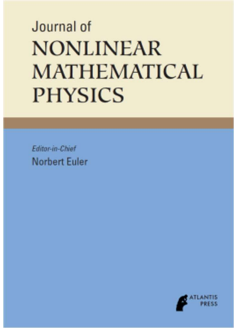

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

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To cite this article: Shuzhi Liu, Lihong Wang, Wei Liu, Deqin Qiu, Jingsong He (2017) The determinant representation of an *N*-fold Darboux transformation for the short pulse equation, Journal of Nonlinear Mathematical Physics 24:2, 183–194, DOI: https://doi.org/10.1080/14029251.2017.1306947

To link to this article: https://doi.org/10.1080/14029251.2017.1306947

Published online: 04 January 2021

Journal of Nonlinear Mathematical Physics, Vol. 24, No. 2 (2017) 183–194

The determinant representation of an N-fold Darboux transformation for the short pulse equation

Shuzhi Liu[1], Lihong Wang[2], Wei Liu[3], Deqin Qiu[4], Jingsong He[1*]

[1] *Department of Mathematics, Ningbo University, Ningbo, Zhejiang 315211, P. R. China*
[2] *School of Mechanical Engineering & Mechanics, Ningbo University, Ningbo 315211, P. R. China*
[3] *College of Mathematic and Information Science, Shandong Technology and Business University, Yantai, 264005, P. R. China*
[4] *College of Mathematics and Statistics, Jishou University, Hunan 416000, P. R. China*
hejingsong@nbu.edu.cn; jshe@ustc.edu.cn

Received 22 October 2016

Accepted 26 January 2017

We present an explicit representation of an N-fold Darboux transformation $\widetilde{T}_N$ for the short pulse equation, by the determinants of the eigenfunctions of its Lax pair. In the course of the derivation of $\widetilde{T}_N$, we show that the quasi-determinant is avoidable, and it is contrast to a recent paper (J. Phys. Soc. Jpn. 81 (2012), 094008) by using this relatively new tool which was introduced to study noncommutative mathematical objectives. $\widetilde{T}_N$ produces new solutions $u^{[N]}$ and $x^{[N]}$ which are expressed by ratios of two corresponding determinants. We also obtain the soliton solutions, which have a variable trajectory, of the short pulse equation from new "seed" solutions.

Keywords: Darboux transformation; short pulse equation; soliton; hodograph transformation.

2000 Mathematics Subject Classification: 5Q51, 35Q60, 37K10, 37K35, 35C08

1. introduction

In recent years, propelled by the demand of the high bit-rate data transmission in one channel, the few-cycle pulses at attosecond scale has been generated successfully in experiments, which arouses extremely the efforts of the research of this field [1, 2, 17]. However, this leads to a serious problem in the research of the nonlinear fibre optics, i.e., the breakdown of the slowly varying envelope approximation (SVEA), which is used to derive the nonlinear Schrödinger (NLS) equation for picosecond pulse propagation in fiber, for pulses which are shorter than 100 femtosecond or few cycle pulses [19]. This fact shows the NLS can not be a model equation of the ultra-short pulse. Thus, it is natural to derive new models for ultra-short pulse by using non-SVEA approach. In this perspective, a new model of ultra-short pulse, i.e.,

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \tag{1.1}$$

has been established at 2004 [5, 32] without the using of the SVEA approach, and which is now called SP equation. Here $u = u(x,t)$ represents the magnitude of the electric field and subscripts x and t denote partial differentiation. As early expectation, Chung and coworkers proved that the NLS equation approximation becomes less accurate as the pulse length shortens, whereas the SP equation provides a better approximation to the solution of Maxwell's equations [5] for this case. The SP equation is an integrable system because it has a Lax pair which is found to be of the

Wadati-Konno-Ichikawa (WKI) type [28]. The Hamiltonian structure, the hierarchy of higher-order flows and a new Lax pair involving pseudo-differential operators of the SP equation are presented in ref. [4], which further shows its the integrable properties. It is very crucial to find that the SP equation can be converted to the sine-Gordon (SG) model by an implicit transformation [29], and thus three kinds of solution, i.e., one-loop, two-loop and breather(pulse) solutions, are constructed from the corresponding solutions of the SG equation. Later, a suitable hodograph transformation [22], which is equivalent essentially to the transformation given by eqs.(8) and (9) in ref. [29], is introduced to establish the connection between the SP and the SG equation. The N-loop and M-breather solutions of the SP equation are constructed by the hodograph transformation from the solitons of the SG equation. To this destination, Matsuno has calculated out analytically a tedious integration of coordinate transformation $x(y,t) = \int^y \cos\phi dy$, i.e. eq. (2.18) of ref. [22], in which ϕ is a solution of the SG equation, and this idea is used to get periodic solutions of the SP equation [23] and the multisoliton solutions of the multi-component SP equation [24]. There exist several extensions of the SP equation including the discrete SP equation [6], the two-component SP [7,34], multi-component SP [24], stochastic SP [18], (2+1)-dimensional SP [31], complex SP [8,21], etc. Very recently, the long-time asymptotical behavior of the solution for the SP equation is given by the Riemann-Hilbert approach [3,35].

Although the hodograph transformation introduced by Matsuno has been successfully used to get several kinds of interesting solutions [22–24] of the SP equation, it is still limited strongly by the difficulty in the integration of $x = x(y,t)$. In general, this integration is not calculable analytically. Therefore it is an interesting problem to solve simultaneously the unknown electric field u and $x(y,t)$ without the appearing of the above irritated integration, in order to get other more solutions. Indeed, this has been done partially by solving an equivalent coupled system of u and $x(y,t)$ (see eqs.(10) and (11) in ref. [30]) with the help of the Darboux transformation (DT) [30]. Note that authors of ref. [30] have avoided successfully the use of any integration and the solution of the SG, but there are several items needed to be improved. In this paper, we shall further study the solution of the SP equation motivated by the following problems.

- Can the quasi-determinant used in derivation of multi-fold DT [30] be avoided? The quasi-determinant is a very powerful and relatively new tool to study the noncommutative mathematical objectives [9], so we suspect naturally whether it is necessary to study a commutative system such as a coupled equations of u and $x(y,t)$ in ref. [30]. Can we derive the N-fold DT of the SP equation using the conventional determinant instead of the quasi-determinant?
- Can we develop a determinant representation $\widetilde{T}_N$ of the N-fold DT for the SP equation? Ref. [30] just provides a determinant form of the $u(N)$ and $x(N)$ generated by the arbitrary N-fold DT. Recently, the determinant representation of the N-fold DT is an effective tool to construct the rogue waves in many soliton equations [13,14,16,26,27,36–38].
- Can we provide an unified determinant expression of the $u(N)$ and $x(N)$ generated by the N-fold DT? Note that there exists two expressions of $u(N)$ and $x(N)$ for two cases when N is an odd or even number respectively [30]. To more details, see eqs. (73) and (74) in ref. [30].
- Construct new solutions of $u(N)$ and $x(N)$ using the DT from different "seed" solutions by comparing with a special "seed" solution $u = 0$ and $x = X$ [30].

The aim of this paper is to solve above problems. The paper is structured as follows. In Section 2, we derive the determinant representation of the N-fold DT without appearing the quasi-determinant,

and then use it to construct an unified expression of the $u(N)$ and $x(N)$ for arbitrary N. In Section 3, we provide explicit formulae of the new soliton solutions generated by the DT from several new "seed" solutions. Finally, conclusions and discussions are given in Section 4.

2. n-order Darboux transformation of the short pulse equation

The DT is a powerful tool to solve integrable equations [10, 11, 20, 25]. In this section, we shall derive a determinant representation $\widetilde{T}_N$ of the N-fold DT for the SP equation, and then use it to construct solutions. First of all, we would like to recall the equivalent equations [30] of the SP equation. The SP equation has a conservation law $w_t = (\frac{1}{2}u^2 w)_x$, here $w^2 = 1 + (u_x)^2$. Then ref. [22] introduced the hodograph transform which maps the independent variables (x,t) into new variables (X,T) through the following system of differential one-form

$$dX = w\,dx + \frac{1}{2}u^2 w\,dt, \qquad dT = dt. \tag{2.1}$$

Set $\frac{1}{w} = \frac{\partial x(X,T)}{\partial X}$, then the SP equation is transformed into the following equivalent coupled equations [30] of u and $x(X,T)$

$$\frac{\partial^2}{\partial X \partial T} x(X,T) + \left(\frac{\partial}{\partial X} u(X,T)\right) u(X,T) = 0, \tag{2.2}$$

$$\frac{\partial^2}{\partial X \partial T} u(X,T) - \left(\frac{\partial}{\partial X} x(X,T)\right) u(X,T) = 0, \tag{2.3}$$

which is the compatibility condition of the following Lax pair

$$\Phi_X = P\Phi, \quad \Phi_T = Q\Phi, \tag{2.4}$$

$$P = \begin{pmatrix} \lambda \frac{\partial}{\partial X} x(X,T) & \lambda \frac{\partial}{\partial X} u(X,T) \\ \lambda \frac{\partial}{\partial X} u(X,T) & -\lambda \frac{\partial}{\partial X} x(X,T) \end{pmatrix}, Q = \begin{pmatrix} \frac{1}{4}\lambda^{-1} & -\frac{1}{2}u(X,T) \\ \frac{1}{2}u(X,T) & -\frac{1}{4}\lambda^{-1} \end{pmatrix}.$$

Here $\Phi(X,T,\lambda) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ be an eigenfunction of the Lax pair equations in (2.4) corresponding to eigenvalue λ, and $\Phi_k = \Phi(X,T,\lambda_k) = \begin{pmatrix} f_{k1} \\ f_{k2} \end{pmatrix}$ be an eigenfunction associated with eigenvalue λ_k. From now on, (2.2) and (2.3) are called the hodograph equivalent short pulse(HESP) equations.

We are now in a position to derive the DT of the HESP equations, i.e. (2.2) and (2.3), instead of the solving directly it. We first construct the one-fold and two-fold with details, then the N-fold DT by iteration and matrix multiplication. Suppose there exists a transformation $\widetilde{T}$ acting on eigenfunction of the SP equation, that is

$$\Phi^{[1]} = \widetilde{T}\Phi, \tag{2.5}$$

and preserving the Lax pair

$$\Phi_X^{[1]} = P^{[1]}\Phi^{[1]}, \quad \Phi_T^{[1]} = Q^{[1]}\Phi^{[1]}, \tag{2.6}$$

where $P^{[1]}$ and $Q^{[1]}$ have the same form as P and Q except that $u(X,T)$ and $x(X,T)$ are replaced by $u^{[1]}(X,T)$ and $x^{[1]}(X,T)$. As given in [10,11,20,25], $\Phi^{[1]}_{XT}=\Phi^{[1]}_{TX}$ implies

$$\widetilde{T}_X+\widetilde{T}P-P^{[1]}\widetilde{T}=0, \qquad \widetilde{T}_T+\widetilde{T}Q-Q^{[1]}\widetilde{T}=0, \tag{2.7}$$

which are governing equations to determine the relations between new solutions $(u^{[1]},x^{[1]})$ and the "seed" solutions (u,x), and also the relations between the elements of matrix $\widetilde{T}$. For example, see $\widetilde{T}_1$ next subsection.

2.1. *One-fold Darboux transformation*

Similar to the widely used form of the DT [10,11,20,25], set the one-fold DT $\widetilde{T}_1$ as

$$\widetilde{T}_1=\begin{pmatrix} a_1^{[1]} & b_1^{[1]} \\ c_1^{[1]} & d_1^{[1]} \end{pmatrix}\lambda+\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.8}$$

Here $a_1^{[1]},b_1^{[1]},c_1^{[1]},d_1^{[1]},a,b,c,d$ are the unknown functions of X and T, which will be given according to the eq.(2.7) and the kernel of $\widetilde{T}_1$. Substituting the matrix of $\widetilde{T}_1$ into eq.(2.7), then comparing the coefficients of λ^j, $j=2,1,0,-1$, then it infers that $\widetilde{T}$ is in the form of

$$\widetilde{T}_1(\lambda)=\begin{pmatrix} a_1^{[1]} & b_1^{[1]} \\ b_1^{[1]} & -a_1^{[1]} \end{pmatrix}\lambda+\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.9}$$

and new solutions are

$$u^{[1]}=u+b_1^{[1]},\ \ x^{[1]}=x+a_1^{[1]}. \tag{2.10}$$

Next, it is a crucial step to express $a_1^{[1]}$ and $b_1^{[1]}$ by the eigenfunctions Φ_k associated with seed solutions u and x. Like other well-known integrable systems [10,11,20,25], this can be implemented by using the kernel of $\widetilde{T}_1$. Specifically, solving algebraic equations $\widetilde{T}_1(\lambda;\lambda_1)\Phi_1|_{\lambda=\lambda_1}=0$ by Cramer's rule, implies

$$a_1^{[1]}=\frac{\begin{vmatrix} \lambda_1 f_{12} & f_{11} \\ \lambda_1 f_{11} & f_{12} \end{vmatrix}}{|W_2|}, \quad b_1^{[1]}=-\frac{\begin{vmatrix} \lambda_1 f_{11} & f_{11} \\ -\lambda_1 f_{12} & f_{12} \end{vmatrix}}{|W_2|}, \tag{2.11}$$

in which

$$W_2=\begin{pmatrix} \lambda_1 f_{11} & \lambda_1 f_{12} \\ -\lambda_1 f_{12} & \lambda_1 f_{11} \end{pmatrix}.$$

Substituting $a_1^{[1]}$ and $b_1^{[1]}$ into eq.(2.9), we get a simple determinant representation of the one-fold DT as

$$\widetilde{T}_1=\widetilde{T}_1(\lambda;\lambda_1)=\frac{1}{|W_2|}\begin{pmatrix} (\widetilde{T}_1)_{11} & (\widetilde{T}_1)_{12} \\ (\widetilde{T}_1)_{21} & (\widetilde{T}_1)_{22} \end{pmatrix}, \tag{2.12}$$

with $(\widetilde{T}_1)_{11}=\begin{vmatrix} \xi_{11} & 1 \\ W_2 & \eta_1 \end{vmatrix}, (\widetilde{T}_1)_{12}=\begin{vmatrix} \xi_{12} & 0 \\ W_2 & \eta_1 \end{vmatrix}, (\widetilde{T}_1)_{21}=(\widetilde{T}_1)_{12}, (\widetilde{T}_1)_{22}=(\widetilde{T}_1)_{11}|_{\lambda\to-\lambda}, \xi_{11}=(\lambda,0), \xi_{12}=(0,\lambda), \eta_1=(f_{11},f_{12})^T$. Furthermore, the determinant representation of new solutions

of the SP equation are

$$u^{[1]}(X,T)=u(X,T)-\frac{\begin{vmatrix}\lambda_1 f_{11} & f_{11}\\ -\lambda_1 f_{12} & f_{12}\end{vmatrix}}{|W_2|},\quad x^{[1]}(X,T)=x(X,T)+\frac{\begin{vmatrix}\lambda_1 f_{12} & f_{11}\\ \lambda_1 f_{11} & f_{12}\end{vmatrix}}{|W_2|}, \tag{2.13}$$

obtained from eq.(2.10) and eq.(2.11). So we have derived the one-fold DT without the using of the quasi-determinant, and this shows it is indeed avoidable in the construction of the DT of the SP equation.

2.2. *Two-fold Darboux transformation*

Through the iteration of $\widetilde{T}_1$, or equivalently matrix multiplication of $\widetilde{T}_1$ given by eq.(2.9), the two-fold DT for the SP equation should be of the following form

$$\widetilde{T}_2(\lambda)=\begin{pmatrix}a_2^{[2]} & b_2^{[2]}\\ -b_2^{[2]} & a_2^{[2]}\end{pmatrix}\lambda^2+\begin{pmatrix}a_2^{[1]} & b_2^{[1]}\\ b_2^{[1]} & -a_2^{[1]}\end{pmatrix}\lambda+\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{2.14}$$

and its kernel is two dimensional. As we have done for one-fold DT, solving $\widetilde{T}_2(\lambda)\Phi_i|_{\lambda=\lambda_i}=0, i=1,2$ by the Cramer's rule, yields

$$a_2^{[2]}=\frac{\begin{vmatrix}\lambda_1^2 f_{12} & \lambda_1 f_{11} & \lambda_1 f_{12} & f_{11}\\ -\lambda_1^2 f_{11} & -\lambda_1 f_{12} & \lambda_1 f_{11} & f_{12}\\ \lambda_2^2 f_{22} & \lambda_2 f_{21} & \lambda_2 f_{22} & f_{21}\\ -\lambda_2^2 f_{21} & -\lambda_2 f_{22} & \lambda_2 f_{21} & f_{22}\end{vmatrix}}{|W_4|},\quad a_2^{[1]}=\frac{\begin{vmatrix}\lambda_1^2 f_{11} & \lambda_1^2 f_{12} & \lambda_1 f_{12} & f_{11}\\ \lambda_1^2 f_{12} & -\lambda_1^2 f_{11} & \lambda_1 f_{11} & f_{12}\\ \lambda_2^2 f_{21} & \lambda_2^2 f_{22} & \lambda_2 f_{22} & f_{21}\\ \lambda_2^2 f_{22} & -\lambda_2^2 f_{21} & \lambda_2 f_{21} & f_{22}\end{vmatrix}}{|W_4|}, \tag{2.15}$$

$$b_2^{[2]}=-\frac{\begin{vmatrix}\lambda_1^2 f_{11} & \lambda_1 f_{11} & \lambda_1 f_{12} & f_{11}\\ \lambda_1^2 f_{12} & -\lambda_1 f_{12} & \lambda_1 f_{11} & f_{12}\\ \lambda_2^2 f_{21} & \lambda_2 f_{21} & \lambda_2 f_{22} & f_{21}\\ \lambda_2^2 f_{22} & -\lambda_2 f_{22} & \lambda_2 f_{21} & f_{22}\end{vmatrix}}{|W_4|},\quad b_2^{[1]}=-\frac{\begin{vmatrix}\lambda_1^2 f_{11} & \lambda_1^2 f_{12} & \lambda_1 f_{11} & f_{11}\\ \lambda_1^2 f_{12} & -\lambda_1^2 f_{11} & -\lambda_1 f_{12} & f_{12}\\ \lambda_2^2 f_{21} & \lambda_2^2 f_{22} & \lambda_2 f_{21} & f_{21}\\ \lambda_2^2 f_{22} & -\lambda_2^2 f_{21} & -\lambda_2 f_{22} & f_{22}\end{vmatrix}}{|W_4|}, \tag{2.16}$$

where $W_4=\begin{pmatrix}\lambda_1^2 f_{11} & \lambda_1^2 f_{12} & \lambda_1 f_{11} & \lambda_1 f_{12}\\ \lambda_1^2 f_{12} & -\lambda_1^2 f_{11} & -\lambda_1 f_{12} & \lambda_1 f_{11}\\ \lambda_2^2 f_{21} & \lambda_2^2 f_{22} & \lambda_2 f_{21} & \lambda_2 f_{22}\\ \lambda_2^2 f_{22} & -\lambda_2^2 f_{21} & -\lambda_2 f_{22} & \lambda_2 f_{21}\end{pmatrix}$. Taking these elements back into eq. (2.14), it provides the determinant representation of the two-fold DT

$$\widetilde{T}_2(\lambda;\lambda_1,\lambda_2)=\frac{1}{|W_4|}\begin{pmatrix}(\widetilde{T}_2)_{11} & (\widetilde{T}_2)_{12}\\ (\widetilde{T}_2)_{21} & (\widetilde{T}_2)_{22}\end{pmatrix}, \tag{2.17}$$

with $(\widetilde{T}_2)_{11}=\begin{vmatrix}\xi_{21} & 1\\ W_4 & \eta_2\end{vmatrix}, (\widetilde{T}_2)_{12}=\begin{vmatrix}\xi_{22} & 0\\ W_4 & \eta_2\end{vmatrix}, (\widetilde{T}_2)_{21}=(\widetilde{T}_2)_{12}|_{\lambda^2\to-\lambda^2}, (\widetilde{T}_2)_{22}=(\widetilde{T}_2)_{11}|_{\lambda\to-\lambda}, \xi_{21}=(\lambda^2,0,\lambda,0), \xi_{22}=(0,\lambda^2,0,\lambda), \eta_2=(f_{11},f_{12},f_{21},f_{22})^T$. At the same time, new solutions of the SP equation generated by the two-fold DT are

$$u^{[2]}(X,T)=u(X,T)+b_2^{[1]},\quad x^{[2]}(X,T)=x(X,T)+a_2^{[1]}. \tag{2.18}$$

2.3. *N-fold DT of the short pulse equation*

Now we present the N-fold DT for the SP equation by the similar method as above. By N-times iteration of $\widetilde{T}_1$, or N-times multiplication of $\widetilde{T}_1$ given by eq.(2.9), the N-fold DT should be the following form

$$\widetilde{T}_N(\lambda)=\begin{pmatrix} a_N^{[N]} & b_N^{[N]} \\ -b_N^{[N]} & a_N^{[N]} \end{pmatrix}\lambda^N+\begin{pmatrix} a_N^{[N-1]} & b_N^{[N-1]} \\ b_N^{[N-1]} & -a_N^{[N-1]} \end{pmatrix}\lambda^{N-1}+\cdots+\begin{pmatrix} a_N^{[1]} & b_N^{[1]} \\ b_N^{[1]} & -a_N^{[1]} \end{pmatrix}\lambda+\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ N \text{ is even,} \tag{2.19}$$

$$\widetilde{T}_N(\lambda)=\begin{pmatrix} a_N^{[N]} & b_N^{[N]} \\ b_N^{[N]} & -a_N^{[N]} \end{pmatrix}\lambda^N+\begin{pmatrix} a_N^{[N-1]} & b_N^{[N-1]} \\ -b_N^{[N-1]} & a_N^{[N-1]} \end{pmatrix}\lambda^{N-1}+\cdots+\begin{pmatrix} a_N^{[1]} & b_N^{[1]} \\ b_N^{[1]} & -a_N^{[1]} \end{pmatrix}\lambda+\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ N \text{ is odd.} \tag{2.20}$$

Solving algebraic equations $\widetilde{T}_N(\lambda)\Phi_j|_{\lambda=\lambda_j}=0, j=1,2,\cdots,N$, it yields an unified determinant representation of the N-fold DT for above two cases, i.e.,

$$\widetilde{T}_N(\lambda)=\frac{1}{|W_{2N}|}\begin{pmatrix} (\widetilde{T}_N)_{11} & (\widetilde{T}_N)_{12} \\ (\widetilde{T}_N)_{21} & (\widetilde{T}_N)_{22} \end{pmatrix}, \tag{2.21}$$

with$(\widetilde{T}_N)_{11}=\begin{vmatrix} \xi_{N1} & 1 \\ W_{2N} & \eta_N \end{vmatrix}, (\widetilde{T}_N)_{12}=\begin{vmatrix} \xi_{N2} & 0 \\ W_{2N} & \eta_N \end{vmatrix}$,
$(\widetilde{T}_N)_{21}=(\widetilde{T}_N)_{12}|_{\lambda^{2j}\to-\lambda^{2j},j\in\mathbb{Z}_+}, (\widetilde{T}_N)_{22}=(\widetilde{T}_N)_{11}|_{\lambda^{2j+1}\to-\lambda^{2j+1},j\in\mathbb{Z}_+}$, $\xi_{N1}=(\lambda^N,0,\lambda^{N-1},0,\cdots,\lambda,0)$, $\xi_{N2}=(0,\lambda^N,0,\lambda^{N-1},\cdots,0,\lambda)$, $\eta_N=(f_{11},f_{12},f_{21},f_{22},\cdots,f_{N1},f_{N2})^T$,

$$W_{2N}=\begin{pmatrix} \lambda_1^N f_{11} & \lambda_1^N f_{12} & \lambda_1^{N-1} f_{11} & \lambda_1^{N-1} f_{12} & \cdots & \lambda_1^2 f_{11} & \lambda_1^2 f_{12} & \lambda_1 f_{11} & \lambda_1 f_{12} \\ (-1)^N\lambda_1^N f_{12} & (-1)^{N-1}\lambda_1^N f_{11} & (-1)^{N-1}\lambda_1^{N-1} f_{12} & (-1)^{N-2}\lambda_1^{N-1} f_{11} & \cdots & \lambda_1^2 f_{12} & -\lambda_1^2 f_{11} & -\lambda_1 f_{12} & \lambda_1 f_{11} \\ \lambda_2^N f_{21} & \lambda_2^N f_{22} & \lambda_2^{N-1} f_{21} & \lambda_2^{N-1} f_{22} & \cdots & \lambda_2^2 f_{21} & \lambda_2^2 f_{22} & \lambda_2 f_{21} & \lambda_2 f_{22} \\ (-1)^N\lambda_2^N f_{22} & (-1)^{N-1}\lambda_2^N f_{21} & (-1)^{N-1}\lambda_2^{N-1} f_{22} & (-1)^{N-2}\lambda_2^{N-1} f_{21} & \cdots & \lambda_2^2 f_{22} & -\lambda_2^2 f_{21} & -\lambda_2 f_{22} & \lambda_2 f_{21} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ \lambda_N^N f_{N1} & \lambda_N^N f_{N2} & \lambda_N^{N-1} f_{N1} & \lambda_N^{N-1} f_{N2} & \cdots & \lambda_N^2 f_{N1} & \lambda_N^2 f_{N2} & \lambda_N f_{N1} & \lambda_N f_{N2} \\ (-1)^N\lambda_N^N f_{N2} & (-1)^{N-1}\lambda_N^N f_{N1} & (-1)^{N-1}\lambda_N^{N-1} f_{N2} & (-1)^{N-2}\lambda_N^{N-1} f_{N1} & \cdots & \lambda_N^2 f_{N2} & -\lambda_N^2 f_{N1} & -\lambda_N f_{N2} & \lambda_N f_{N1} \end{pmatrix}.$$

The determinant representations of new solutions generated by the N-fold DT $\widetilde{T}_N$ are

$$u^{[N]}(X,T)=u(X,T)+b_N^{[1]}, \qquad x^{[N]}(X,T)=x(X,T)+a_N^{[1]}, \tag{2.22}$$

in which

$$a_N^{[1]}=\frac{1}{|W_{2N}|}\begin{vmatrix} \lambda_1^N f_{11} & \lambda_1^N f_{12} & \lambda_1^{N-1} f_{11} & \lambda_1^{N-1} f_{12} & \cdots & \lambda_1^2 f_{11} & \lambda_1^2 f_{12} & \lambda_1 f_{12} & f_{11} \\ (-1)^N\lambda_1^N f_{12} & (-1)^{N-1}\lambda_1^N f_{11} & (-1)^{N-1}\lambda_1^{N-1} f_{12} & (-1)^{N-2}\lambda_1^{N-1} f_{11} & \cdots & \lambda_1^2 f_{12} & -\lambda_1^2 f_{11} & \lambda_1 f_{11} & f_{12} \\ \lambda_2^N f_{21} & \lambda_2^N f_{22} & \lambda_2^{N-1} f_{21} & \lambda_2^{N-1} f_{22} & \cdots & \lambda_2^2 f_{21} & \lambda_2^2 f_{22} & \lambda_2 f_{22} & f_{21} \\ (-1)^N\lambda_2^N f_{22} & (-1)^{N-1}\lambda_2^N f_{21} & (-1)^{n-1}\lambda_2^{N-1} f_{22} & (-1)^{N-2}\lambda_2^{N-1} f_{21} & \cdots & \lambda_2^2 f_{22} & -\lambda_2^2 f_{21} & \lambda_2 f_{21} & f_{22} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ \lambda_N^N f_{N1} & \lambda_N^N f_{n2} & \lambda_N^{N-1} f_{N1} & \lambda_N^{N-1} f_{N2} & \cdots & \lambda_N^2 f_{N1} & \lambda_N^2 f_{N2} & \lambda_N f_{N2} & f_{N1} \\ (-1)^N\lambda_N^N f_{N2} & (-1)^{N-1}\lambda_N^N f_{N1} & (-1)^{N-1}\lambda_N^{N-1} f_{N2} & (-1)^{N-2}\lambda_N^{N-1} f_{N1} & \cdots & \lambda_N^2 f_{N2} & -\lambda_N^2 f_{N1} & \lambda_N f_{N1} & f_{N2} \end{vmatrix},$$

$$b_N^{[1]}=\frac{-1}{|W_{2N}|}\begin{vmatrix} \lambda_1^N f_{11} & \lambda_1^N f_{12} & \lambda_1^{N-1} f_{11} & \lambda_1^{N-1} f_{12} & \cdots & \lambda_1^2 f_{11} & \lambda_1^2 f_{12} & \lambda_1 f_{11} & f_{11} \\ (-1)^N\lambda_1^N f_{12} & (-1)^{N-1}\lambda_1^N f_{11} & (-1)^{N-1}\lambda_1^{N-1} f_{12} & (-1)^{N-2}\lambda_1^{N-1} f_{11} & \cdots & \lambda_1^2 f_{12} & -\lambda_1^2 f_{11} & -\lambda_1 f_{12} & f_{12} \\ \lambda_2^N f_{21} & \lambda_2^N f_{22} & \lambda_2^{N-1} f_{21} & \lambda_2^{N-1} f_{22} & \cdots & \lambda_2^2 f_{21} & \lambda_2^2 f_{22} & \lambda_2 f_{21} & f_{21} \\ (-1)^N\lambda_2^N f_{22} & (-1)^{N-1}\lambda_2^N f_{21} & (-1)^{N-1}\lambda_2^{N-1} f_{22} & (-1)^{N-2}\lambda_2^{N-1} f_{21} & \cdots & \lambda_2^2 f_{22} & -\lambda_2^2 f_{21} & -\lambda_2 f_{22} & f_{22} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ \lambda_N^N f_{N1} & \lambda_N^N f_{N2} & \lambda_N^{N-1} f_{N1} & \lambda_N^{N-1} f_{N2} & \cdots & \lambda_N^2 f_{N1} & \lambda_N^2 f_{N2} & \lambda_N f_{N1} & f_{N1} \\ (-1)^N\lambda_N^N f_{N2} & (-1)^{N-1}\lambda_N^N f_{N1} & (-1)^{N-1}\lambda_N^{N-1} f_{N2} & (-1)^{N-2}\lambda_N^{N-1} f_{N1} & \cdots & \lambda_N^2 f_{N2} & -\lambda_N^2 f_{N1} & -\lambda_N f_{N2} & f_{N2} \end{vmatrix}.$$

Note that $a_N^{[1]}$ and $b_N^{[1]}$ are determined from algebraic equations $\widetilde{T}_N(\lambda)\Phi_j|_{\lambda=\lambda_j}=0, j=1,2,\cdots,N$. Above two formulas will be used to construct the N-soliton solution of the SP equation from a suitable "seed" solution in next section.

3. solutions of short pulse equation

We shall present soliton solutions $u^{[N]}(X,T)$ of the HESP equations in (X,T)-plane, and then convert it to the corresponding solutions $u^{[N]}(x,t)$ of the SP equation in the (x,t)-plane by the hodograph transformation. In other words, $u^{[N]}(x,t)$ is an equivalent solution of $u^{[N]}(X,T)$ according to the hodograph transformation, although it can not be written out explicitly because $x=x(X,T)$ in the hodograph transformation is implicit. However it can be plotted in (x,t)-plane through a parametric form using a symbolic computation program. It is trivial to point out that solutions $u^{[N]}(X,T)$ can not produce the solutions of the SP equation if the conditions of the hodograph transformation are unsatisfied.

- Case a: The "seed" solution $u=0$ and $\frac{\partial x}{\partial X}=1$ of the HESP equations.

For this case, the hodograph transformation is true, and then the solutions $u^{[N]}(X,T)$ of the HESP equations can produce the solutions $u^{[N]}(x,t)$ of the SP equation in (x,t)-plane through this transformation. A simple "seed" to satisfy above condition is $x=X$, then the eigenfunction of the Lax pair (2.4) is

$$\Phi(X,T,\lambda)=\begin{pmatrix} e^{\lambda X+\frac{T}{4\lambda}} \\ e^{-\lambda X-\frac{T}{4\lambda}} \end{pmatrix}. \tag{3.1}$$

Therefore $\Phi_k=\Phi(X,T,\lambda_k)=\begin{pmatrix} f_{k1} \\ f_{k2} \end{pmatrix}$ infers $f_{k1}=e^{\lambda_k X+\frac{T}{4\lambda_k}}, f_{k2}=e^{-\lambda_k X-\frac{T}{4\lambda_k}}$, for $k=1,2,\cdots,N$.

Setting $N=1$, substituting f_{11} and f_{12} into eq.(2.22), two single-soliton solutions generated by the one-fold DT of the SP equation are

$$u^{[1]}(X,T)=-\frac{1}{\lambda_1\cosh(2\lambda_1 X+\frac{T}{2\lambda_1})}, \tag{3.2}$$

$$x^{[1]}(X,T)=X-\frac{\tanh(2\lambda_1 X+\frac{T}{2\lambda_1})}{\lambda_1}, \tag{3.3}$$

which are the same as expressions in (99) of reference [30]. The $u^{[1]}(X,T)$ in (X,T)-plane is a dark soliton which is shown in Fig.1(a). The corresponding $u^{[1]}(x,t)$ is a loop solion of the SP equation in (x,t)-plane as Fig.2(a).

Setting $N=2$, substituting f_{k1} and $f_{k2}(k=1,2)$ into eq.(2.22), two double-soliton solutions generated by the two-fold DT of the SP equation are

$$u^{[2]}(X,T)=-\frac{2(\lambda_1^2-\lambda_2^2)(-\lambda_3 e_3+\lambda_2 e_4)}{\lambda_1\lambda_2((\lambda_1-\lambda_2)^2 e_1+(\lambda_1+\lambda_2)^2 e_2-4\lambda_1\lambda_2)}, \tag{3.4}$$

$$x^{[2]}(X,T)=X-\frac{(\lambda_1^2-\lambda_2^2)((\lambda_1-\lambda_2)e_5+(\lambda_1+\lambda_2)e_6)}{\lambda_1\lambda_2((\lambda_1-\lambda_2)^2 e_1+(\lambda_1+\lambda_2)^2 e_2-4\lambda_1\lambda_2)}, \tag{3.5}$$

with $e_1=\cosh(\frac{(\lambda_1+\lambda_2)(4\lambda_1\lambda_2 X+T)}{2\lambda_1\lambda_2}), e_2=\cosh(\frac{(\lambda_1-\lambda_2)(-4\lambda_1\lambda_2 X+T)}{2\lambda_1\lambda_2}), e_3=\cosh(2\lambda_1 X+\frac{T}{2\lambda_1}), e_4=\cosh(2\lambda_2 X+\frac{T}{2\lambda_2}), e_5=\sinh(\frac{(\lambda_1+\lambda_2)(4\lambda_1\lambda_2 X+T)}{2\lambda_1\lambda_2}), e_6=\sinh(\frac{(\lambda_1-\lambda_2)(-4\lambda_1\lambda_2 X+T)}{2\lambda_1\lambda_2})$. The $u^{[2]}(X,T)$ is

plotted in Fig.1(b), and the corresponding $u^{[2]}(x,t)$ provides a two-loop solution in (x,t)-plane of the SP equation, see figure 2 of ref. [29].

Similarly, setting $N=3$ in eq.(2.22), it generates a three-soliton solution $u^{[3]}(X,T)$, which is plotted in Fig.1(c). Furthermore, for arbitrary N, eq.(2.22) generates an N-soliton solution of the SP equation taking advantage of the N eigenfunctions defined by eq.(3.1).

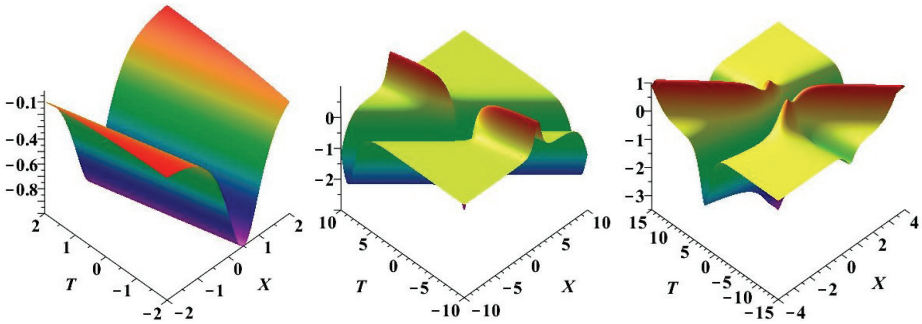


Fig. 1. (Color online)Three soliton-solutions of the SP equation in (X,T)-plane from Case a. Panel (a) is a single-soliton $u^{[1]}(X,T)$ with $\lambda_1=1$. Panel (b) is a double-soliton $u^{[2]}(X,T)$ with $\lambda_1=1$ and $\lambda_2=\frac{1}{2}$. Panel (c) a three-soliton solution $u^{[3]}(X,T)$ with $\lambda_1=1$, $\lambda_2=\frac{1}{2}$ and $\lambda_3=2$.

Next, we would like to consider more general "seed" of x to satisfy $\frac{\partial x}{\partial X}=1$, i.e. $x=X+f(T)$. Here $f(T)$ is an arbitrary smooth function of T. It is trivial to find that $f(T)$ does not affect the solution $u^{[N]}(X,T)$. However $f(T)$ imposes a remarkable constraint on $u^{[N]}(x,t)$ through the hodograph transformation of the SP equation in the (x,t)-plane. Specifically, $f(T)$ changes the trajectory of $u^{[N]}(x,t)$ in (x,t)-plane, but does not destroy its loop at arbitrarily given moment t. This observation is verified by Figs.2 plotted for $u^{[1]}(x,t)$ with $f(T)=T^2, sin(T), e^T, T^3$. The property, i.e. variable trajectory, is a common character of the soliton equations with variable coefficients [12, 15, 33], but is an unusual character for soliton equation with constant coefficients like the SP equation.

- Case b: The "seed" solution $u=0$ and $\frac{\partial x}{\partial X}\neq 1$ of the HESP equations.

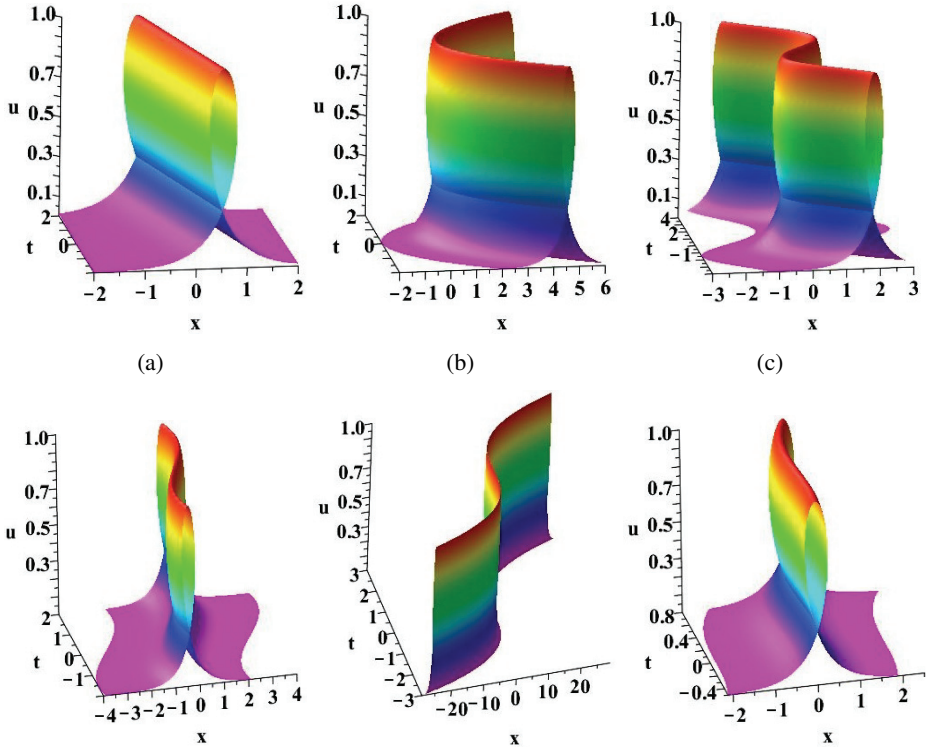
For this case, the hodograph transformation does not hold, and then there does not exist $u^{[N]}(x,t)$ of the SP equation in (x,t)-plane associated with $u^{[N]}(X,T)$ of the HESP equations. To satisfy this condition, set $x=g(X)$ and $\frac{\partial g}{\partial X}\neq 1$, $g(X)$ is an arbitrary smooth function.

Set $u=0, x=\sin X$ in Lax pair (2.4), the two components of eigenfunction associated with λ_k are expressed by $f_{k1}=e^{\lambda_k \sin X+\frac{T}{4\lambda_k}}, f_{k2}=e^{-\lambda_k \sin X-\frac{T}{4\lambda_k}}$, for $k=1,2,\cdots,N$. Setting $N=1$ in eq. (2.22) and using these eigenfunctions, it infers a single-soliton solution

$$u^{[1]}(X,T)=-\frac{2(e_1+e_2)}{\lambda_1(e_3+e_4+1)}, \tag{3.6}$$

$$x^{[1]}(X,T)=\frac{\lambda_1 \sin X e_4+\lambda_1 \sin X e_3+\lambda_1 \sin X-e_4-e_3+1}{\lambda_1(e_3+e_4+1)}, \tag{3.7}$$

of the HESP. Here $e_1=\cosh(\frac{4\lambda_1^2 \sin X+T}{2\lambda_1}), e_2=\sinh(\frac{4\lambda_1^2 \sin X+T}{2\lambda_1}), e_3=\cosh(\frac{4\lambda_1^2 \sin X+T}{\lambda_1}), e_4=\sinh(\frac{4\lambda_1^2 \sin X+T}{\lambda_1})$. The figure of $u^{[1]}(X,T)$ is shown in Fig.3(a). For other choices of $x=e^X$ and

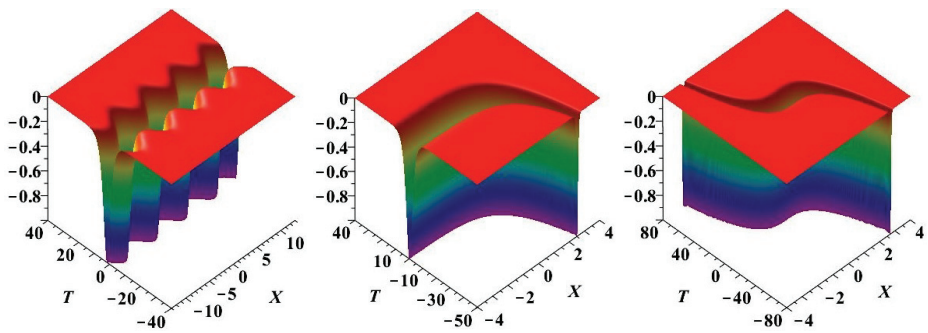


(a) (b) (c)

(d) (e) (f)

Fig. 2. (Color online) Five one-loop solutions $u^{[1]}(x,t)$ of the SP equation in (x,t)-plane with same $\lambda_1 = 1$ but different "seed" solutions associated with $f(T)$ from Case a. From (a) to (e), $f(T) = 0, T^2, sin(T), e^T, T^3$ in order. (f) is a local zoom of (e) around $t = 0$ in order to show clearly its loop.

$x = X^3$, $u^{[1]}(X,T)$ are plotted in Figs.3(b) and 3(c). Once again, we find variable trajectories in Fig.3 of a soliton equation with constant coefficients.



(a) (b) (c)

Fig. 3. (Color online) Three single-soliton solutions $u^{[1]}(X,T)$ of the HESP equations in (X,T)-plane with same $\lambda_1 = 1$ but different "seed" solutions associated with $g(X)$ from Case b. From (a) to (c), $g(X) = sin(X), e^X, X^3$ in order.

4. Conclusions

In this paper, we presented a determinant representation $\widetilde{T}_N$ of the N-fold DT in eq.(2.21) and the new solutions in eq.(2.22) of the SP equation by conventional determinants. If we compare our results with the work in [30] on the DT of the same equation, our results have following advantages and developments.

- We derived a determinant representation of the N-fold DT for the SP equation. In particular, we did not use unconventional tool–quasi-determinant, and obtained different form of new solutions generated by the N-fold DT, which is easy to confirm from eq.(2.22) in this paper and eq.(75) in ref. [30].
- We provided an unified expression of the $u^{[N]}(X,T)$ generated by the N-fold DT no matter N is even or odd, unlike the appearing of two determinants in eq.(75) of ref. [30].
- We generated new solutions in (x,t)-plane of the SP equation with a variable trajectory from new "seed" solution.
- We constructed new solutions in (X,T)-plane of the HESP equations possessing a variable trajectory, which can not produce corresponding solutions in (x,t)-plane of the SP equation.

With respect to the future research related to the SP equation, the HESP equations is a new and useful integrable system, and also has a correspondence of the well-known SG equation, which deserves further study from the point of view of mathematics. We are also interested in new solutions of the sine-Gorden equation from the HESP equations in the near future. In addition, the loop solutions are multi-valued which are not satisfactory from physical point of view, so it is a necessary and interesting problem to find single-valued solution of the SP equation and then use it to model the few-cycle waves [22, 29].

Acknowledgments

This work is supported by the NSF of China under Grant No. 11671219, and the K.C. Wong Magna Fund in Ningbo University. Liu shuzhi is supported by the Scientific Research Foundation of Graduate School of Ningbo University. We thank other members in our group at Ningbo University for their many discussions and suggestions on the paper.

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