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A Vector Fokas-Lenells System from the Coupled Nonlinear Schrödinger Equations

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With the aid of the spectral gradient method of Fuchssteiner, the compatible pair of Hamiltonian operators for the coupled NLS hierarchy is rediscovered. This result enables us to construct a hierarchy, which contains a vector generalization of Fokas-Lenells system. The vector Fokas-Lenells system is shown to be bi-Hamiltonian and to possess a Lax pair.

Keywords: Hamiltonian operators; Lax pair; coupled nonlinear Schrödinger equation; derivative nonlinear Schrödinger equation.

2000 Mathematics Subject Classification: 37K10, 35Q55

1. Introduction

The nonlinear Schrödinger (NLS) equation is one of the most important equations in both mathematics and physics. On the one hand, it has numerous applications in physics such as water waves, optics and plasma physics. On the other hand, the NLS equation has been playing a key role in the development of the soliton theory or the theory of modern integrable systems (see Ref. [1] and the references therein).

The NLS equation has been generalized in various different ways. One particular extension, constructed by Manakov in the seventies of the last century in [20], is the coupled NLS or Manakov model. This system reads as

$$\begin{aligned}q_{1t} &= i(q_{1xx} + \kappa(|q_1|^2 + |q_2|^2)q_1), \\q_{2t} &= i(q_{2xx} + \kappa(|q_1|^2 + |q_2|^2)q_2),\end{aligned}\tag{1.1}$$

where κ is a real parameter. Like the NLS equation itself, the coupled NLS system also is physically relevant and has been studied extensively (see Refs. [2] and [27] for examples).

A more recent generalization of the NLS equation is the following one

$$iu_t - \nu u_{tx} + \gamma u_{xx} + \sigma |u|^2(u + i\nu u_x) = 0, \quad \sigma = \pm 1,$$

which depends on two real parameters γ and ν . Above equation was discovered in 1995 by Fokas [6] and was rederived by Lenells as a model for nonlinear pulse propagation in certain fibres [15]. This equation, referred as Fokas-Lenells (FL) equation, was studied by Fokas and Lenells [16, 17] and others (see Refs. [12, 21, 22, 26, 28] for examples). It is remarked that the bi-Hamiltonian method used by Fokas originated from a fruitful idea of Fokas and Fuchssteiner [8] and has been applied successfully to a number of integrable equations (see Refs. [11, 24]).

The aim of the present paper is to propose a generalization of the coupled NLS equation or Manakov system. We will start with a more general spectral problem or the coupled AKNS spectral problem, which is of the constrained KP type [4, 5, 13, 14, 25]. The bi-Hamiltonian structure for the hierarchy associated with the coupled AKNS spectral problem is known [3, 9, 19], but for our purpose of constructing integrable systems we will rederive them by means of the spectral gradient method of Fuchssteiner [10]. This will be done in the next section. In section 3, we will take up the bi-Hamiltonian method to produce an integrable hierarchy, which includes a generalized Manakov system or vector FL system as a particular flow. In section 4, we will work out a Lax representation for the vector FL system.

2. Spectral Gradient Method and Coupled AKNS Spectral Problem

We consider the following spectral problem

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_x = \begin{pmatrix} 2\lambda & q_1 & q_2 \\ r_1 & -\lambda & 0 \\ r_2 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad (2.1)$$

where λ is the spectral parameter, $q_i, r_i (i = 1, 2)$ are field variables depending on x and t . This spectral problem (2.1) is the matrix version of

$$(\partial - q_1 \partial^{-1} r_1 - q_2 \partial^{-1} r_2) \varphi = 3\lambda \varphi, \quad (2.2)$$

a spectral problem of the constrained KP type [19].

In fact, we introduce the following new variables by

$$\varphi = \phi_1, \quad \partial^{-1} r_1 \varphi = \phi_2, \quad \partial^{-1} r_2 \varphi = \phi_3,$$

then the Eq. (2.2) becomes

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}_x = \begin{pmatrix} 3\lambda & q_1 & q_2 \\ r_1 & 0 & 0 \\ r_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}. \quad (2.3)$$

Through a simple gauge transformation, namely

$$\varphi_1 = e^{\lambda x} \phi_1, \quad \varphi_2 = e^{\lambda x} \phi_2, \quad \varphi_3 = e^{\lambda x} \phi_3,$$

then we arrive at the Eq. (2.1).

The integrable systems associated with (2.1) and their bi-Hamiltonian structure are studied [19, 25], which include the celebrated coupled NLS equation or Manakov system [20] as a particular flow. For our purpose, we adopt Fuchssteiner’s spectral gradient method [10] and rederive the recursion operator and bi-Hamiltonian structure. To this end, we need the adjoint problem of Eq. (2.1), which reads as

$$(\psi_1, \psi_2, \psi_3)_x = -(\psi_1, \psi_2, \psi_3) \begin{pmatrix} 2\lambda & q_1 & q_2 \\ r_1 & -\lambda & 0 \\ r_2 & 0 & -\lambda \end{pmatrix}. \tag{2.4}$$

The directional derivative of $(\phi_1, \phi_2, \phi_3)^T$ determined by Eq. (2.1) in the direction $q_1 + \varepsilon Q$ is given by

$$\begin{aligned} \begin{pmatrix} \phi'_1[Q] \\ \phi'_2[Q] \\ \phi'_3[Q] \end{pmatrix}_x &= \begin{pmatrix} 2\lambda & q_1 & q_2 \\ r_1 & -\lambda & 0 \\ r_2 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} \phi'_1[Q] \\ \phi'_2[Q] \\ \phi'_3[Q] \end{pmatrix} \\ &+ \begin{pmatrix} 2 \langle Q, \lambda_{q_1} \rangle & Q & 0 \\ 0 & - \langle Q, \lambda_{q_1} \rangle & 0 \\ 0 & 0 & - \langle Q, \lambda_{q_1} \rangle \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \end{aligned} \tag{2.5}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between tangent vectors and cotangent vectors, and is defined by

$$\langle \rho, \sigma \rangle = \int_{-\infty}^{+\infty} \rho(x)\sigma(x)dx.$$

Left multiplying Eq. (2.5) by (ψ_1, ψ_2, ψ_3) , then integrating over x from $-\infty$ to $+\infty$, we obtain

$$\begin{aligned} \int_{-\infty}^{+\infty} (\psi_1, \psi_2, \psi_3) \begin{pmatrix} \phi'_1[Q] \\ \phi'_2[Q] \\ \phi'_3[Q] \end{pmatrix}_x dx &= \int_{-\infty}^{+\infty} (\psi_1, \psi_2, \psi_3) \begin{pmatrix} 2\lambda & q_1 & q_2 \\ r_1 & -\lambda & 0 \\ r_2 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} \phi'_1[Q] \\ \phi'_2[Q] \\ \phi'_3[Q] \end{pmatrix} dx \\ &+ \int_{-\infty}^{+\infty} (\psi_1, \psi_2, \psi_3) \begin{pmatrix} 2 \langle Q, \lambda_{q_1} \rangle & Q & 0 \\ 0 & - \langle Q, \lambda_{q_1} \rangle & 0 \\ 0 & 0 & - \langle Q, \lambda_{q_1} \rangle \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} dx. \end{aligned}$$

Assuming that (ψ_1, ψ_2, ψ_3) solves the adjoint problem (2.4) and integrating the left-hand-side of the above equation by parts, we find

$$\int_{-\infty}^{+\infty} (\psi_1, \psi_2, \psi_3) \begin{pmatrix} 2 \langle Q, \lambda_{q_1} \rangle & Q & 0 \\ 0 & - \langle Q, \lambda_{q_1} \rangle & 0 \\ 0 & 0 & - \langle Q, \lambda_{q_1} \rangle \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} dx = 0,$$

or equivalently,

$$\langle Q, \lambda_{q_1} \rangle \int_{-\infty}^{+\infty} (2\psi_1\phi_1 - \psi_2\phi_2 - \psi_3\phi_3)dx + \int_{-\infty}^{+\infty} Q\psi_1\phi_2dx = 0,$$

which holds for arbitrary Q . Thus, we obtain the gradient of λ with respect to q_1 , namely

$$\lambda_{q_1} = -\frac{\psi_1\phi_2}{\int_{-\infty}^{+\infty} (2\psi_1\phi_1 - \psi_2\phi_2 - \psi_3\phi_3)dx}.$$

Similarly, we find the gradients of λ with respect to q_2, r_1, r_2 , and they read as

$$\begin{aligned}\lambda_{q_2} &= -\frac{\psi_1\phi_3}{\int_{-\infty}^{+\infty}(2\psi_1\phi_1 - \psi_2\phi_2 - \psi_3\phi_3)dx}, \\ \lambda_{r_1} &= -\frac{\psi_2\phi_1}{\int_{-\infty}^{+\infty}(2\psi_1\phi_1 - \psi_2\phi_2 - \psi_3\phi_3)dx}, \\ \lambda_{r_2} &= -\frac{\psi_3\phi_1}{\int_{-\infty}^{+\infty}(2\psi_1\phi_1 - \psi_2\phi_2 - \psi_3\phi_3)dx}.\end{aligned}$$

Since we are interested in constructing an eigenvalue problem for the gradients, the constant coefficients may be omitted, namely

$$\begin{pmatrix} \lambda_{q_1} \\ \lambda_{q_2} \\ \lambda_{r_1} \\ \lambda_{r_2} \end{pmatrix} \propto \begin{pmatrix} \psi_1\phi_2 \\ \psi_1\phi_3 \\ \psi_2\phi_1 \\ \psi_3\phi_1 \end{pmatrix}. \tag{2.6}$$

Thus we have

Theorem 2.1. *The gradients of the spectral parameter in the coupled AKNS hierarchy satisfy the equation*

$$\begin{aligned}& \begin{pmatrix} 2q_1\partial^{-1}q_1 & q_1\partial^{-1}q_2 + q_2\partial^{-1}q_1 & R_1 & -q_1\partial^{-1}r_2 \\ q_2\partial^{-1}q_1 + q_1\partial^{-1}q_2 & 2q_2\partial^{-1}q_2 & -q_2\partial^{-1}r_1 & R_2 \\ -(R_1)^* & -r_1\partial^{-1}q_2 & 2r_1\partial^{-1}r_1 & r_1\partial^{-1}r_2 + r_2\partial^{-1}r_1 \\ -r_2\partial^{-1}q_1 & -(R_2)^* & r_2\partial^{-1}r_1 + r_1\partial^{-1}r_2 & 2r_2\partial^{-1}r_2 \end{pmatrix} \begin{pmatrix} \lambda_{q_1} \\ \lambda_{q_2} \\ \lambda_{r_1} \\ \lambda_{r_2} \end{pmatrix} \\ &= 3\lambda \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_{q_1} \\ \lambda_{q_2} \\ \lambda_{r_1} \\ \lambda_{r_2} \end{pmatrix}, \tag{2.7}\end{aligned}$$

where R_1 and R_2 are defined by

$$R_1 = \partial - 2q_1\partial^{-1}r_1 - q_2\partial^{-1}r_2, \quad R_2 = \partial - 2q_2\partial^{-1}r_2 - q_1\partial^{-1}r_1. \tag{2.8}$$

Proof. From Eqs. (2.1) and (2.4), one easily obtains

$$(\psi_1\phi_2)_x = -3\lambda\psi_1\phi_2 + r_1(\psi_1\phi_1 - \psi_2\phi_2) - r_2\psi_3\phi_2, \tag{2.9}$$

$$(\psi_1\phi_3)_x = -3\lambda\psi_1\phi_3 + r_2(\psi_1\phi_1 - \psi_3\phi_3) - r_1\psi_2\phi_3, \tag{2.10}$$

$$(\psi_2\phi_1)_x = 3\lambda\psi_2\phi_1 - q_1(\psi_1\phi_1 - \psi_2\phi_2) + q_2\psi_2\phi_3, \tag{2.11}$$

$$(\psi_3\phi_1)_x = 3\lambda\psi_3\phi_1 - q_2(\psi_1\phi_1 - \psi_3\phi_3) + q_1\psi_3\phi_2. \tag{2.12}$$

Also, we notice that

$$(\psi_1\phi_1 - \psi_2\phi_2)_x = -2r_1\psi_2\phi_1 - r_2\psi_3\phi_1 + 2q_1\psi_1\phi_2 + q_2\psi_1\phi_3, \tag{2.13}$$

$$(\psi_1\phi_1 - \psi_3\phi_3)_x = -2r_2\psi_3\phi_1 - r_1\psi_2\phi_1 + 2q_2\psi_1\phi_3 + q_1\psi_1\phi_2, \tag{2.14}$$

$$(\psi_3\phi_2)_x = r_1\psi_3\phi_1 - q_2\psi_1\phi_2, \tag{2.15}$$

$$(\psi_2\phi_3)_x = r_2\psi_2\phi_1 - q_1\psi_1\phi_3. \tag{2.16}$$

Then substituting Eqs. (2.13)-(2.16) into Eqs. (2.9)-(2.12), we get

$$\begin{pmatrix} (R_1)^* & r_1 \partial^{-1} q_2 & -2r_1 \partial^{-1} r_1 & -r_1 \partial^{-1} r_2 - r_2 \partial^{-1} r_1 \\ r_2 \partial^{-1} q_1 & (R_2)^* & -r_2 \partial^{-1} r_1 - r_1 \partial^{-1} r_2 & -2r_2 \partial^{-1} r_2 \\ -2q_1 \partial^{-1} q_1 & -q_1 \partial^{-1} q_2 - q_2 \partial^{-1} q_1 & -R_1 & q_1 \partial^{-1} r_2 \\ -q_2 \partial^{-1} q_1 - q_1 \partial^{-1} q_2 & -2q_2 \partial^{-1} q_2 & q_2 \partial^{-1} r_1 & -R_2 \end{pmatrix} \begin{pmatrix} \psi_1 \phi_2 \\ \psi_1 \phi_3 \\ \psi_2 \phi_1 \\ \psi_3 \phi_1 \end{pmatrix} \\ = 3\lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_1 \phi_2 \\ \psi_1 \phi_3 \\ \psi_2 \phi_1 \\ \psi_3 \phi_1 \end{pmatrix}, \quad (2.17)$$

which, after left multiplying by the matrix

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

leads to Eq. (2.7), the eigenvalue problem of the spectral gradients. □

The eigenvalue problem (2.7) may be rewritten as

$$B_1 \begin{pmatrix} \lambda_{q_1} \\ \lambda_{q_2} \\ \lambda_{r_1} \\ \lambda_{r_2} \end{pmatrix} = 3\lambda B_0 \begin{pmatrix} \lambda_{q_1} \\ \lambda_{q_2} \\ \lambda_{r_1} \\ \lambda_{r_2} \end{pmatrix},$$

where

$$B_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 2q_1 \partial^{-1} q_1 & q_1 \partial^{-1} q_2 + q_2 \partial^{-1} q_1 & R_1 & -q_1 \partial^{-1} r_2 \\ q_2 \partial^{-1} q_1 + q_1 \partial^{-1} q_2 & 2q_2 \partial^{-1} q_2 & -q_2 \partial^{-1} r_1 & R_2 \\ -(R_1)^* & -r_1 \partial^{-1} q_2 & 2r_1 \partial^{-1} r_1 & r_1 \partial^{-1} r_2 + r_2 \partial^{-1} r_1 \\ -r_2 \partial^{-1} q_1 & -(R_2)^* & r_2 \partial^{-1} r_1 + r_1 \partial^{-1} r_2 & 2r_2 \partial^{-1} r_2 \end{pmatrix}.$$

Therefore, we recover the compatible Hamiltonian operators B_0 and B_1 , and the recursion operator $\mathcal{R} = B_1 B_0^{-1}$ for the coupled AKNS hierarchy [3, 9, 19].

3. Vector Fokas-Lenells Equation

Following the idea of Fokas, Fuchssteiner, Olver and Rosenau, we now recombine the Hamiltonian operators found above section so that new compatible Hamiltonian pair is generated. To this end,

we introduce

$$\beta_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\theta_2 = \begin{pmatrix} R_3 & -q_1\partial^{-1}r_2 & 2q_1\partial^{-1}q_1 & q_1\partial^{-1}q_2 + q_2\partial^{-1}q_1 \\ -q_2\partial^{-1}r_1 & R_4 & q_2\partial^{-1}q_1 + q_1\partial^{-1}q_2 & 2q_2\partial^{-1}q_2 \\ 2r_1\partial^{-1}r_1 & r_1\partial^{-1}r_2 + r_2\partial^{-1}r_1 & -(R_3)^* & -r_1\partial^{-1}q_2 \\ r_2\partial^{-1}r_1 + r_1\partial^{-1}r_2 & 2r_2\partial^{-1}r_2 & -r_2\partial^{-1}q_1 & -(R_4)^* \end{pmatrix},$$

where R_3 and R_4 are defined by

$$R_3 = \gamma\partial - 2q_1\partial^{-1}r_1 - q_2\partial^{-1}r_2, \quad R_4 = \gamma\partial - 2q_2\partial^{-1}r_2 - q_1\partial^{-1}r_1.$$

We further introduce

$$\theta_1 = \beta_0 + i\nu I\partial,$$

where I is a 4×4 identity matrix, and new variables

$$q_k = u_k + i\nu u_{k,x}, \quad r_k = v_k - i\nu v_{k,x}, \quad (k = 1, 2).$$

Then we may construct a hierarchy of equations by

$$\begin{pmatrix} q_1 \\ q_2 \\ r_1 \\ r_2 \end{pmatrix}_{t_n} = (i\theta_2\theta_1^{-1})^n \begin{pmatrix} q_1 \\ q_2 \\ r_1 \\ r_2 \end{pmatrix}_x, \tag{3.1}$$

which in the simplest case reads explicitly as

$$\begin{pmatrix} q_1 \\ q_2 \\ r_1 \\ r_2 \end{pmatrix}_t = i \begin{pmatrix} \gamma u_{1,xx} - 2q_1 u_1 v_1 - q_1 u_2 v_2 - q_2 u_1 v_2 \\ \gamma u_{2,xx} - 2q_2 u_2 v_2 - q_2 u_1 v_1 - q_1 u_2 v_1 \\ -\gamma v_{1,xx} + 2r_1 u_1 v_1 + r_1 u_2 v_2 + r_2 u_2 v_1 \\ -\gamma v_{2,xx} + 2r_2 u_2 v_2 + r_2 u_1 v_1 + r_1 u_1 v_2 \end{pmatrix}. \tag{3.2}$$

If we assume

$$u_1 = -\tilde{u}, \quad v_1 = \frac{1}{2}\tilde{v}, \quad u_2 = v_2 = 0, \quad q_1 = -\tilde{q}, \quad r_1 = \frac{1}{2}\tilde{r},$$

then the Eq. (3.2) reduces to

$$\begin{pmatrix} \tilde{q} \\ \tilde{r} \end{pmatrix}_t = i \begin{pmatrix} \gamma \tilde{u}_{xx} + \tilde{q}\tilde{u}\tilde{v} \\ -\gamma \tilde{v}_{xx} - \tilde{r}\tilde{u}\tilde{v} \end{pmatrix},$$

which is the equation discussed by Lenells and Fokas [17]. Thus, we name the Eq. (3.2) as the vector Fokas-Lenells (vFL) system and the hierarchy (3.1) as the vFL hierarchy.

The vFL system (3.2) is a bi-Hamiltonian system by construction. Indeed, let

$$\beta_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and define the two Hamiltonian operators

$$\begin{aligned} J_2 &= \theta_2 \beta_1, \\ J_1 &= -i\theta_1 \beta_1 \\ &= \begin{pmatrix} 0 & 0 & -i + v\partial & 0 \\ 0 & 0 & 0 & -i + v\partial \\ i + v\partial & 0 & 0 & 0 \\ 0 & i + v\partial & 0 & 0 \end{pmatrix}. \end{aligned} \quad (3.3)$$

Then the vFL system may be reformulated as

$$\begin{pmatrix} q_1 \\ q_2 \\ r_1 \\ r_2 \end{pmatrix}_t = J_2 \begin{pmatrix} \frac{\delta H_1}{\delta q_1} \\ \frac{\delta H_1}{\delta q_2} \\ \frac{\delta H_1}{\delta r_1} \\ \frac{\delta H_1}{\delta r_2} \end{pmatrix} = J_1 \begin{pmatrix} \frac{\delta H_2}{\delta q_1} \\ \frac{\delta H_2}{\delta q_2} \\ \frac{\delta H_2}{\delta r_1} \\ \frac{\delta H_2}{\delta r_2} \end{pmatrix}, \quad (3.4)$$

with the Hamiltonian functionals

$$H_1 = -i \int (u_1 r_{1,x} + u_2 r_{2,x}) dx, \quad (3.5)$$

$$H_2 = \frac{1}{2} \int ((u_1 v_1 + u_2 v_2)(q_1 v_1 + q_2 v_2 + r_1 u_1 + r_2 u_2) - 2\gamma(u_1 v_{1,xx} + u_2 v_{2,xx})) dx. \quad (3.6)$$

By a direct proof, we have proved the Hamiltonian nature of the operators (3.3).

4. A Lax Pair

With the aid of the correspondence between the coupled AKNS spectral problem (2.1) and the eigenvalue equation (2.17), one can construct a spectral problem for the vFL system (3.2). Indeed, we consider the following eigenvalue equation related to the recursion operator [7], i.e.,

$$L^\dagger G_\mu = i\mu G_\mu, \quad (4.1)$$

where \dagger denotes the formal adjoint of an operator, and $L = J_2 J_1^{-1} = i\theta_2 \theta_1^{-1}$. Then from Eq. (4.1), we obtain

$$\theta_2^\dagger G_\mu = \mu \theta_1^\dagger G_\mu, \quad (4.2)$$

where

$$\theta_1^\dagger = \begin{pmatrix} 1 - i\nu\partial & 0 & 0 & 0 \\ 0 & 1 - i\nu\partial & 0 & 0 \\ 0 & 0 & -1 - i\nu\partial & 0 \\ 0 & 0 & 0 & -1 - i\nu\partial \end{pmatrix},$$

$$\theta_2^\dagger = \begin{pmatrix} (R_3)^* & r_1\partial^{-1}q_2 & -2r_1\partial^{-1}r_1 & -r_1\partial^{-1}r_2 - r_2\partial^{-1}r_1 \\ r_2\partial^{-1}q_1 & (R_4)^* & -r_2\partial^{-1}r_1 - r_1\partial^{-1}r_2 & -2r_2\partial^{-1}r_2 \\ -2q_1\partial^{-1}q_1 & -q_1\partial^{-1}q_2 - q_2\partial^{-1}q_1 & -R_3 & q_1\partial^{-1}r_2 \\ -q_2\partial^{-1}q_1 - q_1\partial^{-1}q_2 & -2q_2\partial^{-1}q_2 & q_2\partial^{-1}r_1 & -R_4 \end{pmatrix}.$$

Introducing

$$q_i = \sqrt{\gamma - i\mu\nu}\hat{q}_i, \quad r_i = \sqrt{\gamma - i\mu\nu}\hat{r}_i, \quad (i = 1, 2),$$

then from Eq. (4.2), we have

$$\begin{pmatrix} \hat{R}_1^* & \hat{r}_1\partial^{-1}\hat{q}_2 & -2\hat{r}_1\partial^{-1}\hat{r}_1 & -\hat{r}_1\partial^{-1}\hat{r}_2 - \hat{r}_2\partial^{-1}\hat{r}_1 \\ \hat{r}_2\partial^{-1}\hat{q}_1 & \hat{R}_2^* & -\hat{r}_2\partial^{-1}\hat{r}_1 - \hat{r}_1\partial^{-1}\hat{r}_2 & -2\hat{r}_2\partial^{-1}\hat{r}_2 \\ -2\hat{q}_1\partial^{-1}\hat{q}_1 & -\hat{q}_1\partial^{-1}\hat{q}_2 - \hat{q}_2\partial^{-1}\hat{q}_1 & -\hat{R}_1 & \hat{q}_1\partial^{-1}\hat{r}_2 \\ -\hat{q}_2\partial^{-1}\hat{q}_1 - \hat{q}_1\partial^{-1}\hat{q}_2 & -2\hat{q}_2\partial^{-1}\hat{q}_2 & \hat{q}_2\partial^{-1}\hat{r}_1 & -\hat{R}_2 \end{pmatrix} G_\mu$$

$$= \frac{\mu}{\gamma - i\mu\nu} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} G_\mu, \quad (4.3)$$

where

$$\hat{R}_1 = \partial - 2\hat{q}_1\partial^{-1}\hat{r}_1 - \hat{q}_2\partial^{-1}\hat{r}_2, \quad \hat{R}_2 = \partial - 2\hat{q}_2\partial^{-1}\hat{r}_2 - \hat{q}_1\partial^{-1}\hat{r}_1.$$

We observe that Eq. (4.3) enjoys the same structures as the eigenvalue equation (2.17). This observation implies the following linear spectral problem for the vFL equation

$$\Phi_x = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_x = \begin{pmatrix} 2\hat{\lambda} & \hat{q}_1 & \hat{q}_2 \\ \hat{r}_1 & -\hat{\lambda} & 0 \\ \hat{r}_2 & 0 & -\hat{\lambda} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2i(1-\gamma\xi^2)}{3\nu} & \xi q_1 & \xi q_2 \\ \xi r_1 & -\frac{i(1-\gamma\xi^2)}{3\nu} & 0 \\ \xi r_2 & 0 & -\frac{i(1-\gamma\xi^2)}{3\nu} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = M\Phi, \quad (4.4)$$

where $\xi = \frac{1}{\sqrt{\gamma - i\mu\nu}}$ is the spectral parameter. Next we calculate the t -part of the spectral problem, which is assumed as

$$\Phi_t = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_t = \begin{pmatrix} -E - K & A & B \\ C & E & F \\ G & H & K \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = N\Phi, \quad (4.5)$$

where A, B, C, E, F, G, H, K are functions depending on the field variables.

Requiring that the zero curvature condition $M_t = N_x + [N, M]$ is equivalent to the vFL equation yields the following

$$\begin{aligned} A &= -\frac{u_1}{v}\xi^{-1} + \frac{\gamma q_1}{v}\xi, & B &= -\frac{u_2}{v}\xi^{-1} + \frac{\gamma q_2}{v}\xi, \\ C &= -\frac{v_1}{v}\xi^{-1} + \frac{\gamma r_1}{v}\xi, & G &= -\frac{v_2}{v}\xi^{-1} + \frac{\gamma r_2}{v}\xi, \\ F &= iu_2v_1, & E &= i\left(\frac{1}{3v^2}\xi^{-2} + u_1v_1 - \frac{2\gamma}{3v^2} + \frac{\gamma^2}{3v^2}\xi^2\right), \\ H &= iu_1v_2, & K &= i\left(\frac{1}{3v^2}\xi^{-2} + u_2v_2 - \frac{2\gamma}{3v^2} + \frac{\gamma^2}{3v^2}\xi^2\right). \end{aligned}$$

Thus, we succeeded in finding the spectral problem Eqs. (4.4) and (4.5) for the Eq. (3.2).

It is known that the FL equation is a negative flow of the derivative nonlinear Schrödinger (DNLS) [15]. As a final observation, we comment on the relationship between the vFL equation and two component DNLS equation proposed by Morris and Dodd [23]. In fact, if we take

$$\phi_1 = e^{\frac{2ix}{3v}}\psi_1, \phi_2 = e^{-\frac{ix}{3v}}\psi_2, \phi_3 = e^{\frac{ix}{3v}}\psi_3, \xi = \sqrt{\frac{3v}{\gamma}}\zeta, \quad (4.6)$$

then the Eq. (4.4) becomes

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}_x = \begin{pmatrix} -2i\zeta^2 & \alpha e^{-\frac{ix}{v}}\zeta q_1 & \alpha e^{-\frac{ix}{v}}\zeta q_2 \\ \alpha e^{\frac{ix}{v}}\zeta r_1 & i\zeta^2 & 0 \\ \alpha e^{\frac{ix}{v}}\zeta r_2 & 0 & i\zeta^2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (4.7)$$

where $\alpha = \sqrt{\frac{3v}{\gamma}}$. The same transformation converts the Eq. (4.5) into

$$\Phi_t = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}_t = \begin{pmatrix} -\tilde{E} - \tilde{K} e^{-\frac{ix}{v}}\tilde{A} e^{-\frac{ix}{v}}\tilde{B} \\ e^{\frac{ix}{v}}\tilde{C} & \tilde{E} & \tilde{F} \\ e^{\frac{ix}{v}}\tilde{G} & \tilde{H} & \tilde{K} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix},$$

where

$$\begin{aligned} \tilde{A} &= -\frac{u_1}{\alpha v}\zeta^{-1} + \frac{\gamma q_1 \alpha}{v}\zeta, & \tilde{B} &= -\frac{u_2}{\alpha v}\zeta^{-1} + \frac{\gamma q_2 \alpha}{v}\zeta, \\ \tilde{C} &= -\frac{v_1}{\alpha v}\zeta^{-1} + \frac{\gamma r_1 \alpha}{v}\zeta, & \tilde{G} &= -\frac{v_2}{\alpha v}\zeta^{-1} + \frac{\gamma r_2 \alpha}{v}\zeta, \\ \tilde{F} &= iu_2v_1, & \tilde{E} &= i\left(\frac{1}{3v^2\alpha^2}\zeta^{-2} + u_1v_1 - \frac{2\gamma}{3v^2} + \frac{\gamma}{v}\zeta^2\right), \\ \tilde{H} &= iu_1v_2, & \tilde{K} &= i\left(\frac{1}{3v^2\alpha^2}\zeta^{-2} + u_2v_2 - \frac{2\gamma}{3v^2} + \frac{\gamma}{v}\zeta^2\right). \end{aligned}$$

Now it is easy to see that (4.7) is nothing but the spatial part of the Lax pair for the two component DNLS hierarchy [23] (see also Ref. [18]). Therefore, the vFL system may be regarded as a particular (negative) flow of the two component DNLS hierarchy.

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