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Symmetry reductions and exact solutions of Lax integrable 3-dimensional systems

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We present a complete description of 2-dimensional equations that arise as symmetry reductions of four 3-dimensional Lax-integrable equations: (1) the universal hierarchy equation $u_{yy} = u_z u_{xy} - u_y u_{xz}$; (2) the 3D rdDym equation $u_{ty} = u_x u_{xy} - u_y u_{xx}$; (3) the equation $u_{ty} = u_t u_{xy} - u_y u_{tx}$, which we call modified Veronese web equation; (4) Pavlov's equation $u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}$.

Keywords: Partial differential equations; symmetry reductions, solutions.

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Introduction

We consider four 3-dimensional Lax-integrable^a equations:

- the *universal hierarchy equation* (Sec. 2)

$$u_{yy} = u_z u_{xy} - u_y u_{xz}, \quad (0.1)$$

see [11].

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^aWe say that an equation is *Lax-integrable* if it admits a zero-curvature representation with a non-removable parameter.

- the 3D rdDym equation (Sec. 3)

$$u_{ty} = u_x u_{xy} - u_y u_{xx}, \quad (0.2)$$

see [3, 13, 15].

- the equation (Sec. 4)

$$u_{ty} = u_t u_{xy} - u_y u_{tx}, \quad (0.3)$$

see [1, 6, 8, 16]. In [8] it was shown that the equation at hand is related to a particular case of the 3D Veronese web equation by a Bäcklund transformation. Below we call Eq. (0.3) the *modified Veronese web equation*.

- Pavlov's equation (Sec. 5)

$$u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}, \quad (0.4)$$

see [5, 14].

Some of these equations arise also in [7] as integrable hydrodynamic reductions of multi-dimensional dispersionless PDEs.

All the above listed equations may be obtained as the symmetry reductions of the following Lax-integrable 4-dimensional systems:

$$u_{yz} = u_{tx} + u_x u_{xy} - u_y u_{xx}$$

and

$$u_{ty} = u_z u_{xy} - u_y u_{xz}$$

introduced in [7] and [11], respectively, while the latter two, in turn, are the reductions of

$$u_{yz} = u_{ts} + u_s u_{xz} - u_z u_{xs}.$$

Here we give a complete answer to a natural question: what 2-dimensional equations are the reductions of the 3-dimensional ones? The result comprises 32 equations of which

- sixteen can be solved explicitly,
- one reduces to the Riccati equation,
- five can be linearized by the Legendre transformation,
- while the rest ten are 'nontrivial'.

The latter are presented in Table 1 (in the third column, we exemplify the simplest relations). The first two of these equations can be transformed to the Liouville equation and the Gibbons-Tsarev equation, respectively. The other eight, to our strong opinion, may possess interesting integrability properties and we plan to study them in the nearest future. More detailed, but also concise, information on the reductions may be also found in Table 6.

In Sec. 1, we briefly expose necessary preliminaries (see, e.g., [10]). In Sec. 6, we present the obtained results in a concise form.

Reduction	of Eq.	Relations with the initial equation
$2\Phi = \Phi\Phi_{xz} - \Phi_x\Phi_z$	(0.1)	$u = \frac{\Phi(x,z)}{y}$,
$\Phi_{\xi\xi} = (\xi + \Phi_\xi)\Phi_{\eta\eta} - \Phi_\eta\Phi_{\xi\eta} - 2$	(0.4)	$u = \Phi(\xi, \eta) + t^2\xi - 2t\eta$, $\xi = y, \eta = x + ty$,
$\Phi_{\xi\xi} = \Phi_x\Phi_\xi - \Phi\Phi_{x\xi}$	(0.1)	$u = \Phi(x, \xi)e^{-z}, \xi = ye^{-z}$,
$(1 + \xi\Phi_z)\Phi_{\xi\xi} - \xi\Phi_\xi\Phi_{\xi z} + \Phi_\xi\Phi_z = 0$	(0.1)	$u = \Phi(z, \xi)e^{-x}, \xi = ye^{-x}$,
$\Phi_\eta\Phi_{\xi\eta} - \Phi_\xi\Phi_{\eta\eta} = e^\eta\Phi_{\xi\xi}$	(0.1)	$u = \Phi(\xi, \eta)e^{-x}, \xi = ye^{-z}, \eta = x - z$,
$(\xi + \Phi_\xi)\Phi_{\xi y} - \Phi_y(\Phi_{\xi\xi} + 2) = 0$	(0.2)	$u = \Phi(\xi, y)e^{2t}, \xi = xe^t$,
$\Phi_{\xi t} = 4\Phi\Phi_\xi - \xi\Phi_\xi^2 + 2\xi\Phi\Phi_{\xi\xi}$	(0.2)	$u = \Phi(\xi, t)x^2, \xi = xe^{-y}$,
$\Phi_{\eta\eta} + (\xi + \Phi_\eta)\Phi_{\xi\eta} = \Phi_\eta(2 + \Phi_{\xi\xi})$	(0.2)	$u = \Phi(\xi, \eta)e^{2t}, \xi = xe^{-t}, \eta = y - t$,
$(4\xi^2 - 3\Phi)\Phi_{\xi\xi} - \Phi_{\xi t} - 6\xi\Phi_\xi + \Phi_\xi^2 + 6\Phi = 0$	(0.4)	$u = \Phi(\xi, y)y^3, \xi = \frac{x}{y^2}$,
$\Phi_{\xi\xi} = (\xi - \Phi_\eta)\Phi_{\xi\eta} + (2\eta + \Phi_\xi)\Phi_{\eta\eta} - \Phi_\eta$	(0.4)	$u = \Phi(\xi, \eta)e^{-3t}, \xi = ye^{bt}, \eta = xe^{2t}$

Table 1. ‘Nontrivial’ reductions

1. Preliminaries

Let \mathcal{E} be a differential equation given by

$$F\left(x, \dots, \frac{\partial^{|\sigma|}u}{\partial x^\sigma}, \dots\right) = 0, \tag{1.1}$$

where $u(x)$ is the unknown function in the variables $x = (x^1, \dots, x^n)$. A *symmetry* of \mathcal{E} is a function $\varphi = \varphi(x, \dots, u_\sigma, \dots)$ in the *jet variables* u_σ , σ being a multi-index, $u_\emptyset = u$, that satisfies the *linearized equation*

$$\ell_{\mathcal{E}}(\varphi) \equiv \sum_{\sigma} \frac{\partial F}{\partial u_\sigma} D_\sigma(\varphi) = 0, \tag{1.2}$$

where $D_\sigma = D_{i_1} \circ \dots \circ D_{i_k}$ for $\sigma = i_1 \dots i_k$, while

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\sigma} u_{\sigma i} \frac{\partial}{\partial u_\sigma} \tag{1.3}$$

are the *total derivatives* restricted to \mathcal{E} . Symmetries of \mathcal{E} form a Lie algebra $\text{sym } \mathcal{E}$ over \mathbb{R} with respect to the *Jacobi bracket*

$$\{\varphi, \bar{\varphi}\} = \sum_{\sigma} \left(\frac{\partial \varphi}{\partial u_\sigma} D_\sigma(\bar{\varphi}) - \frac{\partial \bar{\varphi}}{\partial u_\sigma} D_\sigma(\varphi) \right). \tag{1.4}$$

A solution u to Eq. (1.1) is said to be *invariant* with respect to a symmetry $\varphi \in \text{sym } \mathcal{E}$ if it enjoys the equation

$$\varphi\left(x, \dots, \frac{\partial^{|\sigma|}u}{\partial x^\sigma}, \dots\right) = 0. \tag{1.5}$$

The reduction of \mathcal{E} with respect to φ is Eq. (1.1) rewritten in terms of first integrals of Eq. (1.5).

2. The universal hierarchy equation

Recall that the equation is

$$\mathcal{E}_{(0,1)}: \quad u_{yy} = u_z u_{xy} - u_y u_{xz}.$$

2.1. Symmetries

The defining equation for symmetries for this equation is

$$D_y^2(\varphi) = u_z D_x D_y(\varphi) - u_y D_x D_z(\varphi) + u_{xy} D_z(\varphi) - u_{xz} D_y(\varphi). \quad (2.1)$$

Its solutions are

$$\begin{aligned} \varphi_1 &= yu_y + u, \\ \varphi_2(X_2) &= X_2 u_x - X_2' u, \\ \varphi_3(Z_3) &= Z_3 u_z + Z_3' y u_y, \\ \varphi_4(Z_4) &= Z_4 u_y, \\ \varphi_5(X_5) &= X_5, \end{aligned}$$

where X_i are functions of x , Z_i are functions of z and ‘prime’ denotes the derivative with respect to the corresponding variable. The commutator relations are given in Table 2.

	φ_1	$\varphi_2(\bar{X}_2)$	$\varphi_3(\bar{Z}_3)$	$\varphi_4(\bar{Z}_4)$	$\varphi_5(\bar{X}_5)$
φ_1	0	0	0	$\varphi_4(\bar{Z}_4)$	$-\varphi_5(\bar{X}_5)$
$\varphi_2(X_2)$...	$\varphi_2(\bar{X}_2 X_2' - X_2 \bar{X}_2')$	0	0	$\varphi_5(\bar{X}_5 X_2' - X_2 \bar{X}_5')$
$\varphi_3(Z_3)$	$\varphi_3(\bar{Y}_3 Y_3' - Y_3 \bar{Y}_3')$	$\varphi_4(\bar{Y}_4 Y_3' - Y_3 \bar{Y}_4')$	0
$\varphi_4(Z_4)$	0	0
$\varphi_5(X_5)$	0

Table 2. Lie algebra structure of $\text{sym } \mathcal{E}_{(0,1)}$

2.2. Reductions

Thus, the general symmetry of Eq. (0.1) is

$$\varphi = X_2 u_x + (\alpha y + Z_3' y + Z_4) u_y + Z_3 u_z + (\alpha - X_2') u + X_5,$$

where $\alpha \in \mathbb{R}$ is a constant. Thus, invariant with respect to φ solutions are given by the system

$$\frac{dx}{X_2} = \frac{dy}{(\alpha + Z_3')y + Z_4} = \frac{dz}{Z_3} = -\frac{du}{(\alpha - X_2')u + X_5}. \quad (2.2)$$

We consider the following basic cases below:

Case 00 $X_2 = 0, Z_3 = 0$;

Case 01 $X_2 = 0, Z_3 \neq 0$;

Case 10 $X_2 \neq 0, Z_3 = 0$;

Case 11 $X_2 \neq 0, Z_3 \neq 0$.

Let us study them in detail.

2.2.1. Case 00

System (2.2) takes the form

$$\frac{dx}{0} = \frac{dy}{\alpha y + Z_4} = \frac{dz}{0} = -\frac{du}{\alpha u + X_5}.$$

Its integrals are

$$(\alpha y + Z_4)u + X_5 y = \text{const}, \quad x = \text{const}, \quad z = \text{const}$$

and the general solution is given by

$$\Psi((\alpha y + Z)u + Xy, x, z) = 0,$$

where $Z = Z_4, X = X_5$. Hence,

$$u = \frac{\Phi(x, z) - Xy}{\alpha y + Z}. \tag{2.3}$$

To simplify the subsequent exposition, we consider two subcases:

Subcase 00.0 $\alpha = 0$;

Subcase 00.1 $\alpha \neq 0$.

Then we have:

Subcase 00.0 After redenoting $\Phi \mapsto \Phi/Z, Z \neq 0$, we have

$$u = \Phi(x, z) - \frac{Xy}{Z}. \tag{2.4}$$

Substituting to Eq. (0.1), one obtains

$$\frac{1}{Z} \cdot (X\Phi_{xz} - X'\Phi_z) = 0,$$

which leads to the following class of solutions

$$u = \begin{cases} \Phi(x, z), & \text{if } X = 0, \\ XP(z) + Q(x) - \frac{Xy}{Z}, & \text{if } X \neq 0. \end{cases}$$

Subcase 00.1 Making the change $\Phi \mapsto \Phi - XZ$, one gets

$$u = \frac{\Phi}{y+Z} - X.$$

Substituting to (0.1), one arrives to the equation

$$2\Phi = \Phi\Phi_{xz} - \Phi_x\Phi_z.$$

After the change $\Phi = e^\Psi$ we obtain the Liouville equation

$$\Psi_{xz} = 2e^{-\Psi},$$

see, e.g. [4].

2.2.2. Case 01

Now we have

$$\frac{dx}{0} = \frac{dy}{(\alpha + Z'_3)y + Z_4} = \frac{dz}{Z_3} = -\frac{du}{\alpha u + X_5}.$$

The integrals of the system are

$$u \exp\left(\int \frac{\alpha dz}{Z_3}\right) + \int \frac{X_5}{Z_3} \exp\left(\int \frac{\alpha dz}{Z_3}\right) dz = \text{const}, \quad x = \text{const},$$

$$y \exp\left(-\int \frac{\alpha + Z'_3}{Z_3} dz\right) - \int \frac{Z_4}{Z_3} \exp\left(-\int \frac{\alpha + Z'_3}{Z_3} dz\right) dz = \text{const}.$$

We introduce new functions

$$Z = \int \frac{dz}{Z_3}, \quad \bar{Z} = \int \frac{Z_4}{Z_3} \exp\left(-\int \frac{\alpha + Z'_3}{Z_3} dz\right) dz, \quad X = X_5.$$

Note that $Z' \neq 0$, We again distinguish two subcases:

Subcase 01.0 $\alpha = 0$;

Subcase 01.1 $\alpha \neq 0$.

Let us study them.

Subcase 01.0 In this case, the system of integrals transforms to

$$u + XZ = \text{const}, \quad x = \text{const}, \quad yZ' - \bar{Z} = \text{const},$$

and thus

$$\Psi(u + XZ, x, yZ' - \bar{Z}) = 0 \tag{2.5}$$

is the general solution. Consequently,

$$u = \Phi(x, \xi) - XZ,$$

where $\xi = yZ' - \bar{Z}$. Substituting the last expression to Eq. (0.1), we obtain the equation

$$\Phi_{\xi\xi} = X'\Phi_{\xi} - X\Phi_{x\xi}. \quad (2.6)$$

When $X = 0$, we obtain the solutions

$$u = a_1(yZ' - \bar{Z}) + a_0, \quad a_i = a_i(x).$$

If $X \neq 0$ Eq. (2.6) can also be solved and the general solution is

$$u = \Phi\left(yZ' - \bar{Z} + \int \frac{dx}{X}\right) + a(x),$$

where Φ is an arbitrary function in one argument.

Subcase 01.1 We can set $\alpha = 1$ and the general solution is

$$\Psi((u+X)e^{Z,x}, Z'e^{-Z}y - \bar{Z}) = 0.$$

Hence, after the change $Z \mapsto \ln Z$, we have

$$u = \frac{1}{Z}\Phi(x, \xi) - X, \quad (2.7)$$

where $\xi = Z'y/Z^2 - \bar{Z}$. Substituting (2.7) to Eq. (0.1), one obtains

$$\Phi_{\xi\xi} = \Phi_x\Phi_{\xi} - \Phi\Phi_{x\xi}.$$

2.2.3. Case 10

System (2.2) is now of the form

$$\frac{dx}{X_2} = \frac{dy}{\alpha y + Z_4} = \frac{dz}{0} = -\frac{du}{(\alpha - X_2')u + X_5}.$$

Then the integrals are

$$u \exp\left(\int \frac{\alpha - X_2'}{X_2} dx\right) + \int \frac{X_5}{X_2} \exp\left(\int \frac{\alpha - X_2'}{X_2} dx\right) dx = \text{const},$$

$$y \exp\left(-\int \frac{\alpha dx}{X_2}\right) - \int \frac{Z_4}{X_2} \exp\left(-\int \frac{\alpha dx}{X_2}\right) dx = \text{const}, \quad z = \text{const}.$$

Let us introduce the notation

$$\int \frac{dx}{X_2} = X, \quad \frac{X_5}{X_2} \exp\left(\int \frac{\alpha - X_2'}{X_2} dx\right) dx = \bar{X}, \quad Z_4 = Z$$

and consider the subcases

Subcase 10.0 $\alpha = 0$;

Subcase 10.1 $\alpha \neq 0$.

Consider them in detail.

Subcase 10.0 In this case the general solution is given by

$$\Psi(X'u + \bar{X}, y - XZ, z) = 0, \quad X' \neq 0,$$

and thus

$$u = \frac{1}{X'} \Phi(\xi, z) - \bar{X},$$

where $\xi = y - XZ$. Substituting this expression to Eq. (0.1), one obtains

$$(1 + Z\Phi_z)\Phi_{\xi\xi} = Z\Phi_{\xi}\Phi_{\xi z} + Z'\Phi_{\xi}^2.$$

The equation can be solved explicitly. Indeed, dividing by Φ_{ξ}^2 one obtains

$$\frac{\Phi_{\xi\xi}}{\Phi_{\xi}^2} - Z' = Z \frac{\Phi_{\xi}\Phi_{\xi z} - \Phi_z\Phi_{\xi\xi}}{\Phi_{\xi}^2},$$

or

$$-\left(\frac{1}{\Phi_{\xi}}\right)_{\xi} - Z' = Z \left(\frac{\Phi_z}{\Phi_{\xi}}\right)_{\xi}.$$

Hence,

$$-\frac{1}{\Phi_{\xi}} - Z'\xi = Z \frac{\Phi_z}{\Phi_{\xi}} + \varphi,$$

where $\varphi = \varphi(z)$ is an arbitrary function. Thus,

$$Z\Phi_z + (Z'\xi + \varphi)\Phi_{\xi} = -1$$

and

$$\Phi = \Upsilon(\xi - \bar{\varphi}) - \int \frac{dz}{Z}$$

is the general solution, where $\bar{\varphi} = Z \int \frac{\varphi dz}{Z^2}$.

Subcase 10.1 We may set $\alpha = 1$ and then obtain the general solution in the form

$$\Psi(X'ue^X + \bar{X}, e^{-X}(y + Z), z) = 0,$$

or, after the change $X \mapsto \ln X$,

$$u = \frac{\Phi(\xi, z) - \bar{X}}{X'}, \tag{2.8}$$

where $\xi = (y + Z)/X$. Substituting to (0.1), one has

$$(1 + \xi\Phi_z)\Phi_{\xi\xi} - \xi\Phi_{\xi}\Phi_{\xi z} + \Phi_{\xi}\Phi_z = 0.$$

2.2.4. Case 11

We have here

$$\frac{dx}{X_2} = \frac{dy}{(\alpha + Z'_3)y + Z_4} = \frac{dz}{Z_3} = -\frac{du}{(\alpha - X'_2)u + X_5},$$

where $X_2 \neq 0, Z_3 \neq 0$. The integrals are

$$\begin{aligned} u \exp\left(\int \frac{\alpha - X'_2}{X_2} dx\right) + \int \frac{X_5}{X_2} \exp\left(\int \frac{\alpha - X'_2}{X_2} dx\right) dx &= \text{const}, \int \frac{dx}{X_2} - \int \frac{dx}{Z_3} = \text{const}, \\ y \exp\left(-\int \frac{\alpha + Z'_3}{Z_3} dx\right) - \int \frac{Z_4}{Z_3} \exp\left(-\int \frac{\alpha + Z'_3}{Z_3} dx\right) dx &= \text{const}. \end{aligned}$$

As before, we introduce the notation

$$\int \frac{dx}{X_2} = X, \int \frac{X_5}{X_2} \exp\left(\int \frac{\alpha - X'_2}{X_2} dx\right) dx = \bar{X}, \int \frac{dx}{Z_3} = Z, \int \frac{Z_4}{Z_3} \exp\left(-\int \frac{\alpha + Z'_3}{Z_3} dx\right) dx = \bar{Z}$$

and consider two subcases

Subcase 11.0 $\alpha = 0$;

Subcase 11.1 $\alpha \neq 0$.

Then we have:

Subcase 11.0 The general solution is given by

$$\Psi(X'u + \bar{X}, Z'y - \bar{Z}, X - Z) = 0$$

in this case and thus

$$u = \frac{\Phi(\xi, \eta) - \bar{X}}{X'}, \quad \xi = Z'y - \bar{Z}, \quad \eta = X - Z.$$

After substituting to Eq. (0.1), one has

$$\boxed{\Phi_{\xi\xi} = \Phi_{\xi} \Phi_{\eta\eta} - \Phi_{\eta} \Phi_{\xi\eta}.}$$

The equation can be linearized by the Legendre transformation, [12].

Subcase 11.1 Setting $\alpha = 1$, we obtain the general solution

$$\Psi(X'e^X u + \bar{X}, Z'e^{-Z} y - \bar{Z}, X - Z) = 0$$

and, after the change $X \mapsto \ln X, Z \mapsto -\ln Z$, one has

$$u = \frac{\Phi(\xi, \eta) - \bar{X}}{X'}, \quad \xi = Z'y - \bar{Z}, \quad \eta = \ln(XZ). \tag{2.9}$$

Substituting (2.9) to (0.1), we obtain the equation

$$\boxed{\Phi_{\eta} \Phi_{\xi\eta} - \Phi_{\xi} \Phi_{\eta\eta} = e^{\eta} \Phi_{\xi\xi}.}$$

3. The 3D rdDym equation

As it was said above, the equation is

$$\mathcal{E}_{(0,2)}: \quad u_{ty} = u_x u_{xy} - u_y u_{xx}.$$

3.1. Symmetries

Symmetries of Eq. (0.2) are defined by

$$D_t D_y(\varphi) = u_x D_x D_y(\varphi) - u_y D_x^2(\varphi) + u_{xy} D_x(\varphi) - u_{xx} D_y(\varphi). \quad (3.1)$$

Solutions of (3.1) are

$$\begin{aligned} \varphi_1 &= x u_x - 2u, \\ \varphi_2(T_2) &= T_2 u_t + T_2'(x u_x - u) + \frac{1}{2} T_2'' x^2, \\ \varphi_3(Y_3) &= Y_3 u_y, \\ \varphi_4(T_4) &= T_4 u_x + T_4' x, \\ \varphi_5(T_5) &= T_5, \end{aligned}$$

where $Y_i = Y_i(y)$, $T_i = T_i(t)$ and ‘primes’ denote the derivatives. The commutator relations are given in Table 3.

	φ_1	$\varphi_2(\bar{T}_2)$	$\varphi_3(\bar{Y}_3)$	$\varphi_4(\bar{T}_4)$	$\varphi_5(\bar{T}_5)$
φ_1	0	0	0	$\varphi_4(\bar{T}_4)$	$2\varphi_5(\bar{T}_5)$
$\varphi_2(T_2)$...	$\varphi_2(\bar{T}_2 X_2' - T_2 \bar{T}_2')$	0	$\varphi_4(\bar{T}_4 T_2' - T_2 \bar{T}_4')$	$\varphi_5(\bar{T}_5 T_2' - T_2 \bar{T}_5')$
$\varphi_3(Y_3)$	0	0	0
$\varphi_4(T_4)$	$\varphi_5(\bar{T}_4 T_4' - T_4 \bar{T}_4')$	0
$\varphi_5(T_5)$	0

Table 3. Lie algebra structure of $\text{sym } \mathcal{E}_{(0,2)}$

3.2. Reductions

The general symmetry of Eq. (0.2) is

$$\varphi = (\alpha x + T_2' x + T_4) u_x + Y_3 u_y + T_2 u_t - (2\alpha + T_2') u + T_5 + T_4' x + \frac{1}{2} T_2'' x^2,$$

where $\alpha \in \mathbb{R}$ is a constant. Consequently, φ -invariant solutions are defined by the system

$$\frac{dx}{(\alpha + T_2')x + T_4} = \frac{dy}{Y_3} = \frac{dt}{T_2} = \frac{du}{(2\alpha + T_2')u - T_5 - T_4'x - \frac{1}{2} T_2'' x^2}. \quad (3.2)$$

In what follows, we consider the following cases

- Case 00** $Y_3 = 0, T_2 = 0$;
- Case 01** $Y_3 = 0, T_2 \neq 0$;
- Case 10** $Y_3 \neq 0, T_2 = 0$;
- Case 11** $Y_3 \neq 0, T_2 \neq 0$.

3.2.1. Case 00

Eq. (4.2) takes the form

$$\frac{dx}{\alpha x + T_4} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{2\alpha u - T_5 - T_4'x}.$$

As before, two subcases must be considered:

Subcase 00.0 $\alpha = 0$;

Subcase 00.1 $\alpha \neq 0$.

One has the following:

Subcase 00.0 Here we have

$$\frac{dx}{T_4} = \frac{dy}{0} = \frac{dt}{0} = -\frac{du}{T_5 + T_4'x}$$

and the general solution is given by

$$\Psi\left(u + \frac{1}{2}Tx^2 + \bar{T}x, y, t\right) = 0,$$

or

$$u = \Phi(y, t) - \frac{1}{2}Tx^2 - \bar{T}x, \tag{3.3}$$

where $T = T_4'/T_4$, $\bar{T} = T_5/T_4$. Substituting (3.3) in Eq. (2.9), we obtain

$$\Phi_{yt} = T\Phi_y.$$

The general solution is

$$\Phi = \varphi(y)e^{\int T dt} + \psi(t),$$

which leads to the following family of solutions to Eq. (2.9):

$$u = \varphi(y)e^{\int T dt} + \psi(t) - \frac{1}{2}Tx^2 - \bar{T}x.$$

Subcase 00.1 Setting $\alpha = 1$, we obtain

$$\frac{dx}{x + T_4} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{2u - T_5 - T_4'x}.$$

The general solution of this system is

$$u = (x + T)^2\Phi(y, t) + T'(x + T) + \bar{T}, \tag{3.4}$$

where $T = T_4$, $\bar{T} = (T_5 - T_4T_4')/2$. Substituting to (0.2), we obtain the equation

$$\Phi_{yt} = 2\Phi\Phi_y.$$

Integrating over y , we come to the Riccati equation

$$\Phi_t = \Phi^2 + \varphi(t).$$

Thus, to any choice of φ there corresponds a family of solutions to Eq. (0.2).

Examples. Let us consider some particular cases.

(1) If $\varphi = 0$ then

$$\Phi = \frac{1}{\psi - t}.$$

Here and in all the examples below ψ is an arbitrary function of y .

(2) For $\varphi = a^2$, $a = \text{const}$, one has

$$\Phi = a \tan(a(t + \psi)).$$

(3) If $\varphi = -a^2$ then

$$\Phi = a \frac{1 + e^{2a(t+\psi)}}{1 - e^{2a(t+\psi)}}$$

(4) For $\varphi = t^\kappa$, $\kappa \in \mathbb{R}$, one has

$$\Phi = \frac{-\psi J\left(\frac{1}{\kappa+2}, \frac{2t^{\frac{1}{2}\kappa+1}}{\kappa+2}\right) + \psi J\left(\frac{3+\kappa}{\kappa+2}, \frac{2t^{\frac{1}{2}\kappa+1}}{\kappa+2}\right)t^{\frac{1}{2}\kappa+1} - Y\left(\frac{1}{\kappa+2}, \frac{2t^{\frac{1}{2}\kappa+1}}{\kappa+2}\right) + Y\left(\frac{3+\kappa}{\kappa+2}, \frac{2t^{\frac{1}{2}\kappa+1}}{\kappa+2}\right)t^{\frac{1}{2}\kappa+1}}{t\left(\psi J\left(\frac{1}{\kappa+2}, \frac{2t^{\frac{1}{2}\kappa+1}}{\kappa+2}\right) + Y\left(\frac{1}{\kappa+2}, \frac{2t^{\frac{1}{2}\kappa+1}}{\kappa+2}\right)\right)},$$

where

$$J(a, \theta) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+a+1)} \left(\frac{\theta}{2}\right)^{2m+a},$$

$$Y(a, \theta) = \frac{J(a, \theta) \cos(a\pi) - J(-a, \theta)}{\sin(a\pi)}$$

are Bessel functions of the first and second kinds, respectively.

(5) If $\varphi = e^t$, then

$$\Phi = \left(\frac{\psi Y(1, 2e^{\frac{t}{2}})}{\psi Y(0, 2e^{\frac{t}{2}}) + J(0, 2e^{\frac{t}{2}})} + \frac{J(1, 2e^{\frac{t}{2}})}{\psi Y(0, 2e^{\frac{t}{2}}) + J(0, 2e^{\frac{t}{2}})} \right) e^{\frac{t}{2}}$$

(6) For $\varphi = (1-t)/(1+t)$ the solution is

$$\Phi = \frac{2\psi \text{Ei}(1, -2-2t)t + \psi e^{2+2t} + 2t}{2\psi(1+t) \text{Ei}(1, -2-2t) + 2t + \psi e^{2+2t} + 2},$$

where

$$\text{Ei}(a, t) = \int_1^{\infty} \frac{e^{-\theta t} d\theta}{\theta^a}$$

is the exponential integral function.

3.2.2. Case 01

We have

$$\frac{dx}{(\alpha + T_2')x + T_4} = \frac{dy}{0} = \frac{dt}{T_2} = \frac{du}{(2\alpha + T_2')u - T_5 - T_4'x - \frac{1}{2}T_2''x^2},$$

where $T_2 \neq 0$. Its integrals are

$$T'xe^{-\alpha T} - \bar{T} = \text{const}, \quad y = \text{const},$$

$$T'ue^{-2\alpha T} + \bar{\bar{T}} + \left(\alpha\bar{T} + \left(\frac{\bar{T}}{T'} \right)' \right) (T'xe^{-\alpha T} - \bar{T}) - \frac{1}{2} \frac{T''}{(T')^2} (T'xe^{-\alpha T} - \bar{T})^2 = \text{const},$$

where

$$T = \int \frac{dt}{T_2}, \quad \bar{T} = \int (T_4 \cdot (T')^2 \cdot e^{-\alpha T}) dt,$$

$$\bar{\bar{T}} = \int \left(T_5 \cdot (T')^2 \cdot e^{-2\alpha T} + T_4' \cdot T \cdot T' \cdot e^{-\alpha T} + \frac{1}{2} T_2'' \cdot \bar{T}^2 \right) dt.$$

Then the general solution is

$$\Psi \left(T'xe^{-\alpha T} - \bar{T}, y, T'ue^{-2\alpha T} + \bar{\bar{T}} + \left(\alpha\bar{T} + \left(\frac{\bar{T}}{T'} \right)' \right) (T'xe^{-\alpha T} - \bar{T}) - \frac{1}{2} \frac{T''}{(T')^2} (T'xe^{-\alpha T} - \bar{T})^2 \right) = 0,$$

or

$$u = \left(\Phi(\xi, y) - \bar{\bar{T}} - \left(\alpha\bar{T} + \left(\frac{\bar{T}}{T'} \right)' \right) \xi + \frac{1}{2} \frac{T''}{(T')^2} \xi^2 \right) \frac{e^{2\alpha T}}{T'},$$

where

$$\xi = T'xe^{-\alpha T} - \bar{T}.$$

Substituting to Eq. (0.2), one obtains

$$\boxed{(\alpha\xi + \Phi_\xi)\Phi_{\xi y} - \Phi_y(\Phi_{\xi\xi} + 2\alpha) = 0.} \tag{3.5}$$

3.2.3. Case 10

The defining equations are

$$\frac{dx}{\alpha x + T_4} = \frac{dy}{Y_3} = \frac{dt}{0} = \frac{du}{2\alpha u - T_5 - T_4'x}, \tag{3.6}$$

where $Y_3 \neq 0$. Below we consider the following subcases:

Subcase 10.00 $\alpha = 0, T_4 = 0;$

Subcase 10.01 $\alpha = 0, T_4 \neq 0;$

Subcase 10.1 $\alpha \neq 0.$

Subcase 10.00 In this case, $T_5 \neq 0$ and System (3.6) takes the form

$$\frac{dx}{0} = Y' dy = \frac{dt}{0} = -\frac{du}{T_5}.$$

Denote $T_5 = T$. Then the integrals are

$$x = \text{const}, \quad t = \text{const}, \quad u + YT = \text{const}.$$

Then the general solution is given by

$$\Psi(u + YT, x, t) = 0,$$

or

$$u = \Phi(x, t) - YT.$$

Substituting to Eq. (0.2), one obtains

$$-Y'T' = Y'T\Phi_{xx},$$

or, since $Y' = 1/Y_3 \neq 0$,

$$\Phi_{xx} = -\frac{T'}{T}.$$

This delivers us the following family of solutions:

$$u = -\frac{T'}{2T}x^2 + \varphi(t)x + \psi(t) - YT.$$

Subcase 10.01 The defining equations are now

$$\frac{dx}{T_4} = Y' dy = \frac{dt}{0} = -\frac{du}{T_5 + T_4'x}.$$

Let us introduce the notation $T_4 = T$, $T_5/T_4 = \bar{T}$. Then the integrals are

$$x - YT = \text{const}, \quad t = \text{const}, \quad u + \frac{T'}{2T}x^2 + \bar{T}x = \text{const}$$

and the general solution is

$$\Psi\left(u + \frac{T'}{2T}x^2 + \bar{T}x, x - YT, t\right) = 0,$$

or

$$u = \Phi(\xi, t) - \frac{T'}{2T}x^2 - \bar{T}x,$$

where $\xi = x - YT$. Substituting to (0.2), we obtain the linear equation

$$\left(\frac{T'}{T}\xi + \bar{T}\right)\Phi_{\xi\xi} + \Phi_{\xi t} = 0.$$

The general solution of this equation is

$$\Phi = \varphi(\eta)T + \psi(t), \quad \eta = \frac{\xi}{T} - \int \frac{\bar{T}}{T} dt,$$

which gives the family of solutions to (0.2):

$$u = \varphi(\eta)T + \psi(t) - \frac{T'}{2T}x^2 - \bar{T}x.$$

Subcase 10.1 We can assume $\alpha = 1$ and the defining equations become

$$\frac{dx}{x + T_4} = Y' dy = \frac{dt}{0} = \frac{du}{2u - T_5 - T_4'x}.$$

The integrals of this system are

$$(x + T)e^{-Y} = \text{const}, \quad t = \text{const}, \quad \frac{u - \bar{T}}{(x + T)^2} - \frac{T'}{x + T} = \text{const},$$

where $T = T_4$, $\bar{T} = (T_5 - T'T)/2$, and thus the general solution is

$$\Psi \left(\frac{u - \bar{T}}{(x + T)^2} - \frac{T'}{x + T}(x + T), e^{-Y}, t \right) = 0,$$

or

$$u = (x + T)^2 \Phi(\xi, t) + T'(x + T) + \bar{T}, \quad \xi = (x + T)e^{-Y}.$$

Substituting to (0.2), one obtains the equation

$$\Phi_{\xi t} = 4\Phi\Phi_{\xi} - \xi\Phi_{\xi}^2 + 2\xi\Phi\Phi_{\xi\xi}.$$

3.2.4. Case 11

Let us set $Y' = 1/Y_3 \neq 0$ and $T' = 1/T_2 \neq 0$. Then System (3.2) becomes

$$\frac{dx}{(\alpha + T_2')x + T_4} = Y' dy = T' dt = \frac{du}{(2\alpha + T_2')u - T_5 - T_4'x - \frac{1}{2}T_2''x^2}.$$

The integrals are

$$T'xe^{-\alpha T} - \bar{T} = \text{const}, \quad Y - T = \text{const},$$

$$T'ue^{-2\alpha} + \bar{T} + \left(\alpha\bar{T} + \left(\frac{\bar{T}}{T'} \right)' \right) (T'xe^{-\alpha T} - \bar{T}) - \frac{1}{2} \frac{T''}{(T')^2} (T'xe^{-\alpha T} - \bar{T})^2 = \text{const},$$

where, as before,

$$T = \int \frac{dt}{T_2}, \quad \bar{T} = \int (T_4 \cdot (T')^2 \cdot e^{-\alpha T}) dt,$$

$$\bar{T} = \int \left(T_5 \cdot (T')^2 \cdot e^{-2\alpha T} + T_4' \cdot T \cdot T' \cdot e^{-\alpha T} + \frac{1}{2} T_2'' \cdot \bar{T}^2 \right) dt.$$

Thus, the general solution is given by

$$\Psi \left(T'xe^{-\alpha T} - \bar{T}, Y - T, T'ue^{-2\alpha T} + \bar{T} + \left(\alpha\bar{T} + \left(\frac{\bar{T}}{T'} \right)' \right) (T'xe^{-\alpha T} - \bar{T}) - \frac{1}{2} \frac{T''}{(T')^2} (T'xe^{-\alpha T} - \bar{T})^2 \right) = 0,$$

or

$$u = \left(\Phi(\xi, \eta) - \bar{T} - \left(\alpha\bar{T} + \left(\frac{\bar{T}}{T'} \right)' \right) \xi + \frac{1}{2} \frac{T''}{(T')^2} \xi^2 \right) \frac{e^{2\alpha T}}{T'},$$

where

$$\xi = T'xe^{-\alpha T} - \bar{T}, \quad \eta = Y - T.$$

Substituting the last expression to Eq. (0.2), one obtains

$$\Phi_{\eta\eta} + (\alpha\xi + \Phi_{\eta})\Phi_{\xi\eta} = \Phi_{\eta}(2\alpha + \Phi_{\xi\xi}).$$

4. The modified Veronese web equation

The equation is

$$\mathcal{E}_{(0.3)}: \quad u_{ty} = u_t u_{xy} - u_y u_{tx}.$$

4.1. Symmetries

Symmetries of (0.3) are defined by

$$D_t D_y(\varphi) = u_t D_x D_y(\varphi) - u_y D_t D_x(\varphi) + u_{xy} D_t(\varphi) - u_{tx} D_y(\varphi), \quad (4.1)$$

whose solutions are

$$\begin{aligned} \varphi_1(T_1) &= T_1 u_t, \\ \varphi_2(X_2) &= X_2 u_x - X_2' u, \\ \varphi_3(Y_3) &= Y_3 u_y, \\ \varphi_4(X_4) &= X_4, \end{aligned}$$

where $X_i = X_i(x)$, $Y_i = Y_i(y)$, and $T_i = T_i(t)$. The commutator relations in $\text{sym } \mathcal{E}_{(0.3)}$ are given in Table 4.

4.2. Reductions

The general symmetry of Eq. (0.3) is

$$\varphi = X_2 u_x + Y_3 u_y + T_1 u_t - X_2' u + X_4$$

and the corresponding invariant solutions must satisfy the system

$$\frac{dx}{X_2} = \frac{dy}{Y_3} = \frac{dt}{T_1} = \frac{du}{X_2' u - X_4}. \quad (4.2)$$

We consider below the following cases:

	$\varphi_1(\bar{T}_1)$	$\varphi_2(\bar{X}_2)$	$\varphi_3(\bar{Y}_3)$	$\varphi_4(\bar{X}_4)$
$\varphi_1(T_1)$	$\varphi_1(\bar{T}_1 T_1' - T_1 \bar{T}_1')$	0	0	0
$\varphi_2(X_2)$...	$\varphi_2(\bar{X}_2 X_2' - X_2 \bar{X}_2')$	0	$\varphi_4(\bar{X}_4 X_2' - X_2 \bar{X}_4')$
$\varphi_3(Y_3)$	$\varphi_3(\bar{Y}_3 Y_3' - Y_3 \bar{Y}_3')$	0
$\varphi_4(X_4)$	0

Table 4. Lie algebra structure of sym $\mathcal{E}_{(0,3)}$

Case 100 $X_2 \neq 0, Y_3 = 0, Z_4 = 0;$

Case 010 $X_2 = 0, Y_3 \neq 0, Z_4 = 0;$

Case 001 $X_2 = 0, Y_3 = 0, Z_4 \neq 0;$

Case 011 $X_2 = 0, Y_3 \neq 0, Z_4 \neq 0;$

Case 101 $X_2 \neq 0, Y_3 = 0, Z_4 \neq 0;$

Case 110 $X_2 \neq 0, Y_3 \neq 0, Z_4 = 0;$

Case 111 $X_2 \neq 0, Y_3 \neq 0, Z_4 \neq 0;$

and use the notation $1/X_2 = X', 1/Y_3 = Y', 1/Z_4 = Z'$ when it is well defined.

4.2.1. Case 100

The defining equation is

$$\frac{dx}{X_2} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{X_2' u - X_4}$$

The integrals are

$$Xu - \bar{X} = \text{const}, \quad y = \text{const}, \quad t = \text{const},$$

and thus

$$\Psi(Xu - \bar{X}, y, t) = 0$$

is the general solution, where $\bar{X} = \int X_4 X' dx$. Consequently,

$$u = \frac{\Phi(y, t) + \bar{X}}{X}$$

Substituting to (0.3), one obtains

$$\boxed{\Phi_{y,t} = 0.}$$

Hence, $\Phi = \varphi(y) + \psi(t)$ and

$$\boxed{u = \frac{\varphi(y) + \psi(t) + \bar{X}}{X}}$$

is a family of solutions to (0.3).

4.2.2. Case 010

The defining equation is

$$\frac{dx}{0} = \frac{dy}{Y_3} = \frac{dt}{0} = -\frac{du}{X_4}.$$

The integrals are

$$u + \bar{X}Y = \text{const}, \quad x = \text{const}, \quad t = \text{const},$$

where $\bar{X} = X_4$. Then

$$\Psi(u + \bar{X}Y, x, t) = 0$$

is the general solution and

$$u = \Phi(x, t) - \bar{X}Y.$$

Substituting to (0.3), one obtains $Y'(\bar{X}\Phi_{xt} - \bar{X}'\Phi_t) = 0$ and since $Y' \neq 0$,

$$\boxed{\bar{X}\Phi_{xt} - \bar{X}'\Phi_t = 0.}$$

Thus, if $\bar{X} = 0$ we obtain the obvious family of solutions

$$\boxed{u = \Phi(x, t).}$$

If $\bar{X} \neq 0$ the corresponding family of solutions is

$$\boxed{u = \bar{X}\varphi(t) + \psi(x) - \bar{X}Y.}$$

4.2.3. Case 001

The defining equation is

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{T_1} = -\frac{du}{X_4}.$$

Then, again denoting $\bar{X} = X_4$, we get the integrals

$$u + \bar{X}T = \text{const}, \quad x = \text{const}, \quad y = \text{const}$$

and the general solution in the form

$$\Psi(u + \bar{X}T, x, y) = 0,$$

or

$$u = \Phi(x, y) - \bar{X}T.$$

Substituting to (0.3), one obtains

$$\boxed{\bar{X}\Phi_{xy} - \bar{X}'\Phi_y = 0,}$$

since $T' \neq 0$. Then in the case $\bar{X} = 0$ we get the family of solutions

$$u = \Phi(x, y)$$

and when $\bar{X} \neq 0$ the family

$$u = \bar{X}\varphi(y) + \psi(x) - \bar{X}T.$$

4.2.4. Case 011

The defining equation is

$$\frac{dx}{0} = \frac{dy}{Y_3} = \frac{dt}{T_1} = -\frac{du}{X_4}.$$

Its integrals are

$$x = \text{const}, \quad Y - T = \text{const}, \quad u + \bar{X}Y = \text{const}$$

and the general solution is

$$\Psi(u + \bar{X}Y, x, Y - T) = 0,$$

or

$$u = \Phi(x, \xi) - \bar{X}Y, \quad \xi = Y - T.$$

Substituting to Eq. (0.3), one obtains

$$\Phi_{\xi\xi} = \bar{X}\Phi_{x\xi} - \bar{X}'\Phi_{\xi}.$$

If $\bar{X} = 0$ then

$$u = \varphi(x) + \psi(Y - T) - \bar{X}Y.$$

In the case $\bar{X} \neq 0$ the corresponding family is

$$u = \bar{X}\varphi\left(Y - T + \int \frac{dx}{\bar{X}}\right) + \psi(x) - \bar{X}Y.$$

4.2.5. Case 101

The defining equation is

$$\frac{dx}{X_2} = \frac{dy}{0} = \frac{dt}{T_1} = \frac{du}{X_2'u - X_4}.$$

The integrals of this system are

$$X'u + \bar{X} = \text{const}, \quad X - T = \text{const}, \quad y = \text{const},$$

where $\bar{X} = \int (X')^2 X_4 dx$ and the general solution is given by

$$\Psi(X'u - \bar{X}, X - T, y) = 0,$$

or

$$u = \frac{\Phi(y, \xi) + \bar{X}}{X'}, \quad \xi = X - T.$$

After substitution to Eq. (0.3) one obtains

$$(1 + \Phi_\xi) \Phi_{y\xi} = \Phi_y \Phi_{\xi\xi}.$$

The general solution to this equation is

$$\Psi = \Upsilon(\xi + \psi(y)) - \xi,$$

where Υ is an arbitrary function in one argument, and thus we get the family of solutions

$$u = \frac{\Upsilon(X - T + \psi(y)) - X + T + \bar{X}}{X'}$$

to Eq. (0.3).

4.2.6. Case 110

The defining equation is

$$\frac{dx}{X_2} = \frac{dy}{Y_3} = \frac{dt}{0} = \frac{du}{X'_2 u - X_4}.$$

The integrals of this system are

$$X'u - \bar{X} = \text{const}, \quad X - Y = \text{const}, \quad y = \text{const},$$

where $\bar{X} = \int (X')^2 X_4 dx$ and the general solution is given by

$$\Psi(X'u + \bar{X}, X - Y, y) = 0,$$

or

$$u = \frac{\Phi(y, \xi) + \bar{X}}{X'}, \quad \xi = X - Y.$$

Substitution to Eq. (0.3) leads to

$$(1 + \Phi_\xi) \Phi_{t\xi} = \Phi_t \Phi_{\xi\xi}.$$

Similar to the previous case, we solve this equation and obtain the following family of solutions to Eq. (0.3):

$$u = \frac{\Upsilon(X - Y + \psi(t)) - X + Y + \bar{X}}{X'}.$$

4.2.7. Case 111

The defining equation is

$$\frac{dx}{X_2} = \frac{dy}{Y_3} = \frac{dt}{T_1} = \frac{du}{X'_2 u - X_4}.$$

The integrals are

$$X'u - \bar{X} = \text{const}, \quad X - Y = \text{const}, \quad X - T = \text{const},$$

where, as before, $\bar{X} = \int (X')^2 X_4 dx$. This delivers the general solution

$$\Psi(X'u - \bar{X}, X - Y, X - T) = 0,$$

i.e.,

$$u = \frac{\Phi(\xi, \eta) + \bar{X}}{X'}, \quad \xi = X - Y, \quad \eta = X - T.$$

Substituting to (0.3), we obtain the equation

$$\boxed{\Phi_\eta \Phi_{\xi\xi\xi} + (\Phi_\eta - \Phi_\xi - 1) \Phi_{\xi\eta} - \Phi_\xi \Phi_{\eta\eta} = 0.} \quad (4.3)$$

The equation linearizes by the Legendre transformation, [12].

5. Pavlov's equation

The Pavlov equation reads

$$\mathcal{E}_{(0.4)}: \quad u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}.$$

5.1. Symmetries

The defining equation for symmetries of (0.4) is

$$D_y^2(\varphi) = D_t D_x(\varphi) + u_y D_x^2(\varphi) - u_x D_x D_y(\varphi) + u_{xx} D_y(\varphi) - u_{xy} D_x(\varphi). \quad (5.1)$$

Its solutions are

$$\begin{aligned} \varphi_1 &= 2x - yu_x, \\ \varphi_2 &= 3u - 2xu_x - yu_y, \\ \varphi_3(T_3) &= T_3 u_t + T_3'(xu_x + yu_y - u) + \frac{1}{2} T_3''(y^2 u_x - 2xy) - \frac{1}{6} T_3''' y^3, \\ \varphi_4(T_4) &= T_4 u_x - T_4' y, \\ \varphi_5(T_5) &= T_5 u_y + T_5'(yu_x - x) - \frac{1}{2} T_5'' y^2, \\ \varphi_6(T_6) &= T_6, \end{aligned}$$

where T_i are functions of t . The Lie algebra structure in $\text{sym } \mathcal{E}_{(0.4)}$ is given in Table 5.

	φ_1	φ_2	$\varphi_3(\bar{T}_3)$	$\varphi_4(\bar{T}_4)$	$\varphi_5(\bar{T}_5)$	$\varphi_6(\bar{T}_6)$
φ_1	0	φ_1	0	$2\varphi_6(\bar{T}_4)$	$-2\varphi_4(\bar{T}_5)$	0
φ_2	...	0	0	$-2\varphi_4(\bar{T}_4)$	$-\varphi_5(\bar{T}_5)$	$-3\varphi_6(\bar{T}_6)$
$\varphi_3(T_3)$	$\varphi_3(\bar{T}_3 T'_3 - T_3 \bar{T}'_3)$	$\varphi_4(\bar{T}_4 T'_3 - T_3 \bar{T}'_4)$	$\varphi_5(\bar{T}_5 T'_3 - T_3 \bar{T}'_5)$	$\varphi_6(\bar{T}_6 T_3 - T_3 \bar{T}'_6)$
$\varphi_4(T_4)$	0	$\varphi_6(\bar{T}_5 T'_4 - T_4 \bar{T}'_5)$	0
$\varphi_5(T_5)$	$\varphi_4(\bar{T}_5 T'_5 - T_5 \bar{T}'_5)$	0
$\varphi_6(T_6)$	0

Table 5. Lie algebra structure of $\text{sym } \mathcal{E}_{(0,4)}$

5.2. Reductions

The general symmetry of Eq. (0.4) is

$$\varphi = \left(-\alpha y - 2\beta x + T'_3 x + \frac{1}{2} T''_3 y^2 + T_4 + T'_5 y \right) u_x + (-\beta y + T'_3 y + T_5) u_y + T_3 u_t$$

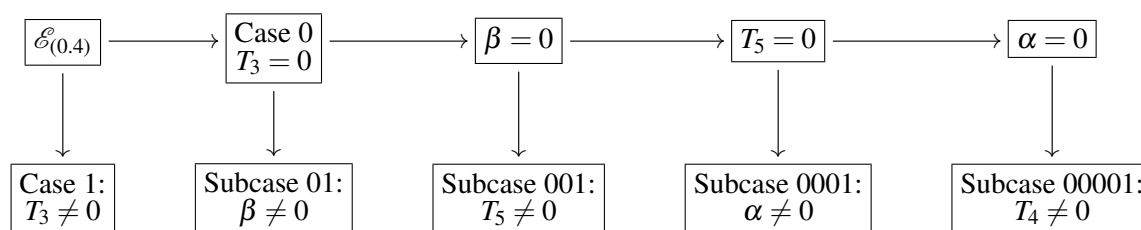
$$+ (3\beta - T'_3) u + 2\alpha x - T''_3 xy - \frac{1}{6} T'''_3 y^3 - T'_5 x - \frac{1}{2} T''_5 y^2 + T_6 - T'_4 y,$$

where $\alpha, \beta \in \mathbb{R}$ are constants. Then the φ -invariant solutions are determined by the system

$$\frac{dx}{(T'_3 - 2\beta)x + (T'_5 - \alpha)y + \frac{1}{2} T''_3 y^2 + T_4} = \frac{dy}{(T'_3 - \beta)y + T_5} = \frac{dt}{T_3}$$

$$= \frac{du}{(T'_3 - 3\beta)u + (T'_5 - 2\alpha)x + T'_4 y + T''_3 xy + \frac{1}{2} T''_5 y^2 + \frac{1}{6} T'''_3 y^3 - T_6}. \quad (5.2)$$

We consider the following tree of options:



5.2.1. Case 0

The defining equations are

$$\frac{dx}{-2\beta x + (T'_5 - \alpha)y + T_4} = \frac{dy}{-\beta y + T_5} = \frac{dt}{0} = \frac{du}{-3\beta u + (T'_5 - 2\alpha)x + T'_4 y + \frac{1}{2} T''_5 y^2 - T_6}.$$

Due to the above picture, consider the following subcases:

Subcase 00001 $\beta = 0, T_5 = 0, \alpha = 0, T_4 \neq 0$;

Subcase 0001 $\beta = 0, T_5 = 0, \alpha \neq 0$;

Subcase 001 $\beta = 0, T_5 \neq 0$;

Subcase 01 $\beta \neq 0$.

Subcase 00001 The defining equations are

$$\frac{dx}{T_4} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{T_4'y - T_6}.$$

Then the integrals are

$$u - (Ty - \bar{T})x = \text{const}, \quad y = \text{const}, \quad t = \text{const},$$

where $T = T_4'/T_4$, $\bar{T} = T_6/T_4$. Thus,

$$\Psi(u - (Ty - \bar{T})x, y, t) = 0$$

is the general solution and

$$u = \Phi(y, t) + (Ty - \bar{T})x.$$

Substituting to Eq. (0.4), one obtains

$$\Phi_{yy} = (T' - T^2)y + T\bar{T} - \bar{T}',$$

which gives the family

$$u = \frac{1}{6}(T' - T^2)y^3 + \frac{1}{2}(T\bar{T} - \bar{T}')y^2 + \varphi(t)y + \psi(t) + (Ty - \bar{T})x.$$

of exact solutions to (0.4).

Subcase 0001 We may assume $\alpha = -1$ and the defining equations become

$$\frac{dx}{y + T_4} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{2x + T_4'y - T_6}.$$

Then the integrals are

$$(y + T_4)u - x^2 - (T_4'y - T_6)x = \text{const}, \quad y = \text{const}, \quad t = \text{const}.$$

Consequently, the general solution is given by

$$\Psi((y + T_4)u - x^2 - (T_4'y - T_6)x, y, t) = 0$$

and thus

$$u = \frac{\Phi(y, t) + x^2 + (T_4'y - T_6)x}{y + T_4}, \tag{5.3}$$

or

$$u = \Phi(y, t) + T'x + \frac{x^2 - \bar{T}x}{y + T},$$

where $T = T_4$, $\bar{T} = T_4T_4' + T_6$. After substituting to (0.4), we obtain the equation

$$\Phi_{yy} = \frac{2\Phi_y}{y + T} + T'' - \frac{\bar{T}'}{y + T} + \frac{\bar{T}^2}{(y + T)^3}.$$

Solving this equation, we obtain the following family of solutions to Eq. (0.4):

$$u = \varphi(t)(y+T)^3 - \frac{1}{2}T''(y+T)^2 + \frac{1}{2}\bar{T}'(y+T) + \frac{2x^2 - 2\bar{T}x + \bar{T}^2}{y+T} + T'x + \psi(t).$$

Subcase 001 The defining equations are

$$\frac{dx}{(T'_5 - \alpha)y + T_4} = \frac{dy}{T_5} = \frac{dt}{0} = \frac{du}{(T'_5 - 2\alpha)x + T'_4y + \frac{1}{2}T''_5y^2 - T_6}.$$

Let us introduce the notation $T = 1/T_5$, $\bar{T} = T_4/T_5$, $\bar{\bar{T}} = T_6/T_5$. Then the integrals acquire the form

$$t = \text{const}, \quad x + \frac{1}{2} \left(\frac{T'}{T} + \alpha T \right) y^2 - \bar{T}y = \text{const},$$

$$u + \left(\frac{1}{6} \frac{T''}{T} + \alpha T' + \frac{2}{3} \alpha^2 T^2 \right) y^3 - \frac{1}{2} (\bar{T}' + 2\alpha T \bar{T}) y^2 + \left(\left(\frac{T'}{T} + 2\alpha T \right) x + \bar{\bar{T}} \right) y = \text{const}.$$

Hence, the general solution is

$$\Psi \left(u + \left(\frac{1}{6} \frac{T''}{T} + \alpha T' + \frac{2}{3} \alpha^2 T^2 \right) y^3 - \frac{1}{2} (\bar{T}' + 2\alpha T \bar{T}) y^2 + \left(\left(\frac{T'}{T} + 2\alpha T \right) x + \bar{\bar{T}} \right) y, \right. \\ \left. x + \frac{1}{2} \left(\frac{T'}{T} + \alpha T \right) y^2 - \bar{T}y, t \right) = 0,$$

or

$$u = \Phi(\xi, t) - \left(\frac{1}{6} \frac{T''}{T} + \alpha T' + \frac{2}{3} \alpha^2 T^2 \right) y^3 + \frac{1}{2} (\bar{T}' + 2\alpha T \bar{T}) y^2 - \left(\left(\frac{T'}{T} + 2\alpha T \right) x + \bar{\bar{T}} \right) y,$$

where

$$\xi = x + \frac{1}{2} \left(\frac{T'}{T} + \alpha T \right) y^2 - \bar{T}y. \tag{5.4}$$

Substituting to Eq. (0.4), one obtains

$$\left(\left(\frac{T'}{T} + 2\alpha T \right) \xi + \bar{T}^2 + \bar{\bar{T}} \right) \Phi_{\xi\xi} - \Phi_{\xi t} - \alpha T \Phi_{\xi} + \bar{T}' + 2\alpha T \bar{T} = 0.$$

Of course, the equation can be solved explicitly, though the final result is too cumbersome: it is easily shown that

$$\Phi_{\xi} = \left(\bar{Z} \left(\xi e^{\int a dt} - \int b e^{\int a dt} dt \right) + \int c e^{\alpha \int T dt} dt \right) e^{-\alpha \int T dt},$$

where \bar{Z} is an arbitrary function in one variable and

$$a = \frac{T'}{T} + 2\alpha T, \quad b = \bar{T}^2 + \bar{\bar{T}}, \quad c = \bar{T}' + 2\alpha T \bar{T}.$$

Thus,

$$\Phi = Z \left(\xi e^{\int a dt} - \int b e^{\int a dt} dt \right) e^{-2\alpha \int T dt} + \left(\int c e^{\alpha \int T dt} dt \right) \xi e^{-\alpha \int T dt} + \varphi(t),$$

and the corresponding family of solutions is

$$u = Z \left(\xi e^{\int a dt} - \int b e^{\int a dt} dt \right) e^{-2\alpha \int T dt} + \left(\int c e^{\alpha \int T dt} dt \right) \xi e^{-\alpha \int T dt} + \varphi(t) - \left(\frac{1}{6} \frac{T''}{T} + \alpha T' + \frac{2}{3} \alpha^2 T^2 \right) y^3 + \frac{1}{2} (\bar{T}' + 2\alpha T \bar{T}) y^2 - \left(\left(\frac{T'}{T} + 2\alpha T \right) x + \bar{T} \right) y$$

with ξ given by (5.4).

Subcase 01 Since $\beta \neq 0$, we can set $\beta = -1$ and the defining equations become

$$\frac{dx}{2x + (T'_5 - \alpha)y + T_4} = \frac{dy}{y + T_5} = \frac{dt}{0} = \frac{du}{3u + (T'_5 - 2\alpha)x + T_4 y + \frac{1}{2} T''_5 y^2 - T_6}. \tag{5.5}$$

Let us introduce the notation $T = T_5$, $\bar{T} = T_4 - T_5(T'_5 - \alpha)$. Then the integrals of (5.5) are

$$t = \text{const}, \quad \frac{x + \frac{1}{2} \bar{T}}{(y + T)^2} + \frac{T' - \alpha}{y + T} = \text{const}$$

$$\frac{u + (x + \frac{1}{2} \bar{T})(T' - 2\alpha) + \frac{1}{3} \bar{T}}{(y + T)^3} + \frac{(T' - \alpha)^2 + \frac{1}{2} \bar{T}'}{(y + T)^2} + \frac{1}{2} \frac{T''}{y + T} = \text{const},$$

where

$$\bar{T} = -\frac{1}{2} (T' - 2\alpha) \bar{T} - (\bar{T}' + T'(T' - \alpha) + T T'') T + \frac{1}{2} T^2 T'' - T_6.$$

Consequently, the general solution is

$$\Psi \left(\frac{u + (x + \frac{1}{2} \bar{T})(T' - 2\alpha) + \frac{1}{3} \bar{T}}{(y + T)^3} + \frac{(T' - \alpha)^2 + \frac{1}{2} \bar{T}'}{(y + T)^2} + \frac{1}{2} \frac{T''}{y + T}, \frac{x + \frac{1}{2} \bar{T}}{(y + T)^2} + \frac{T' - \alpha}{y + T}, t \right) = 0,$$

or

$$u = (y + T)^3 \Phi(\xi, t) - \frac{1}{2} T'' (y + T)^2 - \left((T' - \alpha)^2 + \frac{1}{2} \bar{T}' \right) (y + T) - (T' - 2\alpha) \left(x + \frac{1}{2} \bar{T} \right) - \frac{1}{3} \bar{T},$$

where

$$\xi = \frac{x + \frac{1}{2} \bar{T}}{(y + T)^2} + \frac{T' - \alpha}{y + T}.$$

Substituting to Eq. (0.4), one obtains

$$(4\xi^2 - 3\Phi) \Phi_{\xi\xi} - \Phi_{\xi t} - 6\xi \Phi_{\xi} + \Phi_{\xi}^2 + 6\Phi = 0.$$

5.2.2. Case 1

The defining equation is now

$$\frac{dx}{(T'_3 - 2\beta)x + (T'_5 - \alpha)y + \frac{1}{2}T''_3y^2 + T_4} = \frac{dy}{(T'_3 - \beta)y + T_5} = \frac{dt}{T_3}$$

$$= \frac{du}{(T'_3 - 3\beta)u + (T'_5 - 2\alpha)x + T'_4y + T''_3xy + \frac{1}{2}T''_5y^2 + \frac{1}{6}T'''_3y^3 - T_6}$$

and since $T_3 \neq 0$ we may set $T' = 1/T_3$. The integrals are

$$T'ye^{\beta T} - \bar{T} = \text{const},$$

$$T'xe^{2\beta T} + \frac{1}{2}T'_3\xi^2 - \left(\left(\frac{\bar{T}}{T'} \right)' - \beta\bar{T}' - \alpha k(\beta) \right) \xi - \bar{\bar{T}} = \text{const},$$

$$T'ue^{3\beta T} - T_{00} - T_{10}\xi - T_{01}\eta - T_{20}\xi^2 - T_{11}\xi\eta - T_{30}\xi^3 = \text{const},$$

where

$$\xi = T'ye^{\beta T} - \bar{T},$$

$$\eta = T'xe^{2\beta T} + \frac{1}{2}T'_3\xi^2 - \left(\left(\frac{\bar{T}}{T'} \right)' - \beta\bar{T}' - \alpha k(\beta) \right) \xi - \bar{\bar{T}},$$

$$\bar{T} = \int T_5(T')^2 e^{\beta T} dt,$$

$$\bar{\bar{T}} = \int \left(T_4(T')^2 e^{2\beta T} + (T'_5 - \alpha)\bar{T}T' e^{\beta T} + \frac{1}{2}T''_3(\bar{T})^2 \right) dt,$$

$$T_{00} = \int \left(T''_3\bar{T}\bar{\bar{T}} + \frac{T'''_3\bar{T}^3}{6T'} + \left(T'_4\bar{T}T' + (T'_5 - 2\alpha)T' + \frac{1}{2T''_5(\bar{T})^2} \right) e^{\beta T} - T_6(T')^2 e^{3\beta T} \right) dt,$$

$$T_{10} = \int \left(\frac{T'''_3(\bar{T})^2}{2T'} + T''_3 \left(\bar{\bar{T}} + \bar{T} \left(\left(\frac{\bar{T}}{T'} \right)' - \alpha k(\beta) - \beta\bar{T} \right) \right) \right. \\ \left. + \left(T''_5\bar{T} + (T'_5 - 2\alpha) \left(\left(\frac{\bar{T}}{T'} \right)' - \alpha k(\beta) - \beta\bar{T} \right) \right) e^{\beta T} + T'_4T' e^{2\beta T} \right) dt,$$

$$T_{01} = \int \left(T'''_3\bar{T} + (T'_5 - 2\alpha)T' e^{\beta T} \right) dt,$$

$$T_{20} = \int \left(T''_3 \left(\frac{\bar{T}}{2T'} + \left(\frac{\bar{T}}{T'} \right)' - \alpha k(\beta) - \beta\bar{T} - \frac{1}{2}T'_3\bar{T} \right) + \frac{1}{2} (T''_5 - (T'_5 - 2\alpha)T') e^{\beta T} \right) dt,$$

$$T_{11} = \int T''_3 dt = T'_3,$$

$$T_{30} = \int \left(\frac{T'''_3}{6T'} - \frac{1}{2}T''_3T'_3 \right) dt = \frac{1}{6}T'''_3T_3 - \frac{1}{3}(T'_3)^2,$$

and

$$k(\beta) = \int T' e^{\beta T} dt = \begin{cases} \frac{e^{\beta T}}{\beta}, & \beta \neq 0, \\ T, & \beta = 0. \end{cases}$$

Thus, the general solution is

$$u = \left(\frac{\Phi(\xi, \eta) + T_{00} + T_{10}\xi + T_{01}\eta + T_{20}\xi^2 + T_{11}\xi\eta + T_{30}\xi^3}{T'} \right) e^{3\beta T}. \quad (5.6)$$

Substituting (5.6) to Eq. (0.4), one obtains

$$\Phi_{\xi\xi} = (\beta\xi - \Phi_\eta) \Phi_{\xi\eta} + (2\beta\eta + \alpha\kappa + \Phi_\xi) \Phi_{\eta\eta} - \beta\Phi_\kappa - 2\alpha\kappa,$$

where

$$\kappa = e^{\beta T} - \beta k(\beta),$$

i.e.,

$$\kappa = \begin{cases} 0, & \beta \neq 0, \\ \xi, & \beta = 0. \end{cases}$$

Thus, the reductions are

$$\Phi_{\xi\xi} = (\beta\xi - \Phi_\eta) \Phi_{\xi\eta} + (2\beta\eta + \Phi_\xi) \Phi_{\eta\eta} - \beta\Phi_\eta$$

for $\beta \neq 0$ and

$$\Phi_{\xi\xi} = (\alpha\xi + \Phi_\xi) \Phi_{\eta\eta} - \Phi_\eta \Phi_{\xi\eta} - 2\alpha \quad (5.7)$$

for $\beta = 0$. Note that in the case $\alpha \neq 0$ Eq. (5.7) transforms to the Gibbons-Tsarev equation (see [9])

$$\Phi_{\xi\xi} = \Phi_\xi \Phi_{\eta\eta} - \Phi_\eta \Phi_{\xi\eta} - \alpha$$

by $\Phi \mapsto \Phi - \alpha\xi^2/2$.

6. Summary of results

Below, a concise exposition of the obtained results is given^b in Table 6 on p. 28.

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^bWe use the notation $\infty^{k-\tau}$ to indicate the infinite-dimensional component corresponding to k arbitrary functions in τ and the abbreviation ‘LLT’ means ‘Linearizes by the Legendre transformation’.

Eqn	dim(sym \mathcal{L})	Reductions	Comments
(0.1)	$1 + \infty^{2 \cdot x} + \infty^{2 \cdot z}$	$X\Phi_{xz} - X'\Phi_z = 0$ $2\Phi = \Phi\Phi_{xz} - \Phi_x\Phi_z$ $\Phi_{\xi\xi} = X'\Phi_\xi - X\Phi_{x\xi}$ $\Phi_{\xi\xi} = \Phi_x\Phi_\xi - \Phi\Phi_{x\xi}$ $(1 + Z\Phi_z)\Phi_{\xi\xi} = Z\Phi_\xi\Phi_{\xi z} + Z'\Phi_\xi^2$ $(1 + \xi\Phi_z)\Phi_{\xi\xi} - \xi\Phi_\xi\Phi_{\xi z} + \Phi_\xi\Phi_z = 0$ $\Phi_{\xi\xi} = \Phi_\xi\Phi_{\eta\eta} - \Phi_\eta\Phi_{\xi\eta}$ $\Phi_\eta\Phi_{\xi\eta} - \Phi_\xi\Phi_{\eta\eta} = e^\eta\Phi_{\xi\xi}$	Solves explicitly Transforms to the Liouville eq. Solves explicitly Solves explicitly LLT
(0.2)	$1 + \infty^{1 \cdot y} + \infty^{3 \cdot t}$	$\Phi_{yt} = T\Phi_y$ $\Phi_{yt} = 2\Phi\Phi_y$ $(\alpha\xi + \Phi_\xi)\Phi_{\xi y} - \Phi_y(\Phi_{\xi\xi} + 2\alpha) = 0$ $T\Phi_{xx} = T'$ $T\Phi_{xx} = T'$ $\left(\frac{T'}{T}\xi + \bar{T}\right)\Phi_{\xi\xi} + \Phi_{\xi t} = 0$ $\Phi_{\xi t} = 4\Phi\Phi_\xi - \xi\Phi_\xi^2 + 2\xi\Phi\Phi_{\xi\xi}$ $\Phi_{\eta\eta} + (\alpha\xi + \Phi_\eta)\Phi_{\xi\eta} = \Phi_\eta(2\alpha + \Phi_{\xi\xi})$	Solves explicitly Reduces to the Riccati eq. Solves explicitly Solves explicitly for $\alpha = 0$ Solves explicitly Solves explicitly LLT for $\alpha = 0$
(0.3)	$\infty^{2 \cdot x} + \infty^{1 \cdot y} + \infty^{1 \cdot t}$	$\Phi_{yt} = 0$ $\bar{X}\Phi_{xt} - \bar{X}'\Phi_t = 0$ $\bar{X}\Phi_{xy} - \bar{X}'\Phi_y = 0$ $\Phi_{\xi\xi} = \bar{X}\Phi_{x\xi} - \bar{X}'\Phi_\xi$ $(1 + \Phi_\xi)\Phi_{y\xi} = \Phi_y\Phi_{\xi\xi}$ $(1 + \Phi_\xi)\Phi_{t\xi} = \Phi_t\Phi_{\xi\xi}$ $\Phi_\eta\Phi_{\xi\xi} + (\Phi_\eta - \Phi_\xi - 1)\Phi_{\xi\eta} - \Phi_\xi\Phi_{\eta\eta} = 0$	Solves explicitly Solves explicitly Solves explicitly Solves explicitly Solves explicitly Solves explicitly LLT
(0.4)	$2 + \infty^{4 \cdot t}$	$\Phi_{yy} = (T' - T^2)y + T\bar{T} - \bar{T}'$ $\Phi_{yy} = \frac{2\Phi_y - T'}{y + T} + T'' + \frac{\bar{T}^2}{(y + T)^3}$ $\left(\left(\frac{T'}{T} + 2\alpha T\right)\xi + \bar{T}^2 + \bar{T}\right)\Phi_{\xi\xi}$ $-\Phi_{\xi t} - \alpha T\Phi_\xi + \bar{T}' + 2\alpha T\bar{T} = 0$ $(4\xi^2 - 3\Phi)\Phi_{\xi\xi} - \Phi_{\xi t} - 6\xi\Phi_\xi + \Phi_\xi^2 + 6\Phi = 0$ $\Phi_{\xi\xi} = (\beta\xi - \Phi_\eta)\Phi_{\xi\eta} + (2\beta\eta + \Phi_\xi)\Phi_{\eta\eta} - \beta\Phi_\eta$ $\Phi_{\xi\xi} = (\alpha\xi + \Phi_\xi)\Phi_{\eta\eta} - \Phi_\eta\Phi_{\xi\eta} - 2\alpha$	Solves explicitly Solves explicitly Solves explicitly LLT for $\beta = 0$ Reduces to the Gibbons-Tsarev eq. for $\alpha \neq 0$ LLT for $\alpha = 0$

Table 6. Summary of reductions

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