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## The Elliptic Sinh-Gordon Equation in the Quarter Plane

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We study the elliptic sinh-Gordon equation formulated in the quarter plane by using the so-called Fokas method, which is a significant extension of the inverse scattering transform for the boundary value problems. The method is based on the simultaneous spectral analysis for both parts of the Lax pair and the global algebraic relation that involves all boundary values. In this paper, we address the existence theorem for the elliptic sinh-Gordon equation posed in the quarter plane under the assumption that the boundary values satisfy the global relation. We also present the formal representation of the solution in terms of the unique solution of the matrix Riemann-Hilbert problem defined by the spectral functions.

*Keywords:* Boundary value problem; Integrable system; Sinh-Gordon equation.

2000 Mathematics Subject Classification: 47K15, 35Q55

### 1. Introduction

We study the boundary problem for the elliptic sinh-Gordon equation posed in the quarter plane

$$q_{xx} + q_{yy} = \sinh q, \quad (x, y) \in \Omega, \quad (1.1)$$

where  $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < \infty, 0 < y < \infty\}$ . It is noted that this equation arises as models of interacting charged particles in plasma physics [16]. On the other hand, of interest is the integrability for the equation; the elliptic sinh-Gordon equation is a reduction of the special case of the Toda lattice equations [1]. As a consequence, the usual inverse scattering transform can be used to solve the elliptic sinh-Gordon equation in the entire plane  $\{-\infty < x, y < \infty\}$  [3, 16]. Regarding more complicated domains, the so-called Fokas method is remarkably elegant for solving boundary value problems. The method is widely used to analyze a large class of partial differential equations and hence, it can be considered as a significant generalization of the inverse scattering transform [2, 4, 5, 7] (see also the monograph [9] and recent applications [10, 15]). Recently, the elliptic sinh-Gordon equation posed in the half plane  $\{-\infty < x < \infty, 0 < y < \infty\}$  was studied by applying the Fokas method [14]. It has been shown that the solution of the equation in the half plane exists provided that the boundary values satisfy the global algebraic relation that is the simple but substantial equation involving all boundary values. In the implementation of the Fokas method, the global relation is crucial in proving the existence of the unique solution and characterizing unknown boundary values called the generalized Dirichlet to Neumann map [8].

In this paper, we implement the Fokas method to analyze the elliptic sinh-Gordon equation posed in the quarter plane (1.1). Based on the spectral analysis in the Lax pair, we derive global relation in terms of the spectral functions. We then formulate the matrix Riemann-Hilbert problem

with the jump matrices defined uniquely by the spectral functions. These spectral functions denoted by  $\{a_1(k), b_1(k)\}$  and  $\{a_2(k), b_2(k)\}$  can be determined from the boundary values  $\{q(x, 0), q_y(x, 0)\}$  and  $\{q(0, y), q_x(0, y)\}$ , respectively. Moreover, we show that the given boundary values with appropriate regularity condition, the solution for (1.1) uniquely exists if the boundary values satisfy the global relation. In addition to the existence of the solution, we address the formal representation for the solution in terms of the unique solution of the Riemann-Hilbert problem (see [14, 17, 18] for analogous results).

The outline of the paper is following. In section 2, we introduce the Lax pair for the elliptic sinh-Gordon equation and the regularity assumption for the boundary values as well as relevant notations and formulas. In section 3, we derive the global relation and we then define spectral functions that determine the jump matrices for the Riemann-Hilbert problem. Moreover, we apply the spectral analysis for the Lax pair at  $x = 0$  and  $y = 0$  in order to characterize the boundary values. In section 4, the existence of the solution is discussed by analyzing the the matrix Riemann-Hilbert problem as an inverse problem. We end with concluding remarks in section 5.

## 2. Preliminaries

It is well known that the elliptic sinh-Gordon equation (1.1) can be written as an overdetermined linear system called a Lax pair [3, 14, 16]

$$\mu_x + \omega_1(k) [\sigma_3, \mu] = Q(x, y, k)\mu, \tag{2.1a}$$

$$\mu_y + \omega_2(k) [\sigma_3, \mu] = i\tilde{Q}(x, y, k)\mu, \tag{2.1b}$$

where  $k \in \mathbb{C}$  is a spectral parameter,  $\mu$  is a  $2 \times 2$  matrix-valued eigenfunction and

$$\omega_1(k) = -\frac{1}{2i} \left( k - \frac{1}{4k} \right), \quad \omega_2(k) = -\frac{1}{2} \left( k + \frac{1}{4k} \right), \tag{2.2}$$

$$Q(x, y, k) = \frac{1}{4} \begin{pmatrix} \frac{i}{2k} (\cosh q - 1) & -\left( r + \frac{\sinh q}{2k} \right) \\ r - \frac{\sinh q}{2k} & -\frac{i}{2k} (\cosh q - 1) \end{pmatrix} \tag{2.3}$$

with

$$\tilde{Q}(x, y, k) = Q(x, y, -k), \tag{2.4}$$

$$r(x, y) = iq_x(x, y) + q_y(x, y), \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.5}$$

and the matrix commutator given by  $[\sigma_3, A] = \sigma_3 A - A \sigma_3$ .

Note that the elliptic sinh-Gordon equation (1.1) is the compatible condition for the Lax pair (2.1);  $\mu_{xy} = \mu_{yx}$  in the Lax pair (2.1) implies that the function  $q(x, y)$  solves (1.1) provided that the spectral parameter  $k$  is independent of  $x$  and  $y$ . From the definitions of  $\omega_1(k)$  and  $\omega_2(k)$ , it follows that

$$\operatorname{Re} \omega_1(k) = -\frac{1}{8} \left( 4 + \frac{1}{|k|^2} \right) \operatorname{Im} k, \quad \operatorname{Re} \omega_2(k) = -\frac{1}{8} \left( 4 + \frac{1}{|k|^2} \right) \operatorname{Re} k.$$

Thus, we find

$$\operatorname{Re} \omega_1(k) < 0 \quad \text{for } \operatorname{Im} k > 0, \quad \operatorname{Re} \omega_2(k) < 0 \quad \text{for } \operatorname{Re} k > 0. \tag{2.6}$$

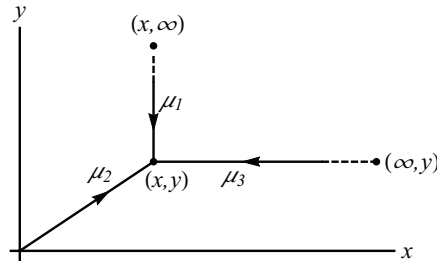


Fig. 1. The three distinct points for the definition of the eigenfunction  $\mu_j$ ,  $j = 1, 2, 3$

Due to the symmetry of  $Q$  and  $\tilde{Q}$ , the eigenfunction possesses the same symmetry, namely,

$$\mu_{11}(x, y, k) = \mu_{22}(x, y, -k), \quad \mu_{21}(x, y, k) = -\mu_{12}(x, y, -k), \quad (2.7)$$

where the subscripts denote the  $(i, j)$ -component of the matrix.

It is convenient to introduce the notation  $\hat{\sigma}_3 A = [\sigma_3, A]$  for the matrix commutator. Using this notation, we let the following notation

$$e^{\hat{\sigma}_3 \xi} A = e^{\sigma_3 \xi} A e^{-\sigma_3 \xi} = \begin{pmatrix} a_{11} & e^{2\xi} a_{12} \\ e^{-2\xi} a_{21} & a_{22} \end{pmatrix}.$$

We also denote boundary values by

$$q(x, 0) = g_0(x), \quad q_y(x, 0) = g_1(x), \quad (2.8a)$$

$$q(0, y) = f_0(y), \quad q_x(0, y) = f_1(y), \quad (2.8b)$$

where we assume that  $g_j, f_j \in H^1(\mathbb{R}^+)$  for  $j = 0, 1$ .

In order to analyze the Lax pair (2.1), we first define a differential 1-form  $W$  given by

$$W(x, y, k) = Q(x, y, k)\mu(x, y, k)dx + i\tilde{Q}(x, y, k)\mu(x, y, k)dy, \quad (2.9)$$

which implies that (2.1) is equivalent to the form

$$d \left[ e^{(\omega_1(k)x + \omega_2(k)y)\hat{\sigma}_3} \mu(x, y, k) \right] = e^{(\omega_1(k)x + \omega_2(k)y)\hat{\sigma}_3} W(x, y, k).$$

Hence, we define eigenfunctions that satisfy both parts of the Lax pair (2.1) as

$$\mu_j(x, y, k) = I + \int_{(x_j, y_j)}^{(x, y)} e^{-(\omega_1(k)(x-\xi) + \omega_2(k)(y-\eta))\hat{\sigma}_3} W_j(\xi, \eta, k), \quad (2.10)$$

where  $(x, y), (x_j, y_j) \in \Omega = \{0 < x < \infty, 0 < y < \infty\}$  and  $W_j$  is the differential form defined by (2.9) with  $\mu_j$ . Since the differential 1-form  $W(x, y, k)$  is closed, the integration in (2.10) does not depend

on paths [6]. In particular, we choose three distinct points  $(x_j, y_j)$  in  $\Omega$ ,  $j = 1, 2, 3$  (see Fig. 1),

$$(x_1, y_1) = (x, \infty), \quad (x_2, y_2) = (0, 0), \quad (x_3, y_3) = (\infty, y).$$

More specifically, the eigenfunctions associated with the points  $(x_j, y_j)$ ,  $j = 1, 2, 3$ , satisfy the following integral equations:

$$\mu_1(x, y, k) = I - i \int_y^\infty e^{-\omega_2(k)(y-\eta)\hat{\sigma}_3} (\tilde{Q}\mu_1)(x, \eta, k) d\eta, \tag{2.11a}$$

$$\begin{aligned} \mu_2(x, y, k) = & I + \int_0^x e^{-\omega_1(k)(x-\xi)\hat{\sigma}_3} (Q\mu_2)(\xi, y, k) d\xi \\ & + i \int_0^y e^{-(\omega_1(k)x + \omega_2(k)(y-\eta))\hat{\sigma}_3} (\tilde{Q}\mu_2)(0, \eta, k) d\eta, \end{aligned} \tag{2.11b}$$

$$\mu_3(x, y, k) = I - \int_x^\infty e^{-\omega_1(k)(x-\xi)\hat{\sigma}_3} (Q\mu_3)(\xi, y, k) d\xi. \tag{2.11c}$$

It should be remarked that the off-diagonal components of the matrix-valued eigenfunctions  $\mu_j$  involve the explicit exponential terms. Thus, according to equations (2.6), let the domains  $D_j$  in the complex  $k$ -plane,  $j = 1, \dots, 4$ , be depicted in Fig. 2 and be defined by

$$\begin{aligned} D_1 &= \{k \in \mathbb{C} : \operatorname{Re} \omega_1(k) < 0\} \cap \{k \in \mathbb{C} : \operatorname{Re} \omega_2(k) < 0\}, \\ D_2 &= \{k \in \mathbb{C} : \operatorname{Re} \omega_1(k) < 0\} \cap \{k \in \mathbb{C} : \operatorname{Re} \omega_2(k) > 0\}, \\ D_3 &= \{k \in \mathbb{C} : \operatorname{Re} \omega_1(k) > 0\} \cap \{k \in \mathbb{C} : \operatorname{Re} \omega_2(k) > 0\}, \\ D_4 &= \{k \in \mathbb{C} : \operatorname{Re} \omega_1(k) > 0\} \cap \{k \in \mathbb{C} : \operatorname{Re} \omega_2(k) < 0\}. \end{aligned}$$

As a result, the domains of analyticity and boundedness for the eigenfunctions can be determined:

- $\mu_1(x, y, k)$  is analytic and bounded for  $k \in (D_2 \cup D_3, D_1 \cup D_4)$ ,
- $\mu_2(x, y, k)$  is analytic and bounded for  $k \in (D_1, D_3)$ ,
- $\mu_3(x, y, k)$  is analytic and bounded for  $k \in (D_3 \cup D_4, D_1 \cup D_2)$ .

For convenience, we write each column of  $\mu_j(x, y, k)$  as the following notations:

$$\mu_1 = \left( \mu_1^{(23)}, \mu_1^{(14)} \right), \quad \mu_2 = \left( \mu_2^{(1)}, \mu_2^{(3)} \right), \quad \mu_3 = \left( \mu_3^{(34)}, \mu_3^{(12)} \right),$$

where the superscripts indicate the analytic and bounded domains  $D_j$ ,  $j = 1, \dots, 4$ , for the columns of the matrix-valued eigenfunctions. Using integration by parts, note that in the appropriate domain

$$\mu_j(x, y, k) = I + O(1/k) \quad \text{as } k \rightarrow \infty. \tag{2.13}$$

Since the matrices  $Q$  and  $\tilde{Q}$  are traceless (i.e.  $\operatorname{trace}(Q) = \operatorname{trace}(\tilde{Q}) = 0$ ), equation (2.13) implies that  $\det \mu_j = 1$ ,  $j = 1, 2, 3$ .

### 3. Spectral analysis

#### 3.1. Spectral functions

The matrix eigenfunctions  $\mu_1$ ,  $\mu_2$  and  $\mu_3$  are both fundamental solutions of the Lax pair (2.1). Note that  $\mu_2(0, 0, k) = I$ . Hence, the eigenfunctions are related by the so-called spectral functions, also

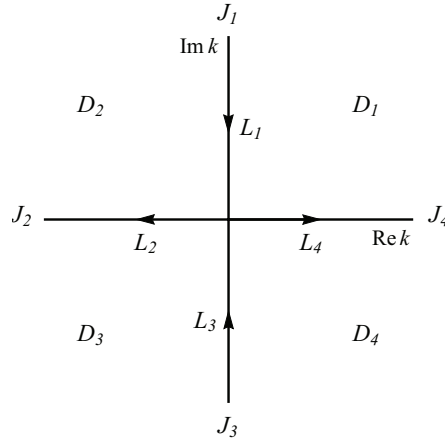


Fig. 2. The domain  $D_j$  and the oriented contour  $L_j$ ,  $j = 1, \dots, 4$

known as the scattering matrices,  $S_1(k)$  and  $S_2(k)$ :

$$\mu_3(x, y, k) = \mu_2(x, y, k)e^{-(\omega_1(k)x + \omega_2(k)y)\hat{\sigma}_3} S_1(k), \quad k \in (\mathbb{R}^+, \mathbb{R}^-), \quad 0 \leq x, y < \infty, \quad (3.1a)$$

$$\mu_1(x, y, k) = \mu_2(x, y, k)e^{-(\omega_1(k)x + \omega_2(k)y)\hat{\sigma}_3} S_2(k), \quad k \in (i\mathbb{R}^+, i\mathbb{R}^-), \quad 0 \leq x, y < \infty. \quad (3.1b)$$

Substituting (3.1b) into (3.1a), of course, equations (3.1) are determined in the form

$$\mu_3(x, y, k) = \mu_1(x, y, k)e^{-(\omega_1(k)x + \omega_2(k)y)\hat{\sigma}_3} (S_2^{-1}(k)S_1(k)), \quad k \in (\partial D_1, \partial D_3). \quad (3.2)$$

Letting  $x = 0$  and  $y = 0$  in (3.1), the spectral functions are given by

$$S_1(k) = \mu_3(0, 0, k), \quad S_2(k) = \mu_1(0, 0, k).$$

From the symmetry (2.7) of the eigenfunctions  $\mu_1$  and  $\mu_3$ , we write the spectral functions  $S_1(k)$  and  $S_2(k)$  as

$$S_1(k) = \begin{pmatrix} a_1(k) & -b_1(-k) \\ b_1(k) & a_1(-k) \end{pmatrix}, \quad S_2(k) = \begin{pmatrix} a_2(k) & -b_2(-k) \\ b_2(k) & a_2(-k) \end{pmatrix}.$$

Since  $\det S_1(k) = \det S_2(k) = 1$ , we find the identities

$$a_1(k)a_1(-k) + b_1(k)b_1(-k) = 1, \quad a_2(k)a_2(-k) + b_2(k)b_2(-k) = 1.$$

Furthermore, we define

$$\Phi(x, k) = \mu_3(x, 0, k), \quad \Psi(y, k) = \mu_1(0, y, k),$$

that is, the functions  $\Phi$  and  $\Psi$  satisfy the following integral equations

$$\Phi(x, k) = I - \int_x^\infty e^{-\omega_1(k)(x-\xi)\hat{\sigma}_3} (Q_0\Phi)(\xi, k) d\xi, \quad k \in (D_1 \cup D_2, D_3 \cup D_4), \quad 0 \leq x < \infty, \quad (3.3a)$$

$$\Psi(y, k) = I - i \int_y^\infty e^{-\omega_2(k)(y-\eta)\hat{\sigma}_3} (\tilde{Q}_0\Psi)(\eta, k) d\eta, \quad k \in (D_2 \cup D_3, D_1 \cup D_4), \quad 0 \leq y < \infty, \quad (3.3b)$$

where  $Q_0(x, k) = Q(x, 0, k)$  and  $\tilde{Q}_0(y, k) = Q(0, y, -k)$ . Note that

$$S_1(k) = \Phi(0, k), \quad S_2(k) = \Psi(0, k),$$

which immediately imply that the spectral functions  $a_1(k)$  and  $b_1(k)$  have analytic continuations for  $\text{Im} k < 0$ , while the spectral functions  $a_2(k)$  and  $b_2(k)$  have analytic continuations for  $\text{Re} k < 0$ . Moreover, due to the symmetry (2.7), the functions  $\Phi$  and  $\Psi$  also can be written as

$$\Phi(x, k) = \begin{pmatrix} \Phi_1(x, k) & -\Phi_2(x, -k) \\ \Phi_2(x, k) & \Phi_1(x, -k) \end{pmatrix}, \quad \Psi(y, k) = \begin{pmatrix} \Psi_1(y, k) & -\Psi_2(y, -k) \\ \Psi_2(x, k) & \Psi_1(y, -k) \end{pmatrix}.$$

We now define the integral representations for the spectral functions below.

**Definition 3.1.** Given  $q(x, 0) = g_0(x)$  and  $q_y(x, 0) = g_1(x)$ , the map

$$\{g_0(x), g_1(x)\} \rightarrow \{a_1(k), b_1(k)\} \tag{3.4}$$

is defined by

$$a_1(k) = 1 - \frac{1}{4} \int_0^\infty \left\{ \frac{i}{2k} (\cosh g_0(\xi) - 1) \Phi_1(\xi, k) - \left( i g_0(\xi) + g_1(\xi) + \frac{\sinh g_0(\xi)}{2k} \right) \Phi_2(\xi, k) \right\} d\xi, \quad \text{Im} k < 0, \tag{3.5a}$$

$$b_1(k) = -\frac{1}{4} \int_0^\infty e^{-2\omega_1(k)\xi} \left\{ \left( i g_0(\xi) + g_1(\xi) - \frac{\sinh g_0(\xi)}{2k} \right) \Phi_1(\xi, k) - \frac{i}{2k} (\cosh g_0(\xi) - 1) \Phi_2(\xi, k) \right\} d\xi, \quad \text{Im} k < 0, \tag{3.5b}$$

where the functions  $\Phi_1$  and  $\Phi_2$  are solutions of the  $x$ -part of the Lax pair (2.1a) with  $y = 0$ , that is,  $\Phi_1$  and  $\Phi_2$  solve the following system of ordinary differential equations:

$$\Phi_{1x} = \frac{1}{4} \left[ \frac{i}{2k} (\cosh g_0(x) - 1) \Phi_1 - \left( i g_0(x) + g_1(x) + \frac{\sinh g_0(x)}{2k} \right) \Phi_2 \right], \tag{3.6a}$$

$$\Phi_{2x} - 2\omega_1(k)\Phi_2 = \frac{1}{4} \left[ \left( i g_0(x) + g_1(x) - \frac{\sinh g_0(x)}{2k} \right) \Phi_1 - \frac{i}{2k} (\cosh g_0(x) - 1) \Phi_2 \right] \tag{3.6b}$$

with  $\lim_{x \rightarrow \infty} (\Phi_1, \Phi_2) = (1, 0)$ .

**Definition 3.2.** Given  $q(0, y) = f_0(y)$  and  $q_x(0, y) = f_1(y)$ , the map

$$\{f_0(y), f_1(x)\} \rightarrow \{a_2(k), b_2(k)\} \tag{3.7}$$

is defined by

$$a_2(k) = 1 - \frac{i}{4} \int_0^\infty \left\{ -\frac{i}{2k} (\cosh f_0(\eta) - 1) \Psi_1(\eta, k) - \left( if_1(\eta) + \dot{f}_0(\eta) - \frac{\sinh f_0(\eta)}{2k} \right) \Psi_2(\eta, k) \right\} d\eta, \quad \text{Re } k < 0, \quad (3.8a)$$

$$b_2(k) = -\frac{i}{4} \int_0^\infty e^{-2\omega_2(k)\eta} \left\{ \left( if_1(\eta) + \dot{f}_0(\eta) + \frac{\sinh f_0(\xi)}{2k} \right) \Psi_1(\eta, k) + \frac{i}{2k} (\cosh f_0(\eta) - 1) \Psi_2(\eta, k) \right\} d\eta, \quad \text{Re } k < 0, \quad (3.8b)$$

where the functions  $\Psi_1$  and  $\Psi_2$  are solutions of the  $y$ -part of the Lax pair (2.1b) with  $x = 0$ , that is,  $\Psi_1$  and  $\Psi_2$  solve the following system of ordinary differential equations:

$$\Psi_{1y} = \frac{i}{4} \left[ -\frac{i}{2k} (\cosh f_0(y) - 1) \Psi_1 - \left( if_0(y) + \dot{f}_1(y) - \frac{\sinh f_0(y)}{2k} \right) \Psi_2 \right], \quad (3.9a)$$

$$\Psi_{2y} - 2\omega_2(k)\Psi_2 = \frac{i}{4} \left[ \left( if_0(y) + \dot{f}_1(y) + \frac{\sinh f_0(y)}{2k} \right) \Psi_1 + \frac{i}{2k} (\cosh f_0(y) - 1) \Psi_2 \right] \quad (3.9b)$$

with  $\lim_{y \rightarrow \infty} (\Psi_1, \Psi_2) = (1, 0)$ .

In what follows we derive the global relation which is the key to applying the Fokas method for boundary value problems. Since the differential 1-form  $W(x, y, k)$  is closed, we know that

$$\int_{\partial\Omega} e^{(\omega_1(k)x + \omega_2(k)y)\hat{\sigma}_3} W(x, y, k) = 0.$$

The above integral can be evaluated explicitly and we find the following global relation

$$\int_0^\infty e^{\omega_1(k)x\hat{\sigma}_3} Q(x, 0, k) \mu(x, 0, k) dx = i \int_0^\infty e^{\omega_2(k)y\hat{\sigma}_3} Q(0, y, -k) \mu(0, y, k) dy. \quad (3.10)$$

Taking  $\mu = \mu_3$  and using (3.2), equation (3.10) yields

$$I - S_1(k) = (I - S_2(k)) S_2^{-1} S_1(k),$$

which implies that  $S_2^{-1}(k)S_1(k) = I$  for  $k \in (D_3, D_1)$ . Therefore, we obtain the global relation in terms of the spectral functions

$$a_1(k) = a_2(k), \quad b_1(k) = b_2(k), \quad k \in D_3. \quad (3.11)$$

Furthermore, substituting  $S_2^{-1}(k)S_1(k) = I$  into equation (3.2), we know that  $\mu_3(x, y, k) = \mu_1(x, y, k)$  for  $k \in (D_3, D_1)$  and hence, we find

$$\mu_1^{(23)}(x, y, k) = \mu_3^{(34)}(x, y, k), \quad k \in D_3, \quad (3.12a)$$

$$\mu_1^{(14)}(x, y, k) = \mu_3^{(12)}(x, y, k), \quad k \in D_1. \quad (3.12b)$$



### 3.2. Spectral analysis at boundary values

In this section we discuss the spectral analysis for the Lax pair at  $x = 0$  and  $y = 0$ , respectively, so that the boundary values can be characterized from the spectral functions.

**Proposition 3.1.** *The inverse map*

$$\{a_1(k), b_1(k)\} \rightarrow \{q(x, 0), q_y(x, 0)\} \tag{3.13}$$

to the map defined in Definition 3.1 is given by

$$\cosh q(x, 0) = 1 - 8i \lim_{k \rightarrow \infty} kM_{11x}^{(x)} - 8 \lim_{k \rightarrow \infty} \left(kM_{21}^{(x)}\right)^2, \tag{3.14a}$$

$$iq_x(x, 0) + q_y(x, 0) = -4i \lim_{k \rightarrow \infty} kM_{21}^{(x)}, \tag{3.14b}$$

where  $M^{(x)}$  is the solution of the matrix Riemann-Hilbert problem:

$$M_-^{(x)}(x, k) = M_+^{(x)}(x, k)J^{(x)}(x, k), \quad k \in \mathbb{R}, \tag{3.15}$$

with the jump matrix  $J^{(x)}$  given by

$$J^{(x)}(x, k) = \begin{pmatrix} 1 & \frac{b_1(-k)}{a_1(k)} e^{-2\omega_1(k)x} \\ \frac{b_1(k)}{a_1(-k)} e^{2\omega_1(k)x} & \frac{1}{a_1(k)a_1(-k)} \end{pmatrix}, \quad k \in \mathbb{R}. \tag{3.16}$$

**Proof.** The proof is based on the spectral analysis to equation (3.1a) with  $y = 0$ :

$$\mu_3(x, 0, k) = \mu_2(x, 0, k)e^{-\omega_1(k)x\hat{\sigma}_3}S_1(k), \quad k \in (\mathbb{R}^+, \mathbb{R}^-), \quad 0 \leq x < \infty. \tag{3.17}$$

Note that the eigenfunction  $\mu_2^{(1)}(x, 0, k)$  is analytic and bounded for  $k \in D_1 \cup D_2$  and the function  $\mu_2^{(3)}(x, 0, k)$  is analytic and bounded for  $k \in D_3 \cup D_4$ . Thus, we formulate the matrix Riemann-Hilbert problem (3.15) with the jump matrix  $J^{(x)}(x, k)$  given by (3.16), where the sectionally meromorphic functions  $M_{\pm}^{(x)}$  are defined by

$$M_+^{(x)}(x, k) = \left( \frac{\mu_2^{(1)}(x, 0, k)}{a_1(-k)}, \mu_3^{(12)}(x, 0, k) \right), \quad \text{Im } k > 0,$$

$$M_-^{(x)}(x, k) = \left( \mu_3^{(34)}(x, 0, k), \frac{\mu_2^{(3)}(x, 0, k)}{a_1(k)} \right), \quad \text{Im } k < 0.$$

Note that  $\det M_{\pm}^{(x)} = 1$  and  $M_{\pm}^{(x)} = I + O(1/k)$  as  $k \rightarrow \infty$ . Thus, we expand the solution  $M^{(x)}$  of the Riemann-Hilbert problem as

$$M^{(x)}(x, k) = I + \frac{M^{(1)}(x)}{k} + \frac{M^{(2)}(x)}{k^2} + O(1/k^2), \quad k \rightarrow \infty. \tag{3.19}$$

Substituting this expansion into the  $x$ -part of the Lax pair (2.1a) with  $y = 0$ , from the  $(2, 1)$ -component at  $O(1)$ , we find

$$iq_x(x, 0) + q_y(x, 0) = -4iM_{21}^{(1)}(x) \tag{3.20}$$

and the  $(1, 1)$ -component at  $O(1/k)$  yields

$$M_{11x}^{(1)}(x) = -\frac{1}{8i} (\cosh q(x, 0) - 1) - \frac{1}{4} (iq_x(x, 0) + q_y(x, 0)) M_{21}^{(1)}(x). \quad (3.21)$$

Simplifying the above equation with (3.20), we obtain

$$\cosh q(x, 0) = 1 - 8iM_{11x}^{(1)}(x) - 8 \left( M_{21}^{(1)}(x) \right)^2. \quad (3.22)$$

and hence equations (3.14) are proved.  $\square$

**Proposition 3.2.** *The inverse map*

$$\{a_2(k), b_2(k)\} \rightarrow \{q(0, y), q_x(0, y)\} \quad (3.23)$$

to the map defined in Definition 3.2 is given by

$$\cosh q(0, y) = 1 + 8 \lim_{k \rightarrow \infty} kM_{11y}^{(y)} + 8 \lim_{k \rightarrow \infty} \left( kM_{21}^{(y)} \right)^2, \quad (3.24a)$$

$$iq_x(0, y) + q_y(0, y) = -4i \lim_{k \rightarrow \infty} kM_{21}^{(y)}, \quad (3.24b)$$

where  $M^{(y)}$  is the solution of the matrix Riemann-Hilbert problem:

$$M_-^{(y)}(y, k) = M_+^{(y)}(y, k)J^{(y)}(y, k), \quad k \in i\mathbb{R}, \quad (3.25)$$

with the jump matrix  $J^{(y)}$  given by

$$J^{(y)}(y, k) = \begin{pmatrix} 1 & \frac{b_2(-k)}{a_2(k)} e^{-2\omega_2(k)y} \\ \frac{b_2(k)}{a_2(-k)} e^{2\omega_2(k)y} & \frac{1}{a_2(k)a_2(-k)} \end{pmatrix}, \quad k \in i\mathbb{R}. \quad (3.26)$$

**Proof.** The proof is similar to that of Proposition 3.1. From the spectral relation (3.1b) with  $x = 0$ , we find the jump matrix  $J^{(y)}(y, k)$  given in (3.26) and the sectionally meromorphic functions  $M_{\pm}^{(y)}$  given by

$$M_+^{(y)}(y, k) = \left( \frac{\mu_2^{(1)}(0, y, k)}{a_2(-k)}, \mu_1^{(14)}(0, y, k) \right), \quad \operatorname{Re} k > 0,$$

$$M_-^{(y)}(y, k) = \left( \mu_1^{(23)}(0, y, k), \frac{\mu_2^{(3)}(0, y, k)}{a_2(k)} \right), \quad \operatorname{Re} k < 0$$

with  $\det M_{\pm}^{(y)} = 1$  and  $M_{\pm}^{(y)} = I + O(1/k)$  as  $k \rightarrow \infty$ . Note that the eigenfunction  $\mu_2^{(1)}(0, y, k)$  is analytic and bounded for  $k \in D_1 \cup D_4$  and the function  $\mu_2^{(3)}(0, y, k)$  is analytic and bounded for  $k \in D_2 \cup D_3$ . In the similar way presented in the proof of Proposition 3.1, equations (3.24) follow.  $\square$

**4. Riemann-Hilbert problem**

Using equations (3.1), the global relation (3.11) and (3.12), we formulate the following matrix Riemann-Hilbert problem:

$$M_-(x, y, k) = M_+(x, y, k)J(x, y, k), \quad k \in \mathcal{L}, \tag{4.1}$$

where the oriented contours  $\mathcal{L} = L_1 \cup L_2 \cup L_3 \cup L_4$  are given by (cf. Fig. 2)

$$L_1 = D_1 \cap D_2, \quad L_2 = D_2 \cap D_3, \tag{4.2a}$$

$$L_3 = D_3 \cap D_4, \quad L_4 = D_4 \cap D_1, \tag{4.2b}$$

and the jump matrices are defined by

$$J_1(x, y, k) = \begin{pmatrix} 1 & 0 \\ \frac{b_2(k)}{a_1(-k)} e^{2\theta(x,y,k)} & 1 \end{pmatrix}, \quad k \in L_1, \tag{4.3a}$$

$$J_2(x, y, k) = \begin{pmatrix} -\frac{b_1(-k)}{a_1(k)} e^{-2\theta(x,y,k)} & 1 \\ 1 & 0 \end{pmatrix}, \quad k \in L_2, \tag{4.3b}$$

$$J_3(x, y, k) = \begin{pmatrix} -\frac{b_2(-k)}{a_1(k)} e^{-2\theta(x,y,k)} & 1 \\ 1 & 0 \end{pmatrix}, \quad k \in L_3, \tag{4.3c}$$

$$J_4(x, y, k) = J_1 J_2^{-1} J_3 = \begin{pmatrix} 1 & 0 \\ \frac{b_1(k)}{a_1(-k)} e^{2\theta(x,y,k)} & 1 \end{pmatrix}, \quad k \in L_4 \tag{4.3d}$$

with  $\theta(x, y, k) = \omega_1(k)x + \omega_2(k)y$ . The matrix-valued functions  $M_{\pm}$  are sectionally meromorphic and defined below

$$M_+(x, y, k) = \begin{cases} \left( \frac{\mu_2^{(1)}}{a_1(-k)}, \mu_1^{(14)} \right), & k \in D_1, \\ \left( \mu_1^{(23)}, \frac{\mu_2^{(3)}}{a_1(k)} \right), & k \in D_3, \end{cases} \tag{4.4a}$$

$$M_-(x, y, k) = \begin{cases} \left( \mu_1^{(23)}, \mu_3^{(12)} \right), & k \in D_2, \\ \left( \mu_3^{(34)}, \mu_1^{(14)} \right), & k \in D_4. \end{cases} \tag{4.4b}$$

Note that  $\det M_{\pm} = 1$  and  $M_{\pm} = I + O(1/k)$  as  $k \rightarrow \infty$ . The Riemann-Hilbert problem (4.1) can be solved by a Cauchy-type integral equation. Indeed, letting  $\tilde{J} = I - J$ , equation (4.1) becomes

$$M_+(x, y, k) - M_-(x, y, k) = M_+(x, y, k)\tilde{J}(x, y, k). \tag{4.5}$$

Applying the Plemelj formula [9], the solution  $M$  of the Riemann-Hilbert problem (4.1) can be expressed as

$$M(x, y, k) = I + \frac{1}{2i\pi} \int_{\mathcal{L}} M_+(x, y, k') \tilde{J}(x, y, k') \frac{dk'}{k' - k}.$$

Note that

$$M(x, y, k) = I - \frac{1}{2ik\pi} \int_{\mathcal{L}} M_+(x, y, k') \tilde{J}(x, y, k') dk' + O(1/k^2), \quad k \rightarrow \infty.$$

Then the solution of the elliptic sinh-Gordon equation in the quarter plane can be obtained in terms of the unique solution of the Riemann-Hilbert problem. In this respect, we expand the solution  $M$  of the Riemann-Hilbert problem (4.1) as

$$M(x, y, k) = I + \frac{M^{(1)}(x, y)}{k} + \frac{M^{(2)}(x, y)}{k^2} + O(1/k^2), \quad k \rightarrow \infty. \quad (4.6)$$

Substituting this expansion into the  $x$ -part of the Lax pair (2.1a), the  $(2, 1)$ -component at  $O(1)$  implies

$$iq_x(x, y) + q_y(x, y) = -4iM_{21}^{(1)}(x, y) \quad (4.7)$$

and the  $(1, 1)$ -component at  $O(1/k)$  yields

$$M_{11x}^{(1)}(x, y) = -\frac{1}{8i} (\cosh q(x, y) - 1) - \frac{1}{4} (iq_x(x, y) + q_y(x, y)) M_{21}^{(1)}(x, y).$$

Simplifying the above equation with (4.7), we obtain the reconstruction formula for the solution of (1.1) given by

$$\cosh q(x, y) = 1 - 8iM_{11x}^{(1)}(x, y) - 8 \left( M_{21}^{(1)} \right)^2. \quad (4.8)$$

Similarly, if we substitute the expansion (4.6) into the  $y$ -part of the Lax pair, the solution is equivalently given by

$$\cosh q(x, y) = 1 + 8M_{11y}^{(1)}(x, y) + 8 \left( M_{21}^{(1)} \right)^2. \quad (4.9)$$

We now state the existence theorem for the elliptic sinh-Gordon equation in the quarter plane.

**Theorem 4.1.** *Assume that the functions  $g_j(x), f_j(y) \in H^1(\mathbb{R}^+)$ ,  $j = 0, 1$ , with the sufficiently small  $H^1$  norms. Let the functions  $a_1(k), b_1(k), a_2(k)$  and  $b_2(k)$  be given by (3.5) and (3.8) in Definitions 3.1 and 3.2, respectively. Suppose that given  $g_0(x)$  and  $g_1(x)$ , there exist functions  $f_0(y)$  and  $f_1(y)$  such that the global relation is satisfied*

$$a_1(k) = a_2(k) \quad \text{and} \quad b_1(k) = b_2(k) \quad k \in D_3. \quad (4.10)$$

Let  $M(x, y, k)$  be the solution of the following matrix Riemann-Hilbert (RH) problem

$$M_-(x, y, k) = M_+(x, y, k)J(x, y, k), \quad k \in \mathcal{L}, \quad (4.11)$$

where  $\det(M_\pm) = 1$ ,  $M_\pm = I + O(1/k)$  as  $k \rightarrow \infty$ , the oriented contours  $\mathcal{L}$  are defined in (4.2) and the jump matrices  $J$  are given in (4.3).

Then the Reimann-Hilbert problem is uniquely solvable and the function  $q(x, y)$  defined by

$$iq_x + q_y = -4i \lim_{k \rightarrow \infty} kM_{21}, \quad \cosh q(x, y) = 1 - 8i \lim_{k \rightarrow \infty} kM_{11x} - 8 \lim_{k \rightarrow \infty} (kM_{21})^2 \quad (4.12)$$

solves the elliptic sinh-Gordon equation (1.1) satisfying the boundary conditions

$$q(x, 0) = g_0(x), \quad q_y(x, 0) = g_1(x), \quad (4.13a)$$

$$q(0, y) = f_0(y), \quad q_x(0, y) = f_1(y). \quad (4.13b)$$

**Proof.** By applying the vanishing lemma and the dressing method discussed in [9, 14], it can be proved that the Riemann-Hilbert problem (4.11) is uniquely solved and that  $q(x, y)$  defined in (4.12) solves the elliptic sinh-Gordon equation (1.1) (see also before Theorem).

We will prove that  $q(x, y)$  given in (4.12) satisfies the boundary values. In this respect, it requires to show that the Riemann-Hilbert problem (4.11) with  $y = 0$  and  $x = 0$  are equivalent to the Riemann-Hilbert problems (3.15) and (3.25) given in Propositions 3.1 and 3.2, respectively.

Regarding equation (4.13a), define

$$M^{(x)}(x, k) = \begin{cases} M(x, 0, k), & k \in D_1, \\ M(x, 0, k)J_1^{-1}(x, 0, k), & k \in D_2, \\ M(x, 0, k)J_3(x, 0, k)F(x, k), & k \in D_3, \\ M(x, 0, k)F(x, k), & k \in D_4, \end{cases} \quad (4.14)$$

where

$$F(x, k) = \begin{pmatrix} 1 & \frac{b_1(-k)}{a_1(k)} e^{-2\omega_1(k)x} \\ 0 & 1 \end{pmatrix}. \quad (4.15)$$

We denote  $M(x, 0, k)$  and  $M^{(x)}(x, k)$  for  $k \in D_j$ ,  $j = 1, \dots, 4$ , by  $M_j(x, 0, k)$  and  $M_j^{(x)}(x, k)$ , respectively. Then, we can write (4.11) and (4.14) as

$$M_2(x, 0, k) = M_1 J_1(x, 0, k), \quad M_2(x, 0, k) = M_3 J_2(x, 0, k), \quad (4.16a)$$

$$M_4(x, 0, k) = M_3 J_3(x, 0, k), \quad M_4(x, 0, k) = M_1 J_4(x, 0, k), \quad (4.16b)$$

and

$$M_1^{(x)}(x, k) = M_1(x, 0, k), \quad M_2^{(x)}(x, k) = M_1 J_1^{-1}(x, 0, k), \quad (4.17a)$$

$$M_3^{(x)}(x, k) = M_3 J_3(x, 0, k)F(x, k), \quad M_4^{(x)}(x, k) = M_4(x, 0, k)F(x, k). \quad (4.17b)$$

Combining (4.17) with (4.16), we find the following jump conditions

$$M_2^{(x)}(x, k) = M_1^{(x)}(x, k), \quad M_3^{(x)}(x, k) = M_2^{(x)}(x, k)J_4(x, 0, k)F(x, k), \quad (4.18a)$$

$$M_4^{(x)}(x, k) = M_3^{(x)}(x, k), \quad M_4^{(x)}(x, k) = M_1^{(x)}(x, k)J_4(x, 0, k)F(x, k). \quad (4.18b)$$

Note that  $J_4(x, 0, k)F(x, k) = J^{(x)}(x, k)$ , where  $J^{(x)}(x, k)$  is given in (3.16), and no jumps occur along the contours  $L_1$  and  $L_3$ . Defining

$$M^{(x)}(x, k) = M_+^{(x)}(x, k) \quad k \in D_1 \cup D_2,$$

$$M^{(x)}(x, k) = M_-^{(x)}(x, k) \quad k \in D_3 \cup D_4,$$

we know that equations (4.16) are equivalent to the Riemann-Hilbert problem (3.15) with the jump matrix  $J^{(x)}(x, k)$ . Thus, the proof that the function  $q(x, y)$  satisfies the boundary values (4.13a) immediately follows from evaluating (4.12) at  $y = 0$ .

In a similar way, equation (4.13b) can be proved. □

## 5. Concluding remarks

In conclusion, we have studied the boundary value problem for the elliptic sinh-Gordon equation formulated in the quarter plane by using the Fokas method. Based on the Lax pair formulation we have derived the global relation that involves all boundary values  $\{q(x,0), q_y(x,0)\}$  and  $\{q(0,y), q_x(0,y)\}$ . Furthermore, if the boundary values satisfy the global relation, we have presented the existence of the unique solution for the elliptic sinh-Gordon equation in the quarter plane. It also has been shown that the solution can be expressed in terms of the unique solution of the Riemann-Hilbert problem with the jump matrices defined by the spectral functions  $\{a_1(k), b_1(k)\}$  and  $\{a_2(k), b_2(k)\}$ .

It should be remarked that using the global relation, we have determined the spectral functions  $\{a_1(k), b_1(k)\}$  and  $\{a_2(k), b_2(k)\}$  in terms of the boundary values  $\{q(x,0), q_y(x,0)\}$  and  $\{q(0,y), q_x(0,y)\}$ . However, it is not necessary to prescribe all boundary values for well-posed boundary value problems. Thus, it should be required to characterize unknown boundary values, called the Dirichlet to Neumann map [8, 11]. This characterization can be done by analyzing the global relation as was done in [12, 13] and we will discuss this issue in the near future.

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