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Automorphisms of the q -deformed algebra $su_q(1, 1)$ and d -Orthogonal polynomials of q -Meixner type

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*This paper is dedicated to Professor Youssef Ben Cheikh
on the occasion of his sixtieth birthday.*

Starting from an operator given as a product of q -exponential functions in irreducible representations of the positive discrete series of the q -deformed algebra $su_q(1, 1)$, we express the associated matrix elements in terms of d -orthogonal polynomials. An algebraic setting allows to establish some properties : recurrence relation, generating function, lowering operator, explicit expression and d -orthogonality relations of the involved polynomials which are reduced to the orthogonal q -Meixner polynomials when $d = 1$. If $q \uparrow 1$, these polynomials tend to some d -orthogonal polynomials of Meixner type.

Keywords: d -orthogonal polynomials, q -deformed algebra $su_q(1, 1)$, q -coherent states, q -hypergeometric functions, q -Meixner polynomials, linear functional vector.

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1. Introduction

Let $(P_n)_{n \geq 0}$ be a polynomial sequence with complex coefficients of n -th degree (i.e $\deg P_n = n$) and $(u_n)_{n \geq 0}$ the corresponding dual sequence defined by

$$\langle u_n, P_m \rangle = \delta_{nm}, \quad n, m = 0, 1, \dots$$

where $\langle u, f \rangle$ is the action of a linear functional u on a polynomial f and δ_{nm} the Kronecker delta.

For a positive integer d , the polynomials $P_n(x)$ are called d -orthogonal with respect to the linear d -dimensional functional vector $\mathcal{U} = {}^t(u_0, u_1, \dots, u_{d-1})$ [12,17] if they satisfy the following *vector orthogonality relations*

$$\begin{cases} \langle u_i, P_m P_n \rangle = 0, & n \geq md + i + 1, \\ \langle u_i, P_m P_n \rangle \neq 0, & n = md + i, \end{cases} \quad (1.1)$$

for each integer $i \in \{0, 1, \dots, d-1\}$.

When $d = 1$, we return to the well known notion of orthogonality.

Recall that the polynomials $P_n(x)$ are d -orthogonal if and only if they satisfy a recurrence relation of order $d + 1$ of the type

$$xP_n(x) = \sum_{i=0}^{d+1} \gamma_{i,n+1} P_{n+1-i}(x), \tag{1.2}$$

where $\gamma_{0,n+1} \gamma_{d+1,n+1} \neq 0$ and by convention $P_{-n} = 0$, $n \geq 1$. The result for $d = 1$ is reduced to the so-called Favard Theorem.

During the last three decades, numerous explicit examples of d -orthogonal polynomials and multiple orthogonal polynomials have been intensively studied and developed by many authors [1,2,5,11,16,19]. However, only in the past few years, some works dealing with the connection between d -orthogonal polynomials, multiple orthogonal polynomials and Lie algebras were introduced. Indeed, by means of an algebraic approach, multivariate Charlier and Meixner polynomials, d -orthogonal Charlier, Al-Salam Carlitz and Krawtchouk polynomials appeared as matrix elements of operators in Lie algebras [6,8,9,18]. In the present paper, we shall identify and study some d -orthogonal polynomials generalizing the q -Meixner polynomials which are presented as matrix elements of a suitable operator of the q -deformed algebra $su_q(1, 1)$. Note that the connection between the orthogonal q -Meixner polynomials and the q -deformed algebra $su_q(1, 1)$ has been the subject of many papers [3,13,15].

The outline of the paper is structured as follows. In section 2, we recall basic facts about $su_q(1, 1)$ algebra and its irreducible representations of the positive discrete series. Moreover, we define a set of q -coherent states and we establish some useful identities in $su_q(1, 1)$. Section 3 is devoted to introduce an operator S that shall be studied along with the associated matrix elements which will be expressed in terms of d -orthogonal polynomials. When $d = 1$, the obtained results are reduced to the q -Meixner polynomials. An algebraic approach allows us to derive some properties: recurrence relation, generating function and lowering operator. In section 4, we focus our study to a family of d -orthogonal polynomials of q -Meixner type that will be expressed in terms of q -hypergeometric functions and we determine explicitly a linear d -dimensional functional vector insuring the d -orthogonality of the involved polynomials. Moreover, we show in section 5 how these polynomials are reduced to some d -orthogonal polynomials of Meixner type when $q \uparrow 1$.

In the remainder, we assume that q is a real number such that $0 < q < 1$.

2. The q -deformed algebra $\mathfrak{su}_q(\mathbf{1}, \mathbf{1})$

In this section, we present a few basic elements of q -calculus that shall be needed throughout the paper (the interested reader may consult [7]) and we review basic facts about $su_q(1, 1)$ algebra concerning its positive series representations and the associated q -coherent states.

2.1. Elements of q -analysis

The basic hypergeometric series is defined by

$${}_r\phi_s \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} \middle| q; z \right) := \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n}{(q, b_1, b_2, \dots, b_s; q)_n} \left((-1)^n q^{\binom{n}{2}} \right)^{1+s-r} z^n,$$

with $\binom{n}{2} = n(n-1)/2$, $(a; q)_n$ stands for the q -shifted factorial

$$(a; q)_n = \begin{cases} 1, & n = 0, \\ (1-a)(1-aq) \cdots (1-aq^{n-1}), & n = 1, 2, \dots \end{cases}$$

and $(a_1, a_2, \dots, a_r)_n = \prod_{i=1}^r (a_i; q)_n$.

The q -shifted factorials satisfy a number of identities :

$$\begin{cases} (q; q)_{n-k} = \frac{(-1)^k (q; q)_n q^{\binom{k}{2} - nk}}{(q^{-n}; q)_k}, \\ (q^{-n}; q)_k = 0, \text{ if } k > n, \end{cases} \quad (2.1)$$

$$(a; q)_{nr+i} = (a; q)_i (aq^i; q)_{nr} = (a; q)_i \prod_{j=0}^{r-1} (aq^{i+j}; q^n)_n, \quad (2.2)$$

where n, k, r and i are non-negative integers.

The q -binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

The little q -exponential, denoted by $e_q(z)$, is defined as

$$e_q(z) = {}_1\phi_0 \left(\begin{matrix} 0 \\ - \end{matrix} \middle| q; z \right) = \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_{\infty}},$$

and the big q -exponential, denoted $E_q(z)$, is given by

$$E_q(z) = {}_0\phi_0 \left(\begin{matrix} - \\ - \end{matrix} \middle| q; -z \right) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q; q)_n} z^n = (-z; q)_{\infty}.$$

It follows that $e_q(z)E_q(-z) = 1$.

The q -binomial theorem states that

$$\frac{(xz; q)_{\infty}}{(z; q)_{\infty}} = \sum_{n=0}^{\infty} \frac{(x; q)_n}{(q; q)_n} z^n. \quad (2.3)$$

With the help of (2.1) we show that

$$E_q(t) {}_1\phi_1 \left(\begin{matrix} 0 \\ q^{\beta} \end{matrix} \middle| q; -(1-q)^2 q^{\frac{\beta-1}{2}} zt \right) = \sum_{n=0}^{\infty} {}_1\phi_1 \left(\begin{matrix} q^{-n} \\ q^{\beta} \end{matrix} \middle| q; (1-q)^2 q^{\frac{\beta+1}{2}} z \right) \frac{q^{\binom{n}{2}}}{(q; q)_n} t^n. \quad (2.4)$$

The q -difference operator D_q is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{x}.$$

D_q satisfies the following useful property

$$D_{1/q}^n f(x) = \frac{1}{x^n} \sum_{k=0}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q f(q^{-k}x). \quad (2.5)$$

2.2. $\mathfrak{su}_q(\mathbf{1}, \mathbf{1})$ algebra and its positive discrete representation

The q -deformed algebra $su_q(1, 1)$ is defined as the associative algebra generated by the elements J_-, J_+, q^{J_0} and q^{-J_0} subject to the defining relations

$$\begin{cases} [J_-, J_+] = [2J_0]_q, \\ [J_0, J_{\pm}] = \pm J_{\pm}, \\ q^{J_0} J_{\pm} q^{-J_0} = q^{\pm 1} J_{\pm}, \\ q^{J_0} q^{-J_0} = q^{-J_0} q^{J_0}, \end{cases} \quad (2.6)$$

where

$$[A, B] := AB - BA, \quad [a]_q := \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}},$$

and satisfying the involution relations

$$J_+^* = J_-, \quad (q^{J_0})^* = q^{J_0}.$$

In the remainder of the paper, we are interested in the positive discrete representations of $su_q(1, 1)$ labelled by a positive number β and acting on a Hilbert space \mathcal{H}_β in the following manner: If $|n\rangle$ is an orthonormal basis (i.e $\langle n|m\rangle = \delta_{nm}$), we have

$$\begin{cases} J_- |n\rangle = \alpha_n |n-1\rangle, \\ J_+ |n\rangle = \alpha_{n+1} |n+1\rangle, \\ q^{\pm J_0} |n\rangle = q^{\pm(n+\frac{\beta}{2})} |n\rangle, \end{cases} \quad (2.7)$$

where

$$\alpha_n = \frac{q^{-(n+\frac{\beta-3}{2})/2}}{(1-q)} \sqrt{(1-q^n)(1-q^{\beta+n-1})}. \quad (2.8)$$

The action by powers on the basis $|n\rangle$ are given by

$$\begin{cases} J_-^i |n\rangle = \frac{\alpha_n!}{\alpha_{n-i}!} |n-i\rangle, \quad i \leq n, \\ J_-^i |n\rangle = 0, \quad i > n, \\ J_+^n |m\rangle = \frac{\alpha_{m+n}!}{\alpha_m!} |m+n\rangle, \end{cases} \quad (2.9)$$

where $\alpha_n!$ is the sequence defined by

$$\alpha_n! := \prod_{k=1}^n \alpha_k = \frac{q^{-n(\beta+n-2)/4}}{(1-q)^n} \sqrt{(q; q)_n (q^\beta; q)_n}, \quad \alpha_0! = 1. \quad (2.10)$$

It is obvious to see that

$$\frac{J_+^n |0\rangle}{\alpha_n!} = |n\rangle. \quad (2.11)$$

By induction on n we show according to (2.6) that

$$\begin{aligned} q^{-J_0} J_-^n &= (qJ_-)^n q^{-J_0}, \\ J_-^n J_+ - J_+ J_-^n &= c_q(1 - q^n)(q^{J_0} J_-^{n-1} - J_-^{n-1} q^{-J_0}), \end{aligned}$$

where

$$c_q = \frac{-\sqrt{q}}{(1 - q)^2}.$$

It follows that for every formal power series f such that $f(J_-)$ exists

$$\begin{cases} q^{-J_0} f(J_-) = f(qJ_-) q^{-J_0}, \\ J_+ f(J_-) = f(J_-) J_+ - c_q q^{J_0} D_q f(J_-) + c_q D_q f(J_-) q^{-J_0}, \end{cases} \quad (2.12)$$

and by conjugation we get

$$\begin{cases} f(J_+) q^{-J_0} = q^{-J_0} f(qJ_+), \\ f(J_+) J_- = J_- f(J_+) - c_q D_q f(J_+) q^{J_0} + c_q q^{-J_0} D_q f(J_+). \end{cases} \quad (2.13)$$

Since $D_q e_q(-z) = -e_q(-z)$ and $e_q(-qz) = (1 + z)e_q(-z)$ we get from (2.13),

$$e_q(-J_+) q^{-J_0} E_q(J_+) = q^{-J_0} + q^{-J_0} J_+, \quad (2.14)$$

$$E_q(J_+) J_- e_q(-J_+) = J_- - c_q q^{J_0} + c_q q^{-J_0} (1 + J_+)^{-1}. \quad (2.15)$$

2.3. q -coherent states

The concept of coherent states and their applications can be traced to early literature in the field of quantum optics and was widened for more general applications, primarily in quantum mechanics [14]. In our work, we introduce the notion of q -coherent states associated with $su_q(1, 1)$ as an algebraic tool which will be exploited in order to establish basic properties of some d -orthogonal polynomials.

Let z be a complex number. By $|z\rangle$, we denote the q -coherent state defined as

$$|z\rangle := \sum_{n=0}^{\infty} \frac{z^n}{\alpha_n!} |n\rangle. \quad (2.16)$$

Its expansion coefficients are $\langle n|z\rangle = \frac{z^n}{\alpha_n!}$. Using (2.10) and (2.11) we show that

$$|z\rangle = {}_1\phi_1 \left(\begin{matrix} 0 \\ q^\beta \end{matrix} \middle| q; -(1 - q)^2 q^{\frac{\beta-1}{2}} z J_+ \right) |0\rangle. \quad (2.17)$$

The state $|z\rangle$ can be looked upon as an eigenstate of the operator J_- . Indeed we have

$$J_- |z\rangle = z |z\rangle,$$

and for a power series f ,

$$f(J_-)|z\rangle = f(z)|z\rangle. \tag{2.18}$$

For q -coherent states $|z_1\rangle$ and $|z_2\rangle$, the inner product is

$$\langle z_1|z_2\rangle = {}_1\phi_1\left(\begin{matrix} 0 \\ q^\beta \end{matrix} \middle| q; -(1-q)^2 q^{\frac{\beta-1}{2}} \overline{z_1 z_2}\right).$$

3. Matrix elements of an operator and d -orthogonal polynomials

Let r, d be two positive integers such that $d = 2r - 1$ and let a_1, a_2, \dots, a_r be complex numbers in $\mathbb{C} - \{0\}$. The operator S which will be the subject of our study in the remainder of the paper is defined by

$$S = E_q(J_+) \prod_{i=1}^r e_q(a_i J_-). \tag{3.1}$$

It is clear that S is invertible and

$$S^{-1} = \prod_{i=1}^r E_q(-a_i J_-) e_q(-J_+). \tag{3.2}$$

We define the matrix elements of S by $\psi_{n,k} = \langle k|S|n\rangle$.

To establish a recurrence relation satisfied by $\psi_{n,k}$ we need the following obvious results.

$$F_q(qz) = P(z)F_q(z) \quad \text{and} \quad D_q F_q(z) = Q(z)F_q(z), \tag{3.3}$$

where F_q, P, Q are given by

$$F_q(z) = \prod_{i=1}^r e_q(a_i z), \quad P(z) = \prod_{i=1}^r (1 - a_i z), \quad Q(z) = \frac{1 - P(z)}{z}. \tag{3.4}$$

3.1. Recurrence relation

Starting from the matrix element $\langle k|q^{-J_0}S|n\rangle$, we have from (2.7)

$$\langle k|q^{-J_0}S|n\rangle = q^{-(k+\frac{\beta}{2})} \psi_{n,k}. \tag{3.5}$$

On the other hand

$$\langle k|q^{-J_0}S|n\rangle = \langle k|S(S^{-1}q^{-J_0}S)|n\rangle. \tag{3.6}$$

According to (2.12),(2.13),(2.14),(3.2) and (3.3), we have

$$\begin{aligned} S^{-1}q^{-J_0}S &= F_q(J_-)^{-1} \left(e_q(-J_+)q^{-J_0}E_q(J_+) \right) F_q(J_-) \\ &= F_q(J_-)^{-1}q^{-J_0}F_q(J_-) + F_q(J_-)^{-1}q^{-J_0} \left(J_+F_q(J_-) \right) \\ &= F_q(J_-)^{-1}F_q(qJ_-)q^{-J_0} + F_q(J_-)^{-1}q^{-J_0}F_q(J_-)J_+ \\ &\quad - c_q F_q(J_-)^{-1}D_q F_q(J_-) + c_q F_q(J_-)^{-1}q^{-J_0}D_q F_q(J_-)q^{-J_0} \\ &= P(J_-)q^{-J_0} + P(J_-)q^{-J_0}J_+ - c_q Q(J_-) + c_q Q(qJ_-)P(J_-)q^{-2J_0}. \end{aligned}$$

Therefore (3.6) becomes

$$\begin{aligned} \langle k|q^{-J_0}S|n\rangle &= \langle k|SP(J_-)q^{-J_0}|n\rangle + \langle k|SP(J_-)q^{-J_0}J_+|n\rangle \\ &\quad - c_q \langle k|SQ(J_-)|n\rangle + c_q \langle k|SQ(qJ_-)P(J_-)q^{-2J_0}|n\rangle. \end{aligned} \quad (3.7)$$

After writing the polynomials $P(z)$, $Q(z)$, $Q(qz)P(z)$ under the form,

$$P(z) = \sum_{i=0}^d \xi_i z^i, \quad Q(z) = \sum_{i=0}^d \eta_i z^i, \quad Q(qz)P(z) = \sum_{i=0}^d \mu_i z^i,$$

with $\mu_d \neq 0$, $\eta_i = \xi_i = 0$, $i > r$, we obtain successively according to (2.9)

$$\begin{aligned} P(J_-)q^{-J_0}|n\rangle &= q^{-(n+\frac{\beta}{2})} \sum_{i=0}^d \xi_i \frac{\alpha_n!}{\alpha_{n-i}!} |n-i\rangle, \\ P(J_-)q^{-J_0}J_+|n\rangle &= q^{-(n+1+\frac{\beta}{2})} \alpha_{n+1} \sum_{i=0}^d \xi_i \frac{\alpha_{n+1}!}{\alpha_{n+1-i}!} |n+1-i\rangle, \\ Q(J_-)|n\rangle &= \sum_{i=0}^d \eta_i \frac{\alpha_n!}{\alpha_{n-i}!} |n-i\rangle, \\ Q(qJ_-)P(J_-)q^{-2J_0} &= q^{-(2n+\beta)} \sum_{i=0}^d \mu_i \frac{\alpha_n!}{\alpha_{n-i}!} |n-i\rangle. \end{aligned}$$

Hence we get from (3.7)

$$\begin{aligned} \langle k|q^{-J_0}S|n\rangle &= q^{-(n+\frac{\beta}{2})} \sum_{i=0}^d \xi_i \frac{\alpha_n!}{\alpha_{n-i}!} \psi_{n-i,k} + q^{-(n+1+\frac{\beta}{2})} \alpha_{n+1} \sum_{i=0}^d \xi_i \frac{\alpha_{n+1}!}{\alpha_{n+1-i}!} \psi_{n+1-i,k} \\ &\quad - c_q \sum_{i=0}^d \eta_i \frac{\alpha_n!}{\alpha_{n-i}!} \psi_{n-i,k} + c_q q^{-(2n+\beta)} \sum_{i=0}^d \mu_i \frac{\alpha_n!}{\alpha_{n-i}!} \psi_{n-i,k}. \end{aligned} \quad (3.8)$$

Comparing (3.5) and (3.8), we obtain

Proposition 3.1. *The matrix elements $\psi_{n,k}$ satisfy the following recurrence relation of order $d+1 = 2r$.*

$$q^{-k} \psi_{n,k} = q^{-(n+1)} \alpha_{n+1} \psi_{n+1,k} + \sum_{i=0}^d \beta_{n,i} \psi_{n-i,k}, \quad (3.9)$$

where $\beta_{n,i}$ are complex numbers, with $\beta_{n,d} \neq 0$.

From this relation one can express $\psi_{n,k}$ recursively, starting from $\psi_{0,k}$. Indeed, putting $n = 0$ (respectively $n = 1$) in (3.9), we get

$$\begin{aligned} \psi_{1,k} &= \frac{q}{\alpha_1} (q^{-k} - \beta_{0,0}) \psi_{0,k}, \\ \psi_{2,k} &= \frac{q^3}{\alpha_1 \alpha_2} ((q^{-k} - \beta_{1,0})(q^{-k} - \beta_{0,0}) - \beta_{1,1}) \psi_{0,k}. \end{aligned}$$

Repeating this process we arrive at the following:

Corollary 3.2. *The matrix elements $\Psi_{n,k}$ are expressed under the form*

$$\Psi_{n,k} = \Psi_{0,k} P_n(q^{-k}), \tag{3.10}$$

where $P_n(q^{-k})$ is a polynomial of degree n in the argument q^{-k} and satisfying the recurrence relation of order $d + 1$ given by

$$q^{-k} P_n(q^{-k}) = q^{-(n+1)} \alpha_{n+1} P_{n+1}(q^{-k}) + \sum_{i=0}^d \beta_{n,i} P_{n-i}(q^{-k}),$$

with the initial conditions $P_0(q^{-k}) = 1, P_n(q^{-k}) = 0, n < 0$.

According to (1.2), the polynomials $P_n(q^{-k})$ are d -orthogonal.

3.2. Generating function

In order to calculate the (formal) generating function $F(z, k)$ of the d -orthogonal polynomials $P_n(q^{-k})$ defined by

$$F(z, k) := \sum_{n=0}^{\infty} \frac{P_n(q^{-k})}{\alpha_n!} z^n,$$

we consider the expression of $\langle k|S|z \rangle$. On the one hand, we have from (2.16)

$$\langle k|S|z \rangle = \langle k|S| \sum_{n=0}^{\infty} \frac{z^n}{\alpha_n!} |n \rangle \rangle = \sum_{n=0}^{\infty} \frac{\Psi_{n,k}}{\alpha_n!} z^n. \tag{3.11}$$

On the other hand, taking into account of (2.4),(2.10),(2.17) and (2.18), we get successively

$$\begin{aligned} \langle k|S|z \rangle &= \langle k|E_q(J_+) \prod_{i=1}^r e_q(a_i J_-) |z \rangle \\ &= \prod_{i=1}^r e_q(a_i z) \langle k|E_q(J_+) {}_1\phi_1 \left(\begin{matrix} 0 \\ q^\beta \end{matrix} \middle| q; -(1-q)^2 q^{\frac{\beta-1}{2}} z J_+ \right) |0 \rangle \\ &= \prod_{i=1}^r e_q(a_i z) \sum_{m=0}^{\infty} \frac{q^{\binom{m}{2}}}{(q; q)_m} {}_1\phi_1 \left(\begin{matrix} q^{-m} \\ q^\beta \end{matrix} \middle| q; (1-q)^2 q^{\frac{\beta+1}{2}} z \right) \langle k|J_+^m |0 \rangle \\ &= \prod_{i=1}^r e_q(a_i z) \sum_{m=0}^{\infty} \frac{\alpha_m! q^{\binom{m}{2}}}{(q; q)_m} {}_1\phi_1 \left(\begin{matrix} q^{-m} \\ q^\beta \end{matrix} \middle| q; (1-q)^2 q^{\frac{\beta+1}{2}} z \right) \langle k|m \rangle \\ &= \prod_{i=1}^r e_q(a_i z) \frac{\alpha_k! q^{\binom{k}{2}}}{(q; q)_k} {}_1\phi_1 \left(\begin{matrix} q^{-k} \\ q^\beta \end{matrix} \middle| q; (1-q)^2 q^{\frac{\beta+1}{2}} z \right) \\ &= \prod_{i=1}^r e_q(a_i z) \frac{q^{k(k-\beta)/4}}{(1-q)^k} \sqrt{\frac{(q^\beta; q)_k}{(q; q)_k}} {}_1\phi_1 \left(\begin{matrix} q^{-k} \\ q^\beta \end{matrix} \middle| q; (1-q)^2 q^{\frac{\beta+1}{2}} z \right). \end{aligned}$$

It follows by virtue of (3.11) that the matrix elements $\Psi_{n,k}$ are generated by

$$\sum_{n=0}^{\infty} \frac{\Psi_{n,k}}{\alpha_n!} z^n = \frac{q^{k(k-\beta)/4}}{(1-q)^k} \sqrt{\frac{(q^\beta; q)_k}{(q; q)_k}} \prod_{i=1}^r e_q(a_i z) {}_1\phi_1 \left(\begin{matrix} q^{-k} \\ q^\beta \end{matrix} \middle| q; (1-q)^2 q^{\frac{\beta+1}{2}} z \right).$$

From the previous calculus we get

$$\psi_{0,k} = \frac{\alpha_k! q^{\binom{k}{2}}}{(q; q)_k} = \frac{q^{k(k-\beta)/4}}{(1-q)^k} \sqrt{\frac{(q^\beta; q)_k}{(q; q)_k}}. \quad (3.12)$$

Using the relation (3.10) we arrive at the following:

Proposition 3.3. *The d -orthogonal polynomials $P_n(q^{-k})$ are generated by*

$$\sum_{n=0}^{\infty} \frac{P_n(q^{-k})}{\alpha_n!} z^n = \prod_{i=1}^r e_q(a_i z) {}_1\phi_1 \left(\begin{matrix} q^{-k} \\ q^\beta \end{matrix} \middle| q; (1-q)^2 q^{\frac{\beta+1}{2}} z \right). \quad (3.13)$$

3.3. Lowering operator

We have obviously from (2.7)

$$\langle k | SJ_- | n \rangle = \alpha_n \psi_{n-1,k}. \quad (3.14)$$

On the other hand, we have

$$\langle k | SJ_- | n \rangle = \langle k | (SJ_- S^{-1}) S | n \rangle. \quad (3.15)$$

According to (2.15)

$$\begin{aligned} SJ_- S^{-1} &= E_q(J_+) J_- e_q(-J_+) \\ &= J_- - c_q q^{J_0} + c_q q^{-J_0} (1 + J_+)^{-1} \\ &= J_- - c_q q^{J_0} + c_q q^{-J_0} \sum_{m=0}^{\infty} (-1)^m J_+^m. \end{aligned}$$

Hence (3.15) becomes

$$\begin{aligned} \langle k | SJ_- | n \rangle &= \langle k | J_- S | n \rangle - c_q \langle k | q^{J_0} S | n \rangle + c_q \langle k | q^{-J_0} (1 + J_+)^{-1} S | n \rangle \\ &= \alpha_{k+1} \psi_{n,k+1} - c_q q^{k+\frac{\beta}{2}} \psi_{n,k} + c_q q^{-(k+\frac{\beta}{2})} \sum_{m=0}^k (-1)^m \frac{\alpha_k!}{\alpha_{k-m}!} \psi_{n,k-m}. \end{aligned} \quad (3.16)$$

Dividing by $\psi_{0,k}$ and combining (3.14) with (3.16), we obtain

$$\begin{aligned} \alpha_n P_{n-1}(q^{-k}) &= \alpha_{k+1} \frac{\psi_{0,k+1}}{\psi_{0,k}} P_n(q^{-(k+1)}) - c_q q^{k+\frac{\beta}{2}} P_n(q^{-k}) \\ &\quad + c_q q^{-(k+\frac{\beta}{2})} \sum_{m=0}^k (-1)^m \frac{\alpha_k!}{\alpha_{k-m}!} \frac{\psi_{0,k-m}}{\psi_{0,k}} P_n(q^{-(k-m)}). \end{aligned} \quad (3.17)$$

By virtue of (2.1), (2.8) and (3.12), we have

$$\begin{aligned} \alpha_{k+1} \frac{\psi_{0,k+1}}{\psi_{0,k}} &= \frac{q^{\frac{1-\beta}{2}}}{(1-q)^2} (1 - q^{\beta+k}) \\ &= -c_q q^{-\frac{\beta}{2}} (1 - q^{\beta+k}), \end{aligned}$$

and

$$\begin{aligned} \frac{\alpha_k!}{\alpha_{k-m}!} \frac{\Psi_{0,k-m}}{\Psi_{0,k}} &= \frac{(q; q)_k q^{\binom{k-m}{2}}}{(q; q)_{k-m} q^{\binom{k}{2}}} \\ &= (-1)^m q^m (q^{-k}; q)_m. \end{aligned}$$

Thus we get from (3.17)

$$\begin{aligned} \alpha_n P_{n-1}(q^{-k}) &= c_q q^{k+\frac{\beta}{2}} (P_n(q^{-(k+1)}) - P_n(q^{-k})) - c_q q^{-\frac{\beta}{2}} P_n(q^{-(k+1)}) \\ &+ c_q q^{-(k+\frac{\beta}{2})} \sum_{m=0}^{\infty} q^m (q^{-k}; q)_m P_n(q^{m-k}). \end{aligned} \quad (3.18)$$

Using now the notation $T_q^{(m)} f(x) = f(q^m x)$, for any integer m , we announce according to (3.18) that

Proposition 3.4. *The operator σ defined by*

$$\sigma = T_q^{(-1)} - q^\beta D_{1/q} - q^{-k} \sum_{m=0}^{\infty} q^m (q^{-k}; q)_m T_q^{(m)},$$

satisfies the following equality

$$\sigma P_n(q^{-k}) = (1-q) q^{(-2n+\beta+1)/4} \sqrt{(1-q^n)(1-q^{\beta+n-1})} P_{n-1}(q^{-k}).$$

Hence σ is a lowering operator of $P_n(q^{-k})$.

4. d -orthogonal polynomials of q -Meixner type

After replacing z by $\frac{-q^{\frac{1-\beta}{2}}}{c(1-q)^2} z$ and a_i by $a'_i = \frac{-q^{\frac{1-\beta}{2}}}{c(1-q)^2} a_i$ in (3.13) with $c > 0$, we assume in this section that $a'_j = e^{\frac{2ij\pi}{r}}$, $1 \leq j \leq r$. Then we get

$$P(z) = \prod_{j=1}^r (1 - a'_j z) = 1 - z^r \quad \text{and} \quad \prod_{j=1}^r e_q(a'_j z) = e_{q^r}(z^r) = \frac{1}{(z^r; q^r)_\infty}.$$

According to (3.13), it is natural to consider the d -orthogonal polynomials

$M_n(q^{-k}; q^{\beta-1}, c, d; q)$ generated by

$$\sum_{n=0}^{\infty} M_n(q^{-k}; q^{\beta-1}, c, d; q) \frac{z^n}{(q; q)_n} = \frac{1}{(z^r; q^r)_\infty} {}_1\phi_1 \left(\begin{matrix} q^{-k} \\ q^\beta \end{matrix} \middle| q; \frac{-qz}{c} \right), \quad (4.1)$$

which are reduced when $d = 1$ to the q -Meixner polynomials $M_n(q^{-k}; b, c; q)$, where $0 < b < q^{-1}$, $c > 0$ and $b = q^{\beta-1}$ [10].

Recall that $M_n(q^{-k}; b, c; q)$ are defined and generated by

$$M_n(q^{-k}; b, c; q) = {}_2\phi_1 \left(\begin{matrix} q^{-n}, q^{-k} \\ bq \end{matrix} \middle| q; -\frac{q^{n+1}}{c} \right), \quad (4.2)$$

$$\sum_{n=0}^{\infty} M_n(q^{-k}; b, c; q) \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_{\infty}} {}_1\phi_1 \left(\begin{matrix} q^{-k} \\ bq \end{matrix} \middle| q; -\frac{qz}{c} \right) \quad (4.3)$$

and they satisfy the orthogonality relations

$$\sum_{k=0}^{\infty} \frac{(bq; q)_k c^k q^{\binom{k}{2}}}{(-bcq; q)_k (q; q)_k} M_m(q^{-k}, b, c; q) M_n(q^{-k}, b, c; q) = 0, \text{ if } m \neq n. \quad (4.4)$$

$M_n(q^{-k}; q^{\beta-1}, c, d; q)$ are called d -orthogonal polynomials of q -Meixner type.

4.1. Explicit expression

To obtain the explicit expression of the polynomials $M_n(q^{-k}; q^{\beta-1}, c, d; q)$, we can proceed directly by expanding the generating function (4.1). Indeed, we have

$$\sum_{n=0}^{\infty} M_n(q^{-k}; q^{\beta-1}, c, d; q) \frac{z^n}{(q; q)_n} = \sum_{s=0}^{\infty} \sum_{i=0}^k \frac{q^{\binom{i+1}{2}} (q^{-k}; q)_i}{c^i (q; q)_i (q^{\beta}; q)_i (q^r; q^r)_s} z^{i+rs}.$$

Then we get

$$M_n(q^{-k}; q^{\beta-1}, c, d; q) = (q; q)_n \sum_{i=0}^k \frac{q^{\binom{i+1}{2}} (q^{-k}; q)_i}{c^i (q; q)_i (q^{\beta}; q)_i (q^r; q^r)_{\frac{n-i}{r}}}. \quad (4.5)$$

In (4.5) the discrete variable i can take the values such that

$$(n - i)/r = s = 0, 1, \dots$$

For any non-negative integer we can put $n = mr + j$, $m = 0, 1, \dots$ and $j = 0, 1, \dots, r - 1$. Then i can take the values $i = rl + j$, $l = 0, 1, \dots$

Therefore (4.5) can be written in the form

$$M_n(q^{-k}; q^{\beta-1}, c, d; q) = (q; q)_n \sum_{l=0}^m \frac{q^{\binom{rl+j+1}{2}} (q^{-k}; q)_{rl+j}}{c^{rl+j} (q; q)_{rl+j} (q^{\beta}; q)_{rl+j} (q^r; q^r)_{m-l}}, \quad (4.6)$$

which becomes after an easy calculation according to (2.1) and (2.2)

$$\begin{aligned} M_n(q^{-k}; q^{\beta-1}, c, d; q) &= \frac{q^{\binom{j+1}{2}} (q; q)_n (q^{-k}; q)_j}{c^j (q^r; q^r)_m (q; q)_j (q^{\beta}; q)_j} \\ &\times \sum_{l=0}^{\infty} \frac{(q^{-mr}; q^r)_l \prod_{s=0}^{r-1} (q^{-k+j+s}; q^r)_l \left((-1)^l q^{\binom{l}{2}} \right)^{r-1} q^{l \binom{r+1}{2} + rl(m+j)}}{(-c)^{rl} \prod_{s=0}^{r-1} (q^{j+s+1}, q^{\beta+j+s}; q^r)_l}. \end{aligned}$$

Therefore if we denote by $\Delta(a; r, q)$ and $\Delta(a, b; r, q)$ the arrays defined by

$$\begin{cases} \Delta(a; r, q) = (q^{a/r}, q^{(a+1)/r}, \dots, q^{(a+r-1)/r}), \\ \Delta(a, b; r, q) = (\Delta(a; r, q), \Delta(b; r, q)), \end{cases}$$

we get

$$\begin{aligned} M_n(q^{-k}; q^{\beta-1}, c, d; q) &= \frac{q^{\binom{j+1}{2}}(q; q)_n(q^{-k}; q)_j}{c^j(q^r; q^r)_m(q; q)_j(q^\beta; q)_j} \\ &\times {}_{r+2}\phi_{2r} \left(\begin{matrix} q^r, q^{-mr}, \Delta(-k+j; r, q^r) \\ \Delta(j+1, \beta+j; r, q^r) \end{matrix} \middle| q^r; \frac{q^{\binom{r+1}{2}+r(m+j)}}{(-c)^r} \right). \end{aligned} \quad (4.7)$$

In the particular case when $d = r = 1$ (then $m = n$ and $j = 0$), we get from (4.7)

$$\begin{aligned} M_n(q^{-k}, q^{\beta-1}, c; q) &= {}_3\phi_2 \left(\begin{matrix} q, q^{-k}, q^{-n} \\ q, q^\beta \end{matrix} \middle| q; \frac{-q^{n+1}}{c} \right) \\ &= {}_2\phi_1 \left(\begin{matrix} q^{-k}, q^{-n} \\ q^\beta \end{matrix} \middle| q; \frac{-q^{n+1}}{c} \right). \end{aligned}$$

Hence we meet again the q -hypergeometric representation of $M_n(q^{-k}; q^{\beta-1}, c; q)$ given in (4.2).

4.2. d -orthogonality relations

Let us now express explicitly in terms of q -hypergeometric functions the linear d -dimensional functional vector $\mathcal{U} = {}^t(u_0, u_1, \dots, u_{d-1})$ insuring the d -orthogonality of the polynomials $M_n(q^{-k}; q^{\beta-1}, c, d; q)$. The adopted approach is based on the notion of obtaining dual sequence of a polynomial set via inversion coefficients [4].

The main result of this section is the following theorem.

Theorem 4.1. *The polynomials $M_n(q^{-k}; q^{\beta-1}, c, d; q)$ generated by (4.1) are d -orthogonal with respect to the linear d -dimensional functional vector $\mathcal{U} = {}^t(u_0, u_1, \dots, u_{d-1})$ given for every $0 \leq i \leq d-1$ by*

$$\langle u_i, f \rangle = c^i q^{-\binom{i}{2}} (q^\beta; q)_i \sum_{s=0}^{r-1} \sum_{k=0}^{\infty} \frac{q^{\binom{rk+s}{2}}}{(q; q)_{rk+s}} \omega_{i, rk+s} f(q^{-(rk+s)}), \quad (4.8)$$

where f is a polynomial and $\omega_{i, rk+s}$ is given by

(1) for $rk+s \leq i-1$

$$\omega_{i, rk+s} = \frac{(-1)^{rk+s}}{(q; q)_{i-rk-s}} {}_{2r}\phi_r \left(\begin{matrix} \Delta(i+1, \beta+i; r, q^r) \\ \Delta(i-rk-s+1; r, q^r) \end{matrix} \middle| q^r; (-c)^r q^{-ir-\binom{i}{2}} \right),$$

(2) for $rk+s \geq i$ and $1-r \leq s-i \leq 0$

$$\omega_{i, rk+s} = \frac{(-1)^{(r+1)k+s} c^{rk} q^{-k^2 \binom{r}{2} - irk} (q^{i+1}; q)_{rk} (q^{\beta+i}; q)_{rk}}{(q; q)_{i-s} (q^r; q^r)_k} \\ \times {}_{2r+1}\phi_{r+1} \left(\begin{matrix} q^r, \Delta(rk+i+1, \beta+rk+i; r, q^r) \\ q^{r(k+1)}, \Delta(i-s+1; r, q^r) \end{matrix} \middle| q^r; (-c)^r q^{-ir-(2k+1)\binom{r}{2}} \right),$$

(3) for $rk+s \geq i$ and $1 \leq s-i \leq r-1$

$$\omega_{i, rk+s} = \frac{(-1)^{(r+1)k+s+1} c^{r(k+1)} q^{-(k+1)^2 \binom{r}{2} - ir(k+1)} (q^{i+1}; q)_{r(k+1)} (q^{\beta+i}; q)_{r(k+1)}}{(q; q)_{r+i-s} (q^r; q^r)_{k+1}} \\ \times {}_{2r+1}\phi_{r+1} \left(\begin{matrix} q^r, \Delta(r(k+1)+i+1, \beta+r(k+1)+i; r, q^r) \\ q^{r(k+2)}, \Delta(r+i-s+1; r, q^r) \end{matrix} \middle| q^r; (-c)^r q^{-ir-(2k+3)\binom{r}{2}} \right),$$

(4) for $rk+s \geq i$ and $2(1-r) \leq s-i \leq -r$

$$\omega_{i, rk+s} = \frac{(-1)^{(r+1)k+s-1} c^{r(k-1)} q^{-(k-1)^2 \binom{r}{2} - ir(k-1)} (q^{i+1}; q)_{r(k-1)} (q^{\beta+i}; q)_{r(k-1)}}{(q; q)_{-r+i-s} (q^r; q^r)_{k-1}} \\ \times {}_{2r+1}\phi_{r+1} \left(\begin{matrix} q^r, \Delta(r(k-1)+i+1, \beta+r(k-1)+i; r, q^r) \\ q^{rk}, \Delta(-r+i-s+1; r, q^r) \end{matrix} \middle| q^r; (-c)^r q^{-ir-(2k-1)\binom{r}{2}} \right).$$

Example. If $d = r = 1$ (then $i = s = 0$). We get from Theorem 4.1 (case (2)) and with the help of (2.3)

$$\omega_{0,k} = c^k (q^\beta; q)_{k3} \phi_2 \left(\begin{matrix} q, q^{k+1}, q^{\beta+k} \\ q^{k+1}, q \end{matrix} \middle| q; -c \right) \\ = c^k (q^\beta; q)_{k1} \phi_0 \left(\begin{matrix} q^{\beta+k} \\ - \end{matrix} \middle| q; -c \right) \\ = \frac{c^k (q^\beta; q)_k (-cq^{\beta+k}; q)_\infty}{(-c; q)_\infty} \\ = \frac{c^k (q^\beta; q)_k (-cq^\beta; q)_\infty}{(-cq^\beta; q)_k (-c; q)_\infty}.$$

Then we obtain

$$\langle u_0, f \rangle = \frac{(-cq^\beta; q)_\infty}{(-c; q)_\infty} \sum_{k=0}^{\infty} \frac{c^k q^{\binom{k}{2}} (q^\beta; q)_k}{(q; q)_k (-cq^\beta; q)_k} f(q^{-k}).$$

Hence we recognise the orthogonality of q -Meixner polynomials $M_n(x; q^{\beta-1}, c; q)$ given in (4.4).

In order to prove Theorem 4.1, we need the following lemma.

Lemma 4.2. For every polynomial f , we have

$$f(x) = \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} \left[D_{1/q}^n f(x) \right]_{x=1} (x; q)_n. \quad (4.9)$$

Proof. Writing f under the form

$$f(x) = \sum_{m=0}^{\infty} \alpha_m(x; q)_m, \quad (4.10)$$

then applying the operator $D_{1/q}^n$ to each side of (4.10) and using the fact that

$$\begin{cases} D_{1/q}^n(x; q)_m = \frac{q^{-n}(q; q)_m}{(q; q)_{m-n}}(x; q)_{m-n}, & \text{if } m \geq n, \\ D_{1/q}^n(x; q)_m = 0, & \text{if } m < n, \\ [(x; q)_n]_{x=1} = \delta_{n,0}, \end{cases}$$

we get

$$\left[D_{1/q}^n f(x) \right]_{x=1} = q^{-n} \alpha_n(q; q)_n,$$

which finishes the proof.

Proof of Theorem 4.1. We have from (4.1)

$${}_1\phi_1 \left(\begin{matrix} x \\ q^\beta \end{matrix} \middle| q; \frac{-qz}{c} \right) = (z^r; q^r)_\infty \sum_{n=0}^{\infty} M_n(x; q^{\beta-1}, c, d; q) \frac{z^n}{(q; q)_n}.$$

Then by equalizing the coefficients of z^n , we get

$$(x; q)_n = \frac{c^n (q; q)_n (q^\beta; q)_n}{q^{\binom{n+1}{2}}} \sum_{m=0}^{\lfloor n/r \rfloor} \frac{(-1)^m q^{r \binom{m}{2}}}{(q^r; q^r)_m (q; q)_{n-mr}} M_{n-mr}(x; q^{\beta-1}, c, d; q). \quad (4.11)$$

(where $[a]$ is the integer part of a).

Applying the dual sequence $(u_i)_{i \geq 0}$ of $M_n(x; q^{\beta-1}, c, d; q)$ to each member of (4.11) we obtain

$$\begin{cases} \langle u_i, (x; q)_n \rangle = \frac{(-1)^m q^{r \binom{m}{2}} c^n (q; q)_n (q^\beta; q)_n}{q^{\binom{n+1}{2}} (q^r; q^r)_m (q; q)_i} & \text{if } n = mr + i, \\ \langle u_i, (x; q)_n \rangle = 0, & \text{otherwise.} \end{cases} \quad (4.12)$$

With the help of (2.5),(4.9) and (4.12), we get successively for every polynomial f

$$\begin{aligned} \langle u_i, f \rangle &= \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} \left[D_{1/q}^n f(x) \right]_{x=1} \langle u_i, (x; q)_n \rangle \\ &= \sum_{m=0}^{\infty} \frac{q^{mr+i}}{(q; q)_{mr+i}} \left[D_{1/q}^{mr+i} f(x) \right]_{x=1} \langle u_i, (x; q)_{mr+i} \rangle \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^{mr+i} \frac{(-1)^{m+k} q^{\binom{k}{2} + r \binom{m}{2}} (q; q)_{mr+i} (q^\beta; q)_{mr+i} c^{mr+i}}{(q; q)_i (q; q)_k (q; q)_{mr+i-k} (q^r; q^r)_m q^{\binom{mr+i}{2}}} f(q^{-k}) \\ &= \underbrace{\sum_{m=0}^{\infty} \sum_{k=0}^{i-1} \frac{(-1)^{m+k} q^{\binom{k}{2} + r \binom{m}{2}} (q; q)_{mr+i} (q^\beta; q)_{mr+i} c^{mr+i}}{(q; q)_i (q; q)_k (q; q)_{mr+i-k} (q^r; q^r)_m q^{\binom{mr+i}{2}}} f(q^{-k})}_{\mathcal{A}_i(f)} \\ &\quad + \underbrace{\sum_{m=0}^{\infty} \sum_{k=i}^{mr+i} \frac{(-1)^{m+k} q^{\binom{k}{2} + r \binom{m}{2}} (q; q)_{mr+i} (q^\beta; q)_{mr+i} c^{mr+i}}{(q; q)_i (q; q)_k (q; q)_{mr+i-k} (q^r; q^r)_m q^{\binom{mr+i}{2}}} f(q^{-k})}_{\mathcal{B}_i(f)}, \end{aligned}$$

with $\mathcal{A}_0(f) = 0$. Using the identity (2.2), we obtain

$$\mathcal{A}_i(f) = c^i q^{-\binom{i}{2}} (q^\beta; q)_i \sum_{k=0}^{i-1} \frac{q^{\binom{k}{2}}}{(q; q)_k} \omega_{i,k}^{(1)} f(q^{-k}),$$

where

$$\begin{aligned} \omega_{i,k}^{(1)} &= \frac{(-1)^k}{(q; q)_{i-k}} \sum_{m=0}^{\infty} \frac{\prod_{j=0}^{r-1} (q^{i+j+1}, q^{\beta+i+j}; q^r)_m \left((-1)^m q^{r \binom{m}{2}} \right)^{1-r} \left((-c)^r q^{-\binom{i}{2} - ir} \right)^m}{(q^r; q^r)_m \prod_{j=0}^{r-1} (q^{i-k+j+1}; q^r)_m} \\ &= \frac{(-1)^k}{(q; q)_{i-k}} {}_{2r}\phi_r \left(\begin{matrix} \Delta(i+1, \beta+i; r, q^r) \\ \Delta(i-k+1; r, q^r) \end{matrix} \middle| q^r; (-c)^r q^{-\binom{i}{2} - ir} \right) \end{aligned}$$

By virtue of the following transformation:

$$\sum_{m=0}^{\infty} \sum_{k=i}^{mr+i} H(m, k) = \sum_{k=i}^{\infty} \sum_{m=0}^{\infty} H(m + \eta_{i,k}, k), \quad \eta_{i,k} = 1 + \left\lfloor \frac{k-i-1}{r} \right\rfloor$$

and (2.2) we get

$$\mathcal{B}_i(f) = c^i q^{-\binom{i}{2}} (q^\beta; q)_i \sum_{k=i}^{\infty} \frac{q^{\binom{k}{2}}}{(q; q)_k} \omega_{i,k}^{(2)} f(q^{-k}),$$

where

$$\begin{aligned} \omega_{i,k}^{(2)} &= \frac{(-1)^{\eta_{i,k}+k} q^{v_{i,k}} c^{r\eta_{i,k}} (q^{i+1}; q)_{r\eta_{i,k}} (q^{\beta+i}; q)_{r\eta_{i,k}}}{(q^r; q^r)_{\eta_{i,k}} (q; q)_{r\eta_{i,k}+i-k}} \\ &\times \sum_{m=0}^{\infty} \frac{\prod_{j=0}^{r-1} (q^{r\eta_{i,k}+i+j+1}, q^{\beta+r\eta_{i,k}+i+j}; q^r)_m \left((-1)^m q^{r\binom{m}{2}} \right)^{1-r} \left((-c)^r q^{\mu_{i,k}} \right)^m}{(q^{r(1+\eta_{i,k})}; q^r)_m \prod_{j=0}^{r-1} (q^{r\eta_{i,k}+i-k+j+1}; q^r)_m} \\ &= \frac{(-1)^{\eta_{i,k}+k} q^{v_{i,k}} c^{r\eta_{i,k}} (q^{i+1}; q)_{r\eta_{i,k}} (q^{\beta+i}; q)_{r\eta_{i,k}}}{(q^r; q^r)_{\eta_{i,k}} (q; q)_{r\eta_{i,k}+i-k}} \\ &\times {}_{2r+1}\phi_{r+1} \left(\begin{matrix} q^r, \Delta(r\eta_{i,k}+i+1, \beta+r\eta_{i,k}+i; r, q^r) \\ q^{r(1+\eta_{i,k})}, \Delta(r\eta_{i,k}+i-k+1; r, q^r) \end{matrix} \middle| q^r; (-c)^r q^{\mu_{i,k}} \right), \end{aligned}$$

with

$$\begin{aligned} \mu_{i,k} &= -(1 + 2\eta_{i,k}) \binom{r}{2} - ir, \\ v_{i,k} &= -ir\eta_{i,k} - \binom{r}{2} (\eta_{i,k})^2. \end{aligned}$$

Hence we get

$$\langle u_i, f \rangle = c^i q^{-\binom{i}{2}} (q^\beta; q)_i \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}}{(q; q)_k} \omega_{i,k} f(q^{-k}),$$

with

$$\begin{cases} \omega_{i,k} = \omega_{i,k}^{(1)}, & \text{if } 0 \leq k \leq i-1, \\ \omega_{i,k} = \omega_{i,k}^{(2)}, & \text{if } i \leq k. \end{cases}$$

Since $0 \leq i \leq 2r-2$ and $0 \leq s \leq r-1$, then the result follows from the following three cases.

$$\eta_{i,rk+s} = \begin{cases} k, & \text{if } 1-r \leq s-i \leq 0, \\ k+1, & \text{if } 1 \leq s-i \leq r-1, \\ k-1, & \text{if } 2(1-r) \leq s-i \leq -r. \end{cases}$$

5. Link with a d -orthogonal polynomials of Meixner type

Mention that in a paper which is under review, we have considered a family of d -orthogonal polynomials $M_n(k; \beta, c, d)$ generated by

$$\sum_{n=0}^{\infty} M_n(k; \beta, c, d) \frac{z^n}{n!} = e^{z^r} {}_1F_1 \left(\begin{matrix} -k \\ \beta \end{matrix} \middle| \frac{1-c}{c} z \right), \quad (5.1)$$

where ${}_rF_s$ is the hypergeometric function defined by

$${}_rF_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_r)_n}{(b_1)_n \dots (b_s)_n n!} z^n.$$

$(a)_n$ is the pochhammer symbol given by

$$(a)_n = a(a+1)\cdots(a+n-1), \quad (a)_0 = 1.$$

$M_n(k; \beta, c, d)$ reduced to the Meixner polynomials $M_n(k; \beta, c)$ [10] when $d = r = 1$ are called d -orthogonal polynomials of Meixner type. Moreover they are related to the polynomials $M_n(q^{-k}; q^{\beta-1}, c, d; q)$ in the following manner:

Replacing z by $r^{\frac{1}{r}}(1-q)^{\frac{1}{r}}z$ and c by $c' = \frac{cr^{\frac{1}{r}}(1-q)^{-1+\frac{1}{r}}}{(1-c)}$ in (4.1), then the polynomials $M_n(q^{-k}; q^{\beta-1}, c', d; q)$ are generated by

$$\begin{aligned} \sum_{n=0}^{\infty} r^{\frac{n}{r}}(1-q)^{\frac{n}{r}}M_n(q^{-k}; q^{\beta-1}, c', d; q) \frac{z^n}{(q; q)_n} &= \frac{1}{(r(1-q)z^r; q^r)_{\infty}} \\ &\times {}_1\phi_1\left(\begin{matrix} q^{-k} \\ q^{\beta} \end{matrix} \middle| q; \frac{-q(1-c)(1-q)}{c}z\right). \end{aligned} \quad (5.2)$$

On the other hand, we have

$$\lim_{q \rightarrow 1} \frac{(q^a; q)_{mr+s}}{(1-q)^{mr+s}} = (a)_{mr+s}, \quad \lim_{q \rightarrow 1} \frac{(q; q)_{mr+s}}{(1-q)^{mr+s}} = (mr+s)! \quad (5.3)$$

and

$$\lim_{q \rightarrow 1} {}_r\phi_s\left(\begin{matrix} q^{a_1}, \dots, q^{a_r} \\ q^{b_1}, \dots, q^{b_s} \end{matrix} \middle| q; (1-q)^{s+1-r}z\right) = {}_rF_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| (-1)^{s+1-r}z\right). \quad (5.4)$$

Since $\lim_{q \rightarrow 1} \frac{1}{(r(1-q)z^r; q^r)_{\infty}} = e^{z^r}$, then we get from (5.1), (5.2) and (5.4)

$$\lim_{q \rightarrow 1} r^{\frac{n}{r}}(1-q)^{n(\frac{1}{r}-1)}M_n(q^{-k}; q^{\beta-1}, c', d; q) = M_n(k; \beta, c, d). \quad (5.5)$$

When $d = r = 1$, we obtain the classical limit relation

$$\lim_{q \rightarrow 1} M_n(q^{-k}; q^{\beta-1}, \frac{c}{1-c}; q) = M_n(k; \beta, c).$$

Using the identities (4.7), (5.3), (5.4) and (5.5), therefore the hypergeometric representation of the polynomials $M_n(k; \beta, c, d)$ are given by

$$M_n(k; \beta, c, d) = \frac{n!(-k)_j(1-c)^j}{m!j!(\beta)_jc^j} {}_{r+2}F_{2r}\left(\begin{matrix} 1, -m, \Delta(-k+j; r) \\ \Delta(j+1; r), \Delta(\beta+j; r) \end{matrix} \middle| -\left(\frac{1-c}{cr}\right)^r\right), \quad (5.6)$$

where $\Delta(a, r) = \left(\frac{a}{r}, \frac{a+1}{r}, \dots, \frac{a+r-1}{r}\right)$ and $n = mr + j$.

In the particular case $d = r = 1$ (then $m = n$ and $j = 0$), we get

$$\begin{aligned} M_n(k; \beta, c) &= {}_3F_2\left(\begin{matrix} 1, -n, -k \\ 1, \beta \end{matrix} \middle| \frac{c-1}{c}\right) \\ &= {}_2F_1\left(\begin{matrix} -n, -k \\ \beta \end{matrix} \middle| \frac{c-1}{c}\right). \end{aligned}$$

Hence we meet again the hypergeometric representation of the Meixner polynomials $M_n(k; \beta, c)$ [10].

In order to determine the d -orthogonality relations of the polynomials $M_n(x; \beta, c, d)$, we start from (1.1). Indeed we have

$$\langle u_i, M_n M_m \rangle = 0, \text{ if } n \geq md + i + 1.$$

Then we immediately obtain according to (4.8)

$$\sum_{s=0}^{r-1} \sum_{k=0}^{\infty} \frac{q^{\binom{rk+s}{2}}}{(q; q)_{rk+s}} \omega_{i, rk+s} M_n(q^{-(rk+s)}) M_m(q^{-(rk+s)}) = 0, \tag{5.7}$$

with $M_n(q^{-k}) = M_n(q^{-k}; q^{\beta-1}, c, d; q)$.

Thus, by means of (5.3), (5.4), (5.5), (5.7) and Theorem 4.1, the d -orthogonality relations satisfied by $M_n(k; \beta, c, d)$ are expressed under the form

$$\sum_{s=0}^{r-1} \sum_{k=0}^{\infty} \frac{\theta_{i, rk+s}}{(rk+s)!} M_n(rk+s) M_m(rk+s) = 0, \text{ if } n \geq md + i + 1, \tag{5.8}$$

where $\theta_{i, rk+s}$ is given by

(1) for $rk+s \leq i-1$

$$\theta_{i, rk+s} = \frac{(-1)^{rk+s}}{(i-rk-s)!} {}_2rF_r \left(\begin{matrix} \Delta(i+1; r), \Delta(\beta+1; r) \\ \Delta(i-rk-s+1; r) \end{matrix} \middle| - \left(\frac{cr}{1-c} \right)^r \right),$$

(2) for $rk+s \geq i$ and $1-r \leq s-i \leq 0$

$$\theta_{i, rk+s} = \frac{(-1)^{(r+1)k+s} (rk+i)! (\beta+i)_{rk} c^{rk}}{i! k! (i-s)! (1-c)^{rk}} \times {}_{2r+1}F_{r+1} \left(\begin{matrix} 1, \Delta(rk+i+1; r), \Delta(\beta+rk+i; r) \\ k+1, \Delta(i-s+1; r) \end{matrix} \middle| - \left(\frac{cr}{1-c} \right)^r \right),$$

(3) for $rk+s \geq i$ and $1 \leq s-i \leq r-1$

$$\theta_{i, rk+s} = \frac{(-1)^{(r+1)k+s+1} (r(k+1)+i)! (\beta+i)_{r(k+1)} c^{r(k+1)}}{i! (r+i-s)! (1-c)^{r(k+1)}} \times {}_{2r+1}F_{r+1} \left(\begin{matrix} 1, \Delta(r(k+1)+i+1; r), \Delta(\beta+r(k+1)+i; r) \\ k+2, \Delta(r+i-s+1; r) \end{matrix} \middle| - \left(\frac{cr}{1-c} \right)^r \right),$$

(4) for $rk + s \geq i$ and $2(1 - r) \leq d - i \leq -r$

$$\theta_{i, rk+s} = \frac{(-1)^{(r+1)k+s-1} (r(k-1) + i)! (\beta + i)_{r(k-1)} c^{r(k-1)}}{i! (-r + i - s)! (1 - c)^{r(k-1)}} \times {}_{2r+1}F_{r+1} \left(\begin{matrix} 1, \Delta(r(k-1) + i + 1; r), \Delta(\beta + r(k-1) + i; r) \\ k, \Delta(-r + i - s + 1; r) \end{matrix} \middle| - \left(\frac{cr}{1-c} \right)^r \right).$$

Example. If $d = r = 1$ (then $i = s = 0$), we get from case(2) and with the help of the binomial theorem

$$\begin{aligned} \theta_{0,k} &= \frac{c^k (\beta)_k}{k! (1-c)^k} {}_3F_2 \left(\begin{matrix} 1, k+1, \beta+k \\ k+1, 1 \end{matrix} \middle| - \frac{c}{1-c} \right) \\ &= \frac{c^k (\beta)_k}{k! (1-c)^k} {}_1F_0 \left(\begin{matrix} \beta+k \\ - \end{matrix} \middle| - \frac{c}{1-c} \right) \\ &= \frac{(1-c)^\beta c^k (\beta)_k}{k!}. \end{aligned}$$

Then we obtain according to (5.8)

$$\sum_{k=0}^{\infty} \frac{c^k (\beta)_k}{k!} M_n(k; \beta, c) M_m(k; \beta, c) = 0, \text{ if } n \neq m.$$

Hence we recognise the orthogonality of the Meixner polynomials $M_n(k; \beta, c)$ [10].

Remark 5.1. For $n = md + i$, I don't succeed to determine the d -orthogonality relations satisfied by the polynomials $M_n(k; \beta, c)$ and I will try to solve this problem in a future work.

6. Concluding remarks

In this paper, we have considered a suitable operator defined in the q -deformed algebra $su_q(1, 1)$ and showed that the associated matrix elements can be expressed in terms of new d -orthogonal polynomials that generalize the q -Meixner polynomials. Furthermore, it was shown that the basic properties of the polynomials were derived from an algebraic approach.

In the limit $q \uparrow 1$ where $su_q(1, 1)$ contracts to $su(1, 1)$, the polynomials tend to some d -orthogonal polynomials of Meixner type; this has been the subject of another work.

Note that in [11] another sequence of d -orthogonal polynomials generalizing the q -Meixner polynomials was introduced in the context of solving a d -Geronimus problem type.

In a subsequent paper, we will try to investigate under which operator we obtain matrix elements expressible in terms of d -orthogonal polynomials.

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