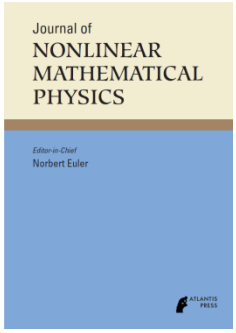

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Generalized Conditional Symmetries, Related Solutions and Conservation Laws of the Grad-Shafranov Equation with Arbitrary Flow

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The generalized conditional symmetry (GCS) method is applied to the case of a generalized Grad-Shafranov equation (GGSE) with incompressible flow of arbitrary direction. We investigate the conditions which yield the GGSE that admits a special class of second-order GCSs. Three GCS generators and the associated families of invariant solutions are pointed out. Several plots of the level sets or flux surfaces of the new solutions are displayed. These results extend the recent solutions with 5 parameters recently obtained on the basis of Lie-point symmetries. They could be useful in the study of plasma equilibrium, of transport phenomena, and of magnetohydrodynamic stability. Futher, by making use of the multiplier's method, three nontrivial conservation laws that are admitted by the concerned equation and which involve arbitrary functions, are highlighted.

Keywords: Grad-Schafranov equation; arbitrary magnetic flux; generalized conditional symmetries; symbolic computation; invariant solutions; conservation laws.

2000 Mathematics Subject Classification: 58J70, 34C14, 35L65

1. Introduction

The concepts of symmetry, invariants and conservation laws are fundamental in the study of dynamical systems, providing a clear connection between the motion equations and their solutions. A symmetry of a system of parial differential eqautions (PDEs) maps whatever solution to another solution of the same system. Several types of symmetries, such as the continuous Lie groups of symmetries and the discrete symmetries, may be obtained algorithmically [2, 8, 25]. In particular, by making use of Lie's algorithm for solving symmetry determining equations, we would be able to discover one-parameter, multi-parameter, and infinite-dimensional symmetry groups. Due to the facts that, for many nonlinear PDEs, this approach does provide limited results only and that many symmetry reductions could not be obtained, several generalizations of the classical symmetry method have been taken into consideration. Let us recall among them the conditional symmetries (CS) method [3], the direct method [6], the generalized conditional symmetry (GCS) method [16, 17, 20, 34]. The latter has been successfully applied in [5, 7, 15, 19, 33] in order to obtain various kinds of exact solutions for nonlinear PDEs arising from various research areas.

An important complement to the full symmetry structure of a PDE system is the knowledge of its conservation laws. These laws do contain important information about the physical properties of the model taken into consideration. They are applied, as for example in: (i) testing the complete integrability of PDEs and in the applying of the Inverse Scattering Transform, (ii) the study of quantitative and qualitative properties of PDEs (Hamiltonian structure, recursion operators) and

(iii) development of numerical methods such as finite element methods [18]. Various methods for constructing the conservation laws have been investigated, including the multiplier method [1], the Lagrangian approach for the evolution equations [9], the relationship between symmetries and conservation laws, irrespective of the existence of a Lagrangian of the system [13], Ibragimov's method [11] for self-adjoint differential equations.

The theory of plasma physics offers a number of nontrivial examples of PDEs which could be successfully treated through symmetry methods. Let us propose the Grad-Shafranov (GS) equation. It is central to almost all magnetic confinement problems and may be derived from ideal magnetohydrodynamic equations assuming a static equilibrium and an azimuthal symmetry. Its widely employed families of analytic solutions with closed flux surfaces are the Solovév solutions [28] and the Hernneger-Maschke ones [21]. However, these, being up-down symmetric and having a limited number of free parameters, cannot describe configurations with a diverted shaping of contemporary tokamaks. The Solovév-like solution is extended in [4, 29] by introducing an arbitrary number of additional terms in the homogeneous part of the solution. Therefore, by exploiting the respective arbitrary number of free parameters, a variety of equilibria with desirable shaping and useful confinement figures have been reported. Further extensions [12,32] to confined plasmas with incompressible flows parallel to the magnetic field and respectively to plasmas with incompressible flows of an arbitrary direction have been achieved on the basis of generalized Grad-Shafranov equation (GGSE). And furthermore, a possible extension that involves various stochastic magnetic field configurations, with or without shear, in turbulent plasmas may be analyzed [22–24, 26, 30].

Our study employs the GGSE [27,31] which describes toroidal configurations with plasma flow non-parallel to the magnetic field, given by:

$$\Delta^* u + \frac{1}{2}\frac{d}{du}\left[\frac{X^2}{1-M_p^2}\right] + R^2\frac{dP_s}{du} + \frac{R^4}{2}\frac{d}{du}\left[\rho\left(\frac{d\Phi}{du}\right)^2\right] = 0, \tag{1.1}$$

where (R, ϕ, z) are the cylindrical coordinates with z axis of symmetry, $u(R, z)$ denotes the poloidal magnetic flux, $M_p(u)$ the poloidal Alfvén-Mach function, $\rho(u)$ the plasma density, $X(u)$ a free function related to poloidal electric current, $I = X/(1-M^2)$, $\Phi(u)$ is the electrostatic potential and $P_s(u)$ is the static pressure; the elliptic operator is defined as $\Delta^* = R\frac{\partial}{\partial R}(\frac{1}{R}\frac{\partial}{\partial R}) + \frac{\partial^2}{\partial z^2}$. For the parallel flow ($\frac{d\Phi}{du} \equiv 0$), (1.1) becomes identical with the GS equation.

Some nonlinear axisymmetric solutions for the GGSE (1.1) with the free-function terms [10]:

$$\frac{1}{2}\frac{d}{du}\left[\frac{X^2}{1-M_p^2}\right] = \frac{k_2}{u^3},\ \frac{dP_s}{du} = \frac{k_1}{u^7},\ \frac{d}{du}\left[\rho\left(\frac{d\Phi}{du}\right)^2\right] = \frac{k_3}{u^{11}}, \tag{1.2}$$

have been obtained by employing the Lie and weak conditional symmetries.

When nonlinear PDES are concerned, an adequate linearization of nonlinear terms represents a specific technique. Therefore, the master Eq. (1.1) can be linearized for suitable physically expressions of the free-function terms as follows [27]:

$$\begin{gathered}\frac{dP_s}{du} = -\frac{P_{sa}}{u_b},\ \frac{1}{2}\frac{d}{du}\left[\frac{X^2}{1-M_p^2}\right] = \gamma_s\frac{P_{sa}R_0^2}{u_b(1+\delta_s^2)},\\ \frac{1}{2}\frac{d}{du}\left[\rho\left(\frac{d\Phi}{du}\right)^2\right] = -\lambda\frac{P_{sa}}{u_b(1+\delta_s^2)R_0^2},\end{gathered} \tag{1.3}$$

where the subscripts a and b show the magnetic axis and the plasma boundary, respectively, R_0 is the major radius of the torus, δ_s relates to the shape of the magnetic surfaces in the vicinity of the magnetic axis, λ relates to the nonparallel component of the flow and γ_s determines the distance of the inner part of the separatrix of the generalized Solovév solution [27] from the z axis. By introducing the dimensionless quantities $x := \frac{R}{R_0}$, $y := \frac{z}{R_0}$, and $U := \frac{u}{u_0}$, where $u_0 := \frac{P_{sa}R_0^4}{2u_b(1+\delta_s^2)}$, Eq.(1.1) takes the linear form:

$$\frac{\partial^2 U}{\partial x^2} - \frac{1}{x}\frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial y^2} + Ex^4 + Fx^2 + G = 0, \tag{1.4}$$

where $E := -2\lambda$, $F := -2(1+\delta_s^2)$ and $G := 2\gamma_s$ are free parameters.

The generalized Solovév solution of Eq. (1.4) with 4 parameters was derived in [27]. This result is recently extended [14], on the basis of Lie-point symmetries, to new exact solutions containing 5 parameters which describe D-shaped toroidal configurations with plasma flow non-parallel to the magnetic field. In the present paper, we will demonstrate some results concerning the structure of the second-order generalized conditional symmetries of Eq. (1.4). Three types of GCS operators and related invariant solutions with more parameters than the ones mentioned above will be pointed out. Our study will be completed by constructing the nontrivial conservation laws admited by the GGSE with arbitrary flow.

The outline of this paper is the following: in Section 2 we expose some basic facts about the GCS method, while in Section 3, we do determine some conditions enabling Eq. (1.4) to admit GCSs. These conditions represent a PDE system which will be solved with the help of the software package Maple. Three classes of group-invariant solutions involving more free parameters than the ones obtained through Lie symmtry method, are pointed out. Several flux surface configurations and the contour plots of magnetic flux associated with the new solutions are displayed. The problem of constructing of nontrivial conservation laws of GGSE, by means of the multiplier's method, will be analyzed in Section 4. The final Section 5 is devoted to conclusions and final remarks about this work.

2. GCS method' summary

Recall some basic facts about the generalized conditional symmetries of nonlinear PDEs. For the case of $(1+1)$-dimensional evolution equations, the main results were pointed out in [15]. Consider a more general $(1+1)$-dimensional PDE of the form:

$$\alpha U_{2t} + K(t,x,U,U_x,...,U_{mx})U_t - R(t,x,U,U_x,...,U_{mx}) = 0, \tag{2.1}$$

where α is a constant, K, R are smooth functions of their arguments and $U_{kx} = \frac{\partial^k U}{\partial x^k}$, $1 \le k \le m$. We also consider the symmetry operator $X = \eta(t,x,U,U_x,U_{2x},...)\frac{\partial}{\partial U}$ and its m-th order prolongation:

$$X^{(m)} = \sum_{k=0}^{m} D_{kx}\eta\frac{\partial}{\partial U_{kx}} + D_t\eta\frac{\partial}{\partial U_t} + D_{2t}\eta\frac{\partial}{\partial U_{2t}}, \tag{2.2}$$

where we make use of the following denotations:

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{m} U_{(k+1)x} \frac{\partial}{\partial U_{kx}},$$
$$D_{(j+1)x} = D_x(D_{jx}),\ D_{0x} = 1,\ j = 0,1,2,...,$$
$$D_t = \frac{\partial}{\partial t} + \sum_{k=0}^{m} U_{(kx)t} \frac{\partial}{\partial U_{kx}},\ D_{2t} = D_t D_t.$$

Definition 1. The vector field (2.2) is said to be a Lie–Bäcklund symmetry of Eq. (2.1) if

$$X^{(m)}[\alpha U_{2t} + K(t,x,U,U_x,...,U_{mx})U_t - N(t,x,U,U_x,...,U_{mx})]\,|_L = 0, \tag{2.3}$$

where L is the set of all the differential consequences of the concerned equation.

Definition 2. The vector field (2.2) is said to be a GCS of Eq. (2.1) if

$$X^{(m)}[\alpha U_{2t} + K(t,x,U,U_x,...,U_{mx})U_t - N(t,x,U,U_x,...,U_{mx})]\,|_{L\cap N} = 0, \tag{2.4}$$

where N denotes the set of all the differential consequences of the equation $\eta = 0$ in respect to x, that is to say:

$$D_{jx}\eta = 0,\ j = 0,1,2,.... \tag{2.5}$$

These general definitions will be applied in the next section of the paper, restricting our investigation to a particular form of Eq. (2.1):

$$U_{2t} = R(t,x,U,U_x,...,U_{mx}). \tag{2.6}$$

3. The GCS method for the Grad-Schafranov equation with arbitrary flow

Let us determine some conditions for Eq. (1.4) such that it could admit second order GCSs. These conditions do lead towards a determining system for the unknown arbitrary functions which would be solved in order to determine the solutions of (1.4) associated with the GCSs

3.1. *The determining system for second-order GCSs*

In this subsection the GCS method enabling to find similarity reductions of PDEs is applied to the master Eq. (1.4). The operator which generates the GCS group admits the second order prolongation:

$$X^{(2)} = \eta \frac{\partial}{\partial U} + (D_x\eta)\frac{\partial}{\partial U_x} + (D_y\eta)\frac{\partial}{\partial U_y} + (D_{2x}\eta)\frac{\partial}{\partial U_{2x}} + (D_{2y}\eta)\frac{\partial}{\partial U_{2y}} + (D_{xy}\eta)\frac{\partial}{\partial U_{xy}}. \tag{3.1}$$

The condition (2.4) written for Eq. (1.4) is:

$$X^{(2)}\left(U_{2x} - \frac{U_x}{x} + U_{2y} + Ex^4 + Fx^2 + G\right)|_{L\cap N} = 0. \tag{3.2}$$

The previous condition is equivalent to the following relation:

$$-\frac{1}{x}D_x\eta + D_{2x}\eta + D_{2y}\eta\,|_{L\cap N} = 0. \tag{3.3}$$

We have also to impose the restriction (2.5), requiring more precisely that

$$D_{jx}\eta = 0,\ j = 1,2.$$

Under these conditions Eq. (1.4) admits the GCSs (3.1) if and only if

$$D_{2y}\eta = 0. \tag{3.4}$$

Remarking that the second order PDE (1.4) depends upon the x variable, we choose to impose that it admits second-order GCSs with the characteristic given by:

$$\eta = U_{2x} - A(U)U_x^2 - B(x,U)U_x - C(x,U). \tag{3.5}$$

We will find the determining system for the unknown functions $A(u)$, $B(x,u)$, $R(r,u)$ which appear in the second order characteristic (3.5).

Taking into account the surface condition $\eta = 0$, we may substitute the derivative U_{2x} by the expression:

$$U_{2x} = A(U)U_x^2 + B(x,U)U_x + C(x,U). \tag{3.6}$$

Consequently, the derivative U_{2y} from master Eq. (1.4) acquires the equivalent form:

$$U_{2y} = \frac{1}{x}U_x - A(U)U_x^2 - B(x,U)U_x - C(x,U) - Ex^4 - Fx^2 - G. \tag{3.7}$$

Starting from the second order GCSs (3.5), the main condition (3.4) becomes:

$$\begin{aligned} &U_{(2x)(2y)} - A^{''}U_y^2U_x^2 - A^{'}U_x^2U_{2y} - 4A'U_xU_yU_{xy} - 2AU_{xy}^2 - 2AU_xU_{x(2y)} \\ &-B_{2U}U_y^2U_x - B_UU_xU_{2y} - 2B_UU_yU_{xy} - BU_{x(2y)} - C_{2U}U_y^2 - C_UU_{2y} = 0. \end{aligned} \tag{3.8}$$

We calculate from (3.7), the derivatives $U_{(2y)x}$ and $U_{(2y)(2x)}$. Then, we substitute them into (3.8) and we make use of (3.6) and (3.7) in order to eliminate U_{2x} and U_{2y}.Therefore, the achieved condition is verified if and only if the coefficient functions of various monomials in derivatives of U are equal to zero. These constraints do lead firstly to $A = 0$, $B_U(x,U) = 0$, $C_{2U} = 0$, that is to say $A = 0$, $B = B(x)$, $C(x,U) = M(x)U + Q(x)$. In addition the still unknown functions $B(x)$, $M(x)$, $Q(x)$ have to satisfy the following ordinary differential equations (ODEs):

$$\begin{aligned} &\frac{2}{x^3} - \frac{B}{x^2} + \frac{B^{'}}{x} - 2BB^{'} - B^{''} - 2M^{'} = 0, \\ &-\frac{2}{x^2}M + \frac{M^{'}}{x} - 2B^{'}M - M^{''} = 0, \\ &-\frac{2Q}{x^2} + \frac{Q^{'}}{x} + 4EBx^3 + 2FBx + EMx^4 + FMx^2 + MG - 2B^{'}Q - Q^{''} - 12Ex^2 - 2F = 0. \end{aligned} \tag{3.9}$$

where “prime” denotes the derivative with respect to x.

The solutions of the system (3.9) do generate the characteristic functions η of the type (3.5) which do correspond to the solutions of Eq. (1.4), some of them expressed in terms of special functions. However, the GCSs approach does lead to some new and interesting solutions for the master Eq. (1.4) that, to our best knowledge, have not been mentioned in literature.

3.2. *Solutions of the Eq. (1.4) associated with second order GCSs*

By solving the determining system (3.9) through the mathematical package Maple, we find three interesting solutions which will be investigated in the next part of the paper.

*Case*1 : The system (3.9) admits the solution:

$$M(x)=0,\ B(x)=\frac{1}{x},\ Q(x)=-Ex^4+\frac{k_1x^2}{2}+k_2, \tag{3.10}$$

where k_1, k_2 represent arbitrary constants. The GCSs are generated now by the operator:

$$X_1=\left(U_{2x}-\frac{U_x}{x}+Ex^4-\frac{k_1x^2}{2}-k_2\right)\frac{\partial}{\partial U}. \tag{3.11}$$

When solving the invariance surface conditiong $\eta=0$, we come to the solution of the GGSE as:

$$U_1(x,y)=-\frac{E}{24}x^6+\frac{k_1}{16}x^4+\frac{k_2[2\ln(x)-1]}{4}x^2+\frac{\beta(y)}{2}x^2+\mu(y). \tag{3.12}$$

By introducing the previous result into the main Eq.(1.4), we do obtain for $\beta(y)$ and $\mu(y)$ an ODE system which admits the solution:

$$\begin{aligned}\beta(y)&=-\left(\frac{k_1}{2}+F\right)y^2+k_3y+k_4,\\ \mu(y)&=-(k_2+G)\frac{y^2}{2}+k_5y+k_6,\end{aligned} \tag{3.13}$$

with k_i, $i=\overline{1,6}$ arbitrary constants.

Consequently, we obtain a 9-parameter family of solutions:

$$\begin{aligned}U_1(x,y)=&-\frac{E}{24}x^6+m_1x^4+\left[2m_2\ln|x|-\left(\frac{F}{2}+4m_1\right)y^2+m_3y+m_4-m_2\right]x^2\\ &-\left(\frac{G}{2}+2m_2\right)y^2+k_5y+k_6,\end{aligned} \tag{3.14}$$

where $k_1=16m_1$, $k_2=4m_2$, $k_3=2m_3$, $k_4=2m_4$.

It is more extensive than the classes of solutions reported in [14] which depend on 5-parameters and describe D-shaped toroidal configurations with plasma flow non-parallel to the magnetic field. The flux surface configuration and the contour plot of magnetic flux corresponding to the solution (3.14) for $E=-2$, $F=-4$, $G=2/5$, $m_2=-20m_1=-25/2$, $k_5=-2m_3=-1/2$, $k_6=4m_4=10$ are represented in Figure 1.

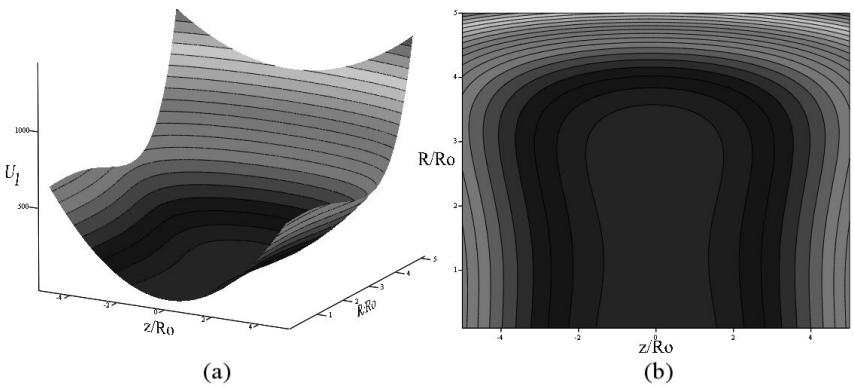

Fig. 1. Equilibrium plasma configutations associated with the solution (3.14) for $E=-2$, $F=-4$, $G=2/5$, $m_2=-20m_1=-25/2$, $k_5=-2m_3=-1/2$, $k_6=4m_4=10$: (**a**) The flux surface configuration. (**b**) The contour plot of magnetic flux.

*Case*2 : The determining system (3.9) also admits the solution:

$$M=0,\ B(x)=\frac{q+r(1+\ln|x|)}{x(q+r\ln|x|)},$$
$$Q(x)=\frac{4c+E\left[r-4(q+r\ln|x|)\right]x^4+2F\left[r-2(q+r\ln|x|)\right]x^2}{4(q+r\ln|x|)}, \tag{3.15}$$

where q, r, c are non-zero parameters.

By solving with respect to (3.15) the invariance surface condition $\eta=U_{2x}-B(x)U_x-Q(x)=0$, we reach to the solution:

$$U_2(y,x)=-\frac{E}{24}x^6-\frac{F}{8}x^4+\left[\left(q-\frac{r}{2}+r\ln(|x|\right)\sigma(y)-\frac{c}{r}\right]\frac{x^2}{2}+\gamma(y), \tag{3.16}$$

which does involve the arbitrary functions $\sigma(y)$ and $\gamma(y)$.

Substituting this result into Eq. (1.4) we derive an ODE system for $\sigma(y)$ and $\gamma(y)$. By solving it the following expressions are obtained:

$$\gamma(y)=-rn_3\frac{y^3}{6}-(rn_4+G)\frac{y^2}{2}+n_1y+n_2,$$
$$\sigma(y)=n_3y+n_4, \tag{3.17}$$

with n_i, $i=\overline{1,4}$ arbitrary constants.

Consequently, the solution (3.16) takes the form:

$$U_2(y,x)=-\frac{E}{24}x^6-\frac{F}{8}x^4+\left[\left(q-\frac{r}{2}+r\ln|x|\right)(n_3y+n_4)-\frac{c}{r}\right]\frac{x^2}{2}$$
$$-rn_3\frac{y^3}{6}-(rn_4+G)\frac{y^2}{2}+n_1y+n_2, \tag{3.18}$$

which admits 9 parameters.

The flux surface configuration and the contour plot of magnetic flux corresponding to the solution (3.18) for $E=1.25$, $F=-1.5$, $G=-1.2$, $n_3=10c=100q=1000r=500$ and $n_1=n_2=n_4=1$ are represented in Figure 2.

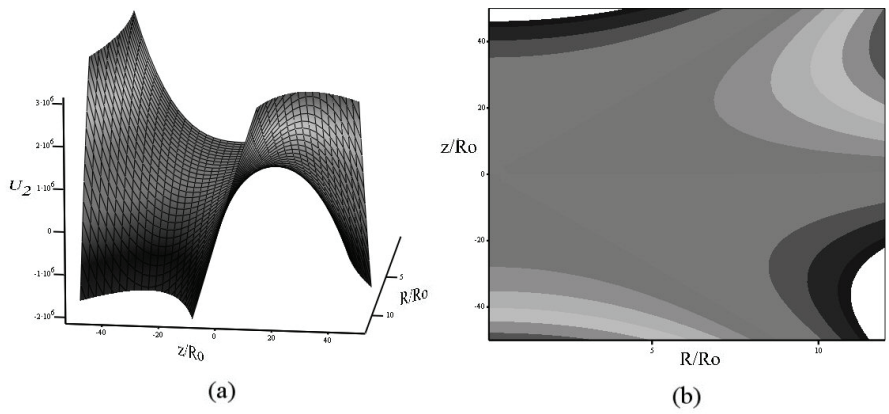

Fig. 2. Equilibrium plasma configutations associated with the solution (3.18) for $E = 1.25$, $F = -1.5$, $G = -1.2$, $n_3 = 10c = 100q = 1000r = 500$ and $n_1 = n_2 = n_4 = 1$: (**a**) The flux surface configuration. (**b**) The contour plot of magnetic flux.

*Case*3 : When the free parameter $G = 2\gamma_s$ from (1.4) becomes zero, the determining system (3.9) admits the solution:

$$Q(x) = 0,\ B(x) = \frac{5}{x},\ M(x) = -\frac{8}{x^2}. \tag{3.19}$$

Under these conditions, the GCS operator becomes:

$$X_3 = \left(U_{2x} - \frac{5U_x}{x} + \frac{8}{x^2}U\right)\frac{\partial}{\partial U}. \tag{3.20}$$

When imposing the condition $\eta = 0$, we come to derive the following solution for the GS equation with arbitrary flow:

$$U_3(y,x) = \zeta(y)x^2 + \chi(y)x^4. \tag{3.21}$$

By substituting it into the governing Eq.(1.4), we reach to the fact that $\zeta(y)$ and $\chi(y)$ ought to satisfy the system of ODEs:

$$\chi'' + E = 0,\ \zeta'' + 8\chi + F = 0,$$

which admits the solution:

$$\begin{aligned}\zeta(y) &= \frac{E}{3}y^4 - \frac{4c_3}{3}y^3 - \frac{8c_4+F}{2}y^2 + c_1y + c_2,\\ \chi(y) &= -\frac{E}{2}y^2 + c_3y + c_4.\end{aligned} \tag{3.22}$$

Consequently, we could associate to the GCS operator (3.20) the following 6-parameter family of solutions of Eq.(1.4):

$$U_3(y,x) = \left[\frac{E}{3}y^4 - \frac{4c_3}{3}y^3 - \frac{8c_4+F}{2}y^2 + c_1y + c_2\right]x^2 + \left[\frac{E}{2}y^2 + c_3y + c_4\right]x^4. \tag{3.23}$$

This is again a generalization of the solutions mentioned in [14]. In order to reduce the number of parameters in the solution (3.23) we can use, for example, make use of the rescaling transformations

$y \to \frac{y}{c_3^{1/5}}$, $x \to \frac{x}{c_3^{1/5}}$. Under such conditions, the solution (3.23) becomes:

$$U_3(y,x) = \left[\frac{\alpha_1}{3}y^4 - \frac{4}{3}y^3 - \left(4\alpha_2 + \frac{\alpha_3}{2}\right)y^2 + \alpha_4 y + \alpha_5\right]x^2 + \left[\frac{\alpha_1}{2}y^2 + y + \alpha_2\right]x^4, \tag{3.24}$$

where $\alpha_1 = \frac{E}{c_3^{6/5}}$, $\alpha_2 = \frac{c_4}{c_3^{4/5}}$, $\alpha_3 = \frac{F}{c_3^{4/5}}$, $\alpha_4 = \frac{c_1}{c_3^{3/5}}$, $\alpha_5 = \frac{c_2}{c_3^{2/5}}$. Two of them do include the physical parameters E and F which had been introduced in Section 1.

The flux surface configuration and the contour plot of magnetic flux corresponding to the solution (3.24) for $E = 10$, $F = 2$, $G = 0$, $\alpha_1 = 22.97$, $\alpha_2 = -17.41$, $\alpha_3 = 3.48$, $\alpha_4 = 0.75$, $\alpha_5 = 13.19$ are displayed in Figure 3.

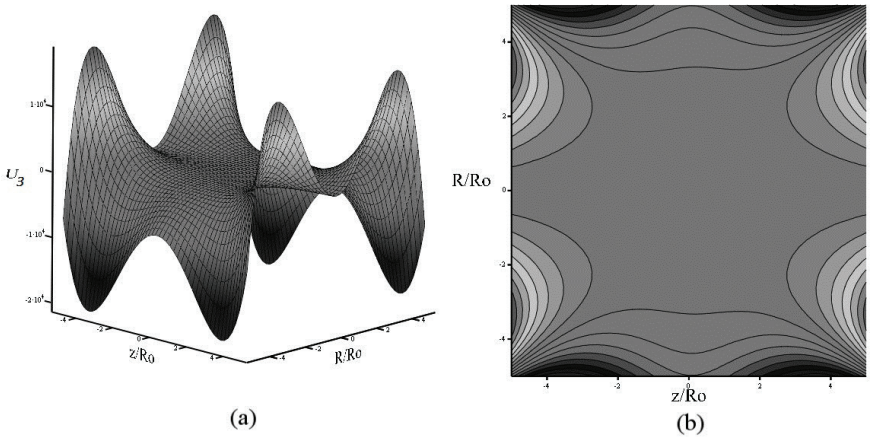

Fig. 3. Equilibrium plasma configutations associated with the solution (3.24) for $E = 10$, $F = 2$, $G = 0$, $\alpha_1 = 22.97$, $\alpha_2 = -17.41$, $\alpha_3 = 3.48$, $\alpha_4 = 0.75$, $\alpha_5 = 13.19$: (**a**) The flux surface configuration. (**b**) The contour plot of magnetic flux.

4. Conservation laws for the Grad-Shafranov equation with arbitrary flow

It is well known that conservation laws are always seen as significant property of a PDE. In many cases, the conservation laws may be regarded as an important indice of iintegrability for a PDE. In this section, the conservation laws of the Eq. (1.4), will be derived by by using the multiplier method.

The first step will be to obtain the conservation laws multipliers. A multiplier Λ of Eq. (1.4) owns the property that:

$$\Lambda[U_{2x} - \frac{1}{x}U_x + U_{2y} + Ex^4 + Fx^2 + G] = D_x P^x + D_y P^y, \tag{4.1}$$

for all functions $U(x,y)$, not only for the solutions of the concerned equation.

Let us consider multipliers of the form $\Lambda = \Lambda(x,\, y, U,\, U_x,\, U_y)$. The right hand side of (4.1) is a divergence expression. Hence, this expression vanishes when applying the Euler operator which

takes the form:

$$E_U = \frac{\partial}{\partial U} - D_x\left(\frac{\partial}{\partial U_x}\right) - D_y\left(\frac{\partial}{\partial U_y}\right) + D_{2x}\left(\frac{\partial^2}{\partial U_{2x}}\right) + D_{2y}\left(\frac{\partial^2}{\partial U_{2y}}\right) + D_{xy}\left(\frac{\partial^2}{\partial U_{xy}}\right) - \ldots.$$

The determining equation for the multiplier $\Lambda = \Lambda(x, y, U, U_x, U_y)$ is:

$$E_U[\Lambda(U_{2x} - \frac{1}{x}U_x + U_{2y} + Ex^4 + Fx^2 + G)] = 0. \tag{4.2}$$

The expansion of (4.2) does generate the following determining system for multipliers:

$$\begin{gathered}
\Lambda_{2p} + \Lambda_{2v} = 0, \\
2\Lambda_U + \Lambda_{xp} + p\Lambda_{pU} + \frac{p\Lambda_{2p}}{x} - (Ex^4 + Fx^2 + G)\Lambda_{2p} + \frac{2}{x}\Lambda_p - \Lambda_{yv} + v\Lambda_{vU} = 0, \\
2\Lambda_U - \Lambda_{xp} - p\Lambda_{pU} + \frac{p\Lambda_{2v}}{x} - (Ex^4 + Fx^2 + G)\Lambda_{2v} + \Lambda_{yv} - v\Lambda_{vU} = 0, \\
\left(\frac{p}{x} - Ex^4 - Fx^2 - G\right)\Lambda_{pv} + \frac{\Lambda_v}{x} + \Lambda_{xv} + p\Lambda_{vU} + \Lambda_{yp} + v\Lambda_{pU} = 0, \\
\left(Ex^4 + Fx^2 + G\right)(\Lambda_U - \Lambda_{xp} - p\Lambda_{pU} - \Lambda_{yv} - v\Lambda_{vU}) - 2x\Lambda_p(2Ex^2 + F) + \\
x^{-1}\left[p(\Lambda_{xp} + \Lambda_{yv} + v\Lambda_{vU}) + p^2\Lambda_{pU} + \Lambda_x\right] - x^{-2}(p\Lambda_p + \Lambda) + \\
\Lambda_{2x} + \Lambda_{2y} + 2(p\Lambda_{xU} + v\Lambda_{yU}) + \Lambda_{2U}(p^2 + v^2) = 0,
\end{gathered} \tag{4.3}$$

where we make use the denotations $v = U_y$, $p = U_x$.

Through the mathematical package Maple, we obtain the general multiplier:

$$\begin{gathered}
\Lambda = (c_1 y + c_2)\left(\frac{3}{4}Gx\ln(x) + \frac{11Ex^5}{48} + \frac{7Fx^3}{16} + p\right) - \left[\frac{3}{4}G(c_1 y + c_2) + c_1 v\right]\frac{x}{2} + \\
\frac{(c_1 y + 2c_2)vy - (c_1 y + c_2)U + 2c_3 v}{2x}.
\end{gathered} \tag{4.4}$$

From (4.4), we are able to point out three independent multipliers:

i) For $c_3 = 1$, $c_2 = c_1 = 0$, we reach to

$$\Lambda_1 = \frac{v}{x}, \tag{4.5}$$

ii) For $c_2 = 1$, $c_1 = c_3 = 0$,we obtain

$$\Lambda_2 = \frac{x}{2}\left(\frac{3}{2}G\ln(x) + \frac{11Ex^4}{24} + \frac{7Fx^2}{8} - \frac{3}{4}G\right) + \frac{2vy - U}{2x} + p, \tag{4.6}$$

iii) For $c_1 = 1$, $c_2 = c_3 = 0$, we generate

$$\Lambda_3 = \frac{x}{2}\left[y\left(\frac{3}{2}G\ln(x) + \frac{11Ex^4}{24} + \frac{7Fx^2}{8} - \frac{3}{4}G\right) - v\right] + y\left(\frac{vy - U}{2x} + p\right). \tag{4.7}$$

By substituting the multipliers in the condition (4.1), expanding the divergence expression, equating the coefficient functions of U_{2x}, U_{2y}, U_{xy} and vanishing the expression without second

derivatives of U, we generate the PDE systems for the components P^x, P^y of the conserved vectors. When solving these systems, three nontrivial conservation laws are generated.

The first conserved vector of Eq. (1.4) associated with the multiplier (4.5) does admit the components:

$$\begin{aligned} P_1^y &= \frac{v^2-p^2}{2x} - p\left[G\ln(x) + \frac{Fx}{2} + \frac{Ex^3}{4} - \int f_U dx - g(y,U)\right] + f(x,y,U), \\ P_1^x &= \frac{vp}{x} + v\left[G\ln(x) + \frac{Fx^2}{2} + \frac{Ex^4}{4} - \int f_U dx - g(y,U)\right] - \int f_y dx + \\ &\quad \int\left[\int 2\left(f_{yU} + v f_{2U}\right)dx + g_y\right]dU + h(y), \end{aligned} \tag{4.8}$$

with $f(x,y,U)$, $g(y,U), h(y)$ arbitrary functions.

Remark. A particular solution of the analyzed model (1.4) provided by the conserved densities (4.8) may be derived, for example, through imposing the condition $f(x,y,U) = g(y,U) = h(y) = 0$ in (4.8) and then through solving the partial differential system which does include the master Eq. (1.4) and the differential constraints $D_x P_1^x = 0$ and $D_y P_1^y = 0$. The result consists in the following particular solution:

$$U(x,y) = \frac{G}{2}x^2\ln(x) - \frac{E}{24}x^6 - \frac{F}{8}x^4 + \frac{(G+2m_1)}{4}x^2 + m_2 y + m_3, \tag{4.9}$$

with m_i, $i = \overline{1,3}$ arbitrary constants. In fact it does represent a particular case of solution (3.14).

The second conserved vector of Eq. (1.4) associated with the multiplier (4.6) does admit the components:

$$\begin{aligned} P_2^y &= \int f_x(x,y,U)dU + fp + \sigma(y,x) + \frac{3}{4}Gxv\ln(x) + \frac{11Ev}{48}x^5 + \frac{7Fv+16EyU}{16}x^3 + \\ &\quad \frac{8FyU-3Gv}{8}x + pv + \frac{(v^2-p^2)y + (2Gy-v)U}{2x}, \\ P_2^x &= -\int \sigma_y(y,x)dx - \int f_y(x,y,U)dU - fv + h(y) + \frac{11E^2}{480}x^{10} + \frac{EF}{12}x^8 + \\ &\quad \frac{GE\left(36\ln(x)-13\right)+21F^2}{288}x^6 + \frac{11Ep}{48}x^5 + \frac{GF\left(6\ln(x)-1\right)-12EU}{32}x^4 + \frac{7Fp}{16}x^3 + \\ &\quad \frac{3\left[G^2\left(\ln(x)-1\right)-2FU\right]}{8}x^2 + \frac{3\left[Gp\left(2\ln(x)-1\right)\right]}{8}xr + \frac{(2yv-U)p}{2x} + \\ &\quad \frac{p^2-v^2}{2} - \frac{GU(3\ln(x)-2)}{2}, \end{aligned} \tag{4.10}$$

with $f(x,y,U)$, $\sigma(y,x)$, $h(y)$ arbitrary functions.

The third conserved vector of Eq. (1.4) associated with the multiplier (4.7) does admit the components:

$$
\begin{aligned}
P_3^y &= p\int f_U(x,y,U)dx + f + pg(y,U) + x^6\frac{35Ep}{288} + \frac{x^5}{48}11Eyv + \frac{px^4}{64}(15F-8Ey^2) + \frac{x^3}{16}7Fyv + \\
&\frac{px^2}{8}\left[G(3\ln(x)-1)-2Fy^2\right] + \frac{x}{8}[2(p^2-v^2)+36Gyv(2\ln(x)-1)] + \frac{y}{4x}\left[y(v^2-p^2)-2vU\right] + \\
&\frac{p}{2}[2yv-(Gy^2+U)\ln(x)], \\
P_3^x &= \int\left[\int -f_{yU}(x,y,U)dx\right]dU - \int g_y(y,U)dU - v\int f_U dx - vg(y,U) + x^{10}\frac{11E^2y}{480} + x^8\frac{EFy}{12} + \\
&\frac{x^6}{288}[EGy(36\ln(x)-13)+21F^2y-35Ev] + \frac{x^5}{48}11Eyp + \frac{x^4}{64}[2GFy(6\ln(x)-1)-8EyU- \\
&15Fv+8Evy^2] + \frac{x^3}{16}7Fyp + \frac{x^2}{8}[3G^2y(\ln(x)-1)-Gv(3\ln(x)-1)+2Fy(vy-U)] + \\
&\frac{x}{8}[3Gyp(2\ln(r)-1)-4pv] + \frac{py}{2x}(vy-U) + \frac{1}{2}[Gy(vy-U)\ln(x)+y(2GU+p^2-v^2)+ \\
&vU\ln(x)],
\end{aligned}
\tag{4.11}
$$

with $f(x,y,U)$, $g(y,U)$ arbitrary functions.

5. Concluding remarks

Group symmetry methods, with their various aspects, do offer opportunities for the study of differential equations arising from modern physical theories. In the present paper, the nonclassical GCS method which enables us to find invariant solutions of PDEs has been applied to Eq. (1.4) that describes toroidal plasma configuration with non-parallel flow which corresponds to physically appropriate choices (1.3) of the free-function terms involved in the nonlinear GGSE (1.1). We investigated the conditions enabling Eq. (1.4) to admit a special class of second order GCSs with the characteristic (3.5). By solving the invariant surface condition $\eta = 0$ in some specific cases, three families of new analytical solutions, not yet reported in literature to our best of knowledge, have been generated: two 9-parametric classes (3.14), (3.18) and a 5-parametric one (3.23). They stand as complements of the 5-parametric solutions obtained recently in [14] through making use of classical Lie symmetry method. For each class of solutions, the specific flux surface configurations as well as the contour plot of magnetic flux have been displayed. Whatever given level set $U = const.$ may be taken into consideration as a plasma boundary.

The group invariant solutions we have obtained may be useful in constructing the magnetic flux and the pressure profile for an equilibrium plasma configuration with a toroidal symmetry. They may be particularly relevant when it comes to the development of some analytical models for tokamak fluid stability as well as in the tests performed for the numerical simulations for tokamak equilibrium. In addition, they can be useful into some steady states of astrophysical interest. For example, the contour lines of the magnetic flux represented in Figure 1 could locally model some arcade-shaped structures generated by two sources within the photospherical plane.

It is known that conservation laws do contain important information about the physical properties of a model under consideration, do provide conserved norms which are used in the analysis of solutions and in the development of numerical methods. Due to this fact, the complete set of three nontrivial conservation laws admitted by (1.4), associated respectively to multipliers $\Lambda_i(x, y, U, U_x, U_y)$ from (4.5), (4.6), (4.7) which admit linear dependences upon the first derivatives U_x, U_y, has

been pointed out. Let us remark that all the components of the conserved vectors involve arbitrary functions. By vanishing the arbitrary functions from the conserved quantities P_1^x, P_1^y in (4.8) and by solving the PDE system that involve Eq. (1.4), $D_x P_1^x = 0$ and $D_y P_1^y = 0$, a 6−parameter particular solution (4.9) of the master Eq. (1.4) is discovered. By choosing the parameter $m_2 = 0$ and a suitable value for m_1, this solution does become identical with a particular one reported in [14].

It would be interesting to generalize the analysis of the present paper to some other choices of the free-function terms that appear into the GGSE (1.1) with arbitrary flow in order to identify higher order generalized conditional symmetries, related equilibrium configurations and new conservation laws. This analysis constitutes the object of a forthcoming paper.

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