–220.
[43] C. Rogers, On a Ermakov-Ray-Reid reduction in 2+1-dimensional transverse magnetogasdynamics, Proceedings International Workshop on Group Analysis and Integrable Systems, Cyprus (2010).
[44] C. Rogers, A novel Ermakov-Painlevé II system. N+1-dimensional coupled NLS and elastodynamic reductions, *Stud. Appl. Math.* **133** (2014) 214–231.
[45] C. Rogers and H. An, Ermakov-Ray-Reid systems in 2+1-dimensional rotating shallow water theory, *Stud. Appl. Math.* **125** (2010) 275–299.
[46] C. Rogers, A.P. Bassom and W.K. Schief, On a Painlevé II model in steady electrolysis: application of a Bäcklund transformation, *J. Math. Anal. Appl.* **240** (1999) 367–381.
[47] C. Rogers, C. Hoenselaers and J.R. Ray, On 2+1-dimensional Ermakov systems, *J. Phys. A: Math. Gen.* **26** (1993) 2625–2633.
[48] C. Rogers, B. Malomed and H. An, Ermakov-Ray-Reid reductions of variational approximations in nonlinear optics, *Stud. Appl. Math.* **129** (2012) 389–413.
[49] C. Rogers, B. Malomed, K. Chow and H. An, Ermakov-Ray-Reid systems in nonlinear optics, *J. Phys. A: Math. Theor.* **43** (2010) 455214 (15 pp).
[50] C. Rogers and W.F. Shadwick, *Bäcklund Transformations and Their Applications* (Academic Press, New York, 1982).
[51] C. Rogers and W.K. Schief, Infinitesimal Bäcklund transformations of K-nets. The 2+1-dimensional sinh-Gordon system, International Meeting on Bäcklind and Darboux transformations, *CRM Proceedings, Lecture Notes, Amer. Math. Soc.* **29** (2001) 385–392.
[52] C. Rogers and W.K. Schief, *Bäcklund and Darboux Transformations. Geometry and Modern Applications in Soliton Theory* (Cambridge Texts in Applied Mathematics, Cambridge University Press, 2002).
[53] C. Rogers and W.K. Schief, The pulsrodon in 2+1-dimensional magneto-gasdynamics. Hamiltonian structure and integrability, *J. Math. Phys.* **52** (2011) 083701.
[54] C. Rogers and W.K. Schief, On the integrability of a Hamiltonian reduction of a 2+1-dimensional non-isothermal rotating gas cloud system, *Nonlinearity* **24** (2011) 3165–3178.
[55] C. Rogers, W.K. Schief and P. Winternitz, Lie-theoretical generalization and discretization of the Pinney equation, *J. Math. Anal. Appl.* **216** (1997) 246–264.
[56] A. Rubino and P. Brandt, Warm-core eddies studied by laboratory experiments and numerical modelling, *J. Phys. Oceanography* **33** (2003) 431–435.
[57] W.K. Schief, A discrete Pinney equation, *Appl. Math. Lett.* **10** (1997) 13–15.
[58] W.K. Schief and C. Rogers, Loewner transformations. Adjoint and binary Darboux connections, *Stud. Appl. Math.* **100** (1998) 391–422.
[59] W.K. Schief, C. Rogers and A. Bassom, Ermakov systems with arbitrary order and dimension: structure and linearization, *J. Phys. A: Math. Gen.* **29** (1996) 903–911.
[60] W.K. Schief, H. An and C. Rogers, Universal and integrable aspects of an elliptic vortex representation in 2+1-dimensional magnetogasdynamics, *Stud. Appl. Math.* **130** (2013) 49–79.

[61] A. Steen, Om Formen for Integralet of den lineaere Differentialligning of anden Orden, *Overs. over d. K. Danske Vidensk. Selsk. Forh* (1874) 1–12.

[62] W.G. Wagner, H.A. Haus and J.H. Marburger, Large-scale self-trapping of optical beams in the paraxial ray approximation, *Phys. Rev.* **175** (1968) 256–266.

[63] T. Xu, B. Tian, C. Zhang, X.H. Meng and X. Lu, Alfvén solitons in the coupled derivative nonlinear Schrödinger system with symbolic computation, *J. Phys. A: Math. & Gen.* **41** (2009) 415201.