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In this paper we present a dynamical study of the exact nonlinear Pollard wave solution to the geophysical water-wave problem in the f-plane approximation. We deduce an exact dispersion relation and we discuss some properties of this solution.

1. Introduction

The early exact deep water-wave solution by Gerstner [13], rediscovered by Rankine [26], (for a modern exposition see the papers [1] and [14]), was further modified to described edge waves propagating along a sloping beach - in an implicit form by Yih [29] (see also Mollo-Christensen [24]) and in an explicit form by Constantin [2] - and free surface waves in a rotating fluid by Pollard [25]. Some steps further to describe edge waves in rotating fluids by modifying Gerstner's exact solution, were made by Mollo-Christensen [23] and Weber [28]. By considering the motion of a fluid in a reference frame rotating with uniform angular velocity, Coriolis effects play an important role. In geophysical fluid dynamics, the rotating reference frame encountered is the Earth. In regions near the Equator, the Coriolis effects are taken only partially into account, the f-plane approximation and the β-plane approximation are commonly used (see [6], [9], and [27], [11], [12], [16], respectively). Constantin [5], [7] found exact Gerstner-like 3-dimensional solutions to the nonlinear governing equations for geophysical water waves in the β-plane approximation near the Equator. Geophysical edge wave solutions along a sloping beach with the shoreline parallel to the Equator were provided in the f-plane approximation by Matioc [22] and by Ionescu-Kruse [18], and in the β-plane approximation by Ionescu-Kruse [19].

In this paper, following Pollard [25], we present the exact nonlinear 3-dimensional solution (2.6) to the geophysical water-wave problem in the f-plane approximation. We deduce the dispersion relation (2.10) for the propagation speed. The fluid particles move on circles which lie in planes slightly tilted from the vertical, the surface wave profile is a smooth trochoid in this plane. The flow (2.6) is rotational, we will see that its vorticity (3.12) is non-zero.

An advantage of solutions in the Lagrangian framework, beside revealing the qualitative properties of the physical fluid motion, is the possibility of a stability/instability analysis by the short-wavelength instability method. This method was applied by Leblanc [20] for Gerstner's waves, by Constantin and Germain [8] for the equatorially trapped waves [5], by Ionescu-Kruse [17] for the edge waves [1]. We mention the potential applicability of the short-wavelength instability method to the Pollard wave solution.

2. Pollard's equatorial waves

For oceanic motion within 2° latitude from the Equator, it is adequate to use the f-plane approximation in the governing equations (see the discussion in [6] and more recently in [9]). The ocean is considered to be an incompressible inviscid fluid of constant density ρ_0 rotating counterclockwise around a vertical axis with a constant angular velocity $\Omega := \frac{f}{2}$, f being the Coriolis parameter. We chose a Cartesian coordinate system with the x- and y-axes along the undisturbed ocean surface and z-axis pointing vertically upwards. The governing equations of fluid motion are the following ([12])

$$
\begin{cases}
u_t + uu_x + wu_y + vu_z - fv = -\frac{p_x}{\rho_0}, \\[2mm]
w_t + uw_x + ww_y + vw_z + fu = -\frac{p_y}{\rho_0}, \\[2mm]
v_t + uv_x + wv_y + vv_z \quad\quad = -\frac{p_z}{\rho_0} - g,
\end{cases}
\tag{2.1}
$$

$$
u_x + w_y + v_z = 0.
\tag{2.2}
$$

Here (u,w,v) is the velocity field of the fluid, p is the pressure, t represents time and $g = 9,8m/s^2$ is the constant gravitational acceleration at the Earth's surface.

The ocean is assumed infinitely deep and of unlimited horizontal extent. The appropriate boundary conditions are the dynamic boundary condition ([4])

$$
p = p_{atm} \quad\quad \text{on the free surface} \quad z = \eta(t,x,y),
\tag{2.3}
$$

p_{atm} being the constant atmospheric pressure, and the kinematic boundary condition

$$
w = \eta_t + u\eta_x + v\eta_y \quad\quad \text{on the free surface} \quad z = \eta(t,x,y).
\tag{2.4}
$$

We also assume that at great depth there is practically no motion, that is,

$$
(u,v,w) \to (0,0,0) \quad \text{as} \quad z \to -\infty.
\tag{2.5}
$$

The 3-dimensional flow described at any time t by the following coordinates of the fluid particles

$$
\begin{cases}
x = q - \frac{akc^2}{g}e^{\frac{k^2c^2}{g}s}\sin[k(q-ct)], \\[3mm]
y = r + f\frac{ac}{g}e^{\frac{k^2c^2}{g}s}\cos[k(q-ct)], \\[3mm]
z = s + ae^{\frac{k^2c^2}{g}s}\cos[k(q-ct)],
\end{cases}
\tag{2.6}
$$

yields the Pollard wave solution in the f-plane approximation. Associated with the flow (2.6) *there is a unique pressure satisfying the geophysical water-wave problem* (2.1)-(2.5) *given by*

$$
p = p(s) = \rho_0 \frac{a^2k^2c^2}{2}\left[e^{2\frac{k^2c^2}{g}s} - e^{2\frac{k^2c^2}{g}s_0}\right] - \rho_0 g(s - s_0) + p_{atm}.
\tag{2.7}
$$

In (2.6) and (2.7), (q,r,s) are the Lagrangian labels, $k > 0$ is a fixed wave number, a is a parameter and c is the wave speed. The label domain is given by real values of q, r and s, with

$$
s \le s_0, \quad s_0 < 0 \text{ being fixed.}
\tag{2.8}
$$

The expression of the wave speed c and a condition on the parameter a are determined so that (2.6) satisfies the equations (2.1)-(2.4). We will get the following *upper bound for the parameter a*

$$a^2 \leq \frac{g^2}{k^4 c^4} \tag{2.9}$$

and the following *dispersion relation for c*

$$c^2 = \frac{f^2 + \sqrt{f^4 + 4g^2k^2}}{2k^2}. \tag{2.10}$$

Let us see that (2.6) is a solution to the problem (2.1)-(2.5). The incompressibility equation (2.2) requires that the determinant of the Jacobian matrix of (2.6) be independent of time. The Jacobian matrix of (2.6) is given by

$$\begin{pmatrix} \frac{\partial x}{\partial q} & \frac{\partial y}{\partial q} & \frac{\partial z}{\partial q} \\[2mm] \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\[2mm] \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \end{pmatrix} = \begin{pmatrix} 1 - \frac{ak^2c^2}{g}e^{\frac{k^2c^2}{g}s}\cos\theta & -f\frac{akc}{g}e^{\frac{k^2c^2}{g}s}\sin\theta & -ake^{\frac{k^2c^2}{g}s}\sin\theta \\[3mm] 0 & 1 & 0 \\[3mm] -\frac{ak^3c^4}{g^2}e^{\frac{k^2c^2}{g}s}\sin\theta & f\frac{ak^2c^3}{g^2}e^{\frac{k^2c^2}{g}s}\cos\theta & 1 + \frac{ak^2c^2}{g}e^{\frac{k^2c^2}{g}s}\cos\theta \end{pmatrix}, \tag{2.11}$$

where

$$\theta := k(q - ct). \tag{2.12}$$

Thus, the determinant D of the Jacobian matrix (2.11),

$$\Delta = 1 - \frac{a^2 k^4 c^4}{g^2} e^{2\frac{k^2 c^2}{g}s}, \tag{2.13}$$

is time independent. We have to impose the condition

$$\Delta \neq 0, \tag{2.14}$$

so that the transformation (2.6) is a local change of coordinates. Taking into account (2.8), the condition (2.14) becomes

$$\frac{a^2 k^4 c^4}{g^2} e^{2\frac{k^2 c^2}{g}s_0} < 1, \tag{2.15}$$

and thus we get (2.9).

The Euler equations (2.1) are written in Lagrangian form as follows

$$\begin{cases} \frac{Du}{Dt} - fv = -\frac{p_x}{\rho_0}, \\[3mm] \frac{Dv}{Dt} + fu = -\frac{p_y}{\rho_0}, \\[3mm] \frac{Dw}{Dt} = -\frac{p_z}{\rho_0} - g, \end{cases} \tag{2.16}$$

where $\frac{D}{Dt}$ denotes the material derivative in time. From (2.6), the particle velocity is given by

$$u = \frac{ak^2c^3}{g}e^{\frac{k^2c^2}{g}s}\cos[k(q-ct)],$$

$$v = f\frac{akc^2}{g}e^{\frac{k^2c^2}{g}s}\sin[k(q-ct)], \qquad (2.17)$$

$$w = akce^{\frac{k^2c^2}{g}s}\sin[k(q-ct)],$$

and the particle acceleration by

$$\frac{Du}{Dt} = \frac{ak^3c^4}{g}e^{\frac{k^2c^2}{g}s}\sin[k(q-ct)],$$

$$\frac{Dv}{Dt} = -f\frac{ak^2c^3}{g}e^{\frac{k^2c^2}{g}s}\cos[k(q-ct)], \qquad (2.18)$$

$$\frac{Dw}{Dt} = -ak^2c^2e^{\frac{k^2c^2}{g}s}\cos[k(q-ct)].$$

Substituting (2.18) into (2.16) yields the system

$$\begin{cases} p_x = -\rho_0\left[\frac{ak^3c^4}{g}e^{\frac{k^2c^2}{g}s}\sin[k(q-ct)] - fv\right] \overset{(2.17)}{=} -\rho_0\left[k^2c^2 - f^2\right]\frac{akc^2}{g}e^{\frac{k^2c^2}{g}s}\sin[k(q-ct)], \\[2mm] p_y = -\rho_0\left[-f\frac{ak^2c^3}{g}e^{\frac{k^2c^2}{g}s}\cos[k(q-ct)] + fu\right] \overset{(2.17)}{=} 0, \\[2mm] p_z = -\rho_0\left[-ak^2c^2e^{\frac{k^2c^2}{g}s}\cos[k(q-ct)] + g\right]. \end{cases} \qquad (2.19)$$

Therefore, from

$$\begin{pmatrix} p_q \\ p_r \\ p_s \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial q} & \frac{\partial y}{\partial q} & \frac{\partial z}{\partial q} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \end{pmatrix}\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}, \qquad (2.20)$$

we get in terms of the Lagrangian labels the following system

$$\begin{cases} p_q = -\rho_0(k^2c^4 - f^2c^2 - g^2)\frac{ak}{g}e^{\frac{k^2c^2}{g}s}\sin[k(q-ct)]\left[1 - \frac{ak^2c^2}{g}e^{\frac{k^2c^2}{g}s}\cos[k(q-ct)]\right], \\[2mm] p_r = 0, \\[2mm] p_s = -\rho_0\left[-(k^2c^4 - f^2c^2 - g^2)\frac{a^2k^4c^4}{g^3}e^{2\frac{k^2c^2}{g}s}\sin^2[k(q-ct)] - \frac{a^2k^4c^4}{g}e^{2\frac{k^2c^2}{g}s} + g\right]. \end{cases} \qquad (2.21)$$

From the symmetry of the second derivatives with respect to the Lagrangian labels, that is, $p_{qs} = p_{sq}$, it follows that

$$k^2c^4 - f^2c^2 - g^2 = 0. \qquad (2.22)$$

The equation (2.22) yields the dispersion relation (2.10). We also conclude that the pressure does not depend on the Lagrangian labels r and q. We find the following pressure function for which (2.21) holds:

$$p = p(s) = \rho_0 \frac{a^2 k^2 c^2}{2} e^{2\frac{k^2 c^2}{g}s} - \rho_0 gs + const. \tag{2.23}$$

The free surface is obtained by setting $s = s_0$ in (2.6). The kinematic boundary condition (2.4), which states that all surface particles remain confined to the surface, simply holds. By requiring (2.3), from (2.23) we obtain the explicit determination of the pressure (2.7) at any particle during the flow (2.6).$\square$

3. Discussion

Let us emphasize some properties of the nonlinear 3-dimensional wave (2.6). The particle orbits while still circular lie in planes slightly tilted from the vertical. Indeed, with (2.22) in view, each particle identified by the Lagrangian labels (q, r, s), follows the circle

$$\begin{cases} (x-q)^2 + (y-r)^2 + (z-s)^2 = \frac{a^2 k^2 c^4}{g^2} e^{2\frac{k^2 c^2}{g}s}, \\[2mm] ay - f\frac{ac}{g}z - ar + f\frac{ac}{g}s = 0, \end{cases} \tag{3.1}$$

centered at (q,r,s), whose radius is $\frac{akc^2}{g}e^{\frac{k^2 c^2}{g}s}$. The motion is identical in all planes parallel to the plane

$$ay - f\frac{ac}{g}z - ar + f\frac{ac}{g}s = 0. \tag{3.2}$$

This plane, having the normal $\mathbf{n}_{pl} = (0, a, -f\frac{ac}{g})$, makes the angle

$$\alpha = \arctan\left(f\frac{c}{g}\right) \tag{3.3}$$

with the vertical axis $\mathbf{n} = (0, 1, 0)$. The motion of another particle is obtained by changing the values of the Lagrangian labels q, r, s, such that q, r, s are in a plane which normal makes the angle α (3.3) with the vertical axis $\mathbf{n} = (0, 1, 0)$, that is, they satisfy

$$-ar + f\frac{ac}{g}s + \mathscr{C} = 0, \tag{3.4}$$

$\mathscr{C}$ being a real constant. By setting $s = s_0$ in (2.6), we get the following surface wave profile

$$\begin{cases} x = q - \frac{akc^2}{g}e^{\frac{k^2 c^2}{g}s_0}\sin[k(q-ct)], \\[3mm] y = \frac{\mathscr{C}}{a} + f\frac{c}{g}s_0 + f\frac{ac}{g}e^{\frac{k^2 c^2}{g}s_0}\cos[k(q-ct)], \\[3mm] z = s_0 + ae^{\frac{k^2 c^2}{g}s_0}\cos[k(q-ct)], \end{cases} \tag{3.5}$$

$q \in \mathbf{R}$ is the parameter of the curve. With $s_0 < 0$, the above formula represents the parametrization of a smooth trochoid in the plane tilted at the angle α to the vertical.

Concerning the dispersion relation (2.10), dispersion relation which is different from the ones obtained in [5], [7], [16], we observe that if the wave travels along the the Equator from west to east, then

$$c = c_+ = \frac{\sqrt{f^2 + \sqrt{f^4 + 4g^2k^2}}}{\sqrt{2}k},$$

(3.6)

if it travels along the Equator from east to west, then

$$c = c_- = -\frac{\sqrt{f^2 + \sqrt{f^4 + 4g^2k^2}}}{\sqrt{2}k}.$$

(3.7)

Thus, the direction of propagation has no influence on the speed of the wave as $c_+ = -c_-$. The dispersion relation (2.10) may be written as

$$c^2 = \frac{g}{k}\left[\frac{1}{2}\frac{f^2}{gk} + \sqrt{1 + \left(\frac{1}{2}\frac{f^2}{gk}\right)^2}\right].$$

(3.8)

If the Coriolis force is ignored, that is, $f = 0$, the dispersion relation reduced to the dispersion relation for deep-water waves

$$c^2 = \frac{g}{k}.$$

(3.9)

As pointed out by Pollard [25], for surface waves on the Earth, $\sqrt{gk} \approx 1$ and $f \approx 10^{-4}$ sec^{-1}, which make $\frac{f}{\sqrt{gk}} \approx 10^{-4}$, and thus, Earth's rotation modifies the dispersion relation for deep-water waves by $O(10^{-8})$.

The waves (2.6) are rotational, they can be generated by irrotational forces in a rotating system. The inverse of the Jacobian matrix (2.11) has the expression

$$\begin{pmatrix} \frac{\partial q}{\partial x} & \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial q}{\partial y} & \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \\ \frac{\partial q}{\partial z} & \frac{\partial r}{\partial z} & \frac{\partial s}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{1 + \frac{ak^2c^2}{g}e^{\frac{k^2c^2}{g}s}\cos\theta}{\Delta} & f\frac{akc}{g\Delta}e^{\frac{k^2c^2}{g}s}\sin\theta & \frac{ak}{\Delta}e^{\frac{k^2c^2}{g}s}\sin\theta \\ 0 & 1 & 0 \\ \frac{ak^3c^4}{g^2\Delta}e^{\frac{k^2c^2}{g}s}\sin\theta & -f\frac{ak^2c^3}{g^2\Delta}e^{\frac{k^2c^2}{g}s}\cos\theta + \frac{a^2k^4c^5}{g^3\Delta}e^{2\frac{k^2c^2}{g}s} & \frac{1 - \frac{ak^2c^2}{g}e^{\frac{k^2c^2}{g}s}\cos\theta}{\Delta} \end{pmatrix}$$

(3.10)

From (2.17) and (3.10), by straightforward calculation, we can express the vorticity

$$\gamma = (\gamma_1, \gamma_2, \gamma_3) = (w_y - v_z, u_z - w_x, v_x - u_y)$$

(3.11)

of the fluid flow in the Lagrangian labels as

$$\gamma_1 = -f\frac{ak^3c^4}{g^2\Delta}e^{\frac{k^2c^2}{g}s}\sin\theta,$$

$$\gamma_2 = \frac{ak^2c}{\Delta}\left(\frac{k^2c^4}{g^2}-1\right)e^{\frac{k^2c^2}{g}s}\cos\theta - \frac{a^2k^4c^3}{g\Delta}\left(\frac{k^2c^4}{g^2}+1\right)e^{\frac{2k^2c^2}{g}s}$$
$$\overset{(2.22)}{=} f^2\frac{ak^2c^3}{g^2\Delta}e^{\frac{k^2c^2}{g}s}\cos\theta - f^2\frac{a^2k^4c^2}{g^3\Delta}e^{\frac{2k^2c^2}{g}s} - \frac{2a^2k^4c^3}{g\Delta}e^{\frac{2k^2c^2}{g}s}, \tag{3.12}$$

$$\gamma_3 = f\frac{ak^2c^2}{g\Delta}e^{\frac{k^2c^2}{g}s}\left(\cos\theta + \frac{ak^2c^2}{g}e^{\frac{k^2c^2}{g}s}\right).$$

In the non-geophysical case $f=0$, the first and the third component of the vorticity are identically zero and the second component is different from zero; if a takes its upper value (2.9), then, this second component becomes the vorticity of a Gerstner wave.

We point out that the fact that all particle paths are closed is a peculiar feature of this rotational 3-dimensional flow – in irrotational 2-dimensional flows with or without underlying currents, there are no closed particle paths, cf. the discussion in the papers [3], [10], [15].

Acknowledgments

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