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Solutions of the buoyancy-drag equation with a time-dependent acceleration

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We perform the analytic study of the buoyancy-drag equation with a time-dependent acceleration $\gamma(t)$ by two methods. We first determine its equivalence class under the point transformations of Roger Liouville, and thus for some values of $\gamma(t)$ define a time-dependent Hamiltonian from which the buoyancy-drag equation can be derived. We then determine the Lie point symmetries of the buoyancy-drag equation, which only exist for values of $\gamma(t)$ including the previous ones, plus additional classes of accelerations for which the equation is reducible to an Abel equation. This allows us to exhibit two régimes for the asymptotic (large time t) solution of the buoyancy-drag equation. It is shown that they describe a mixing zone driven by the Rayleigh–Taylor instability and the Richtmyer–Meshkov instability, respectively.

Keywords: Buoyancy-drag equation; Lie point symmetries; Abel equation.

2010 Mathematics Subject Classification: 22E99, 34Mxx (see also 30Dxx, 32G34), 76Fxx (see also 37-XX, 60Gxx, 60Jxx)

1. Introduction. The buoyancy-drag equation

Interface instabilities such as Richtmyer–Meshkov instability, Kelvin–Helmholtz instability, or Rayleigh–Taylor instability, are known to produce the mixing of two fluids living on each side of a common interface [5, 23]. The Rayleigh–Taylor instability (RTI) plays, however, a special role for at least two reasons. First, from the physical viewpoint, RTI has been evidenced to occur

in various objects/processes with a spatial scale ranging from one millimeter for inertial confinement fusion laser targets to 10^{11} meters for supernovae and typically 10^{18} meters (≈ 30 parsec) in supernova remnants. The second reason is related to the mathematical description. In the linear stage approximation, the formalism of the Richtmyer–Meshkov instability can be recovered from the RTI approach where the acceleration $\gamma(t)$ (t is time) experienced by the interface is restricted to a kick (impulsive acceleration), and, moreover, the dispersion relation of the RTI can be easily derived from the Kelvin–Helmholtz instability in a gravitational acceleration g provided g is formally replaced by $\gamma(t)$ [5]. Because of the broad application of its formalism, we have decided in this paper to study the mixing zone produced by the RTI. Considering a vertical downward acceleration field, we take a two-fluid configuration where the high mass density material (heavy fluid), labeled by the subscript “2” lies above the low mass density medium (light fluid), labeled by the subscript “1” (see [4]). The fluids are assumed to extend vertically from $-\infty$ to $+\infty$ with a velocity of the flow having only a vertical component depending on the position in the horizontal plane (two-dimensional configuration). Because of the RTI, spikes of “2” with length $h_2(t)$ (this length is measured from the initial position of the interface) drop into “1” while bubbles of “1” rise into “2” with elevation $h_1(t)$ (measured also from the initial position of the interface between the two fluids) in the nonlinear régime. The mixing zone corresponds to a dynamical region that is bounded at its bottom by a surface intersecting the spike tips, and its upper border is given by the surface joining the tops of the bubbles. Above (resp. below) the upper (resp. lower) frontier, the location of which depends upon time, the fluid “2” (resp. “1”) is pure. In a zero-dimensional modeling, a spatial average is performed horizontally and the upper and lower boundaries reduce to two horizontal straight lines with respective position $h_1(t)$ above and $h_2(t)$ below the interface and the inner spatial structures of the flow cannot be described in between. Nevertheless, the height $h(t)$ of the mixing zone satisfies $h(t) = h_1(t) + h_2(t)$ and a nonlinear ordinary differential equation (ODE) governing the evolution of $h(t)$ can be derived. This equation is called the buoyancy-drag equation (BDE). Similarly to an equation of motion, the BDE is a second order ordinary differential equation that provides the length $h(t)$ as a function of time

$$\frac{d^2h}{dt^2} = B\gamma(t) - \frac{C}{h} \left(\frac{dh}{dt} \right)^2 \quad (1.1)$$

where B (buoyancy coefficient) and C (drag coefficient) are two positive constants (see [4]), and where $\gamma(t)$ is the time-dependent acceleration experienced by the interface. More precisely, one has $0 \leq B \leq 1$ and $1 \leq C \leq 4$ typically.

Although the BDE was initially obtained from phenomenological viewpoints and was considered as an engineering model describing the motion of the heavy fluid [1], it has been derived by Dimonte [11] and Dimonte and Schneider [12] from the initial study by Davies and Taylor [10]. This equation has been also derived by Srebro et al. [34] from an extension of Layzer approach [19] by Hecht, Alon and Shvarts [16]. Additionally, a more theoretical approach shows that the BDE comes out from a Lagrangian formalism with the inclusion of an additional generalized force term to represent the effect of dissipation [15, 33]. Through this formulation, the total energy is preserved and this conservation law can be used for estimating global turbulent kinetic energy and dissipation rate in the mixing zone.

The motivation of the present paper is to find values of C and, more importantly, $B\gamma(t)$ allowing at least a partial integration of (1.1). The case of a time-independent (constant) acceleration has been studied earlier in [8] and [4]. Although only the asymptotic (large value of time) form of the

solution was found by Cheng, Glimm and Sharp [8], the general solution of the BDE for a constant acceleration has been derived for the first time a few years later [4]. In the present work, our goal is not to obtain the general solution of (1.1) for any time-dependent acceleration but, instead, to find classes of accelerations $\gamma(t)$ for which the BDE can be fully or partially integrated.

There exist three general methods to investigate a given second order ordinary differential equation. The first one corresponds to the Painlevé analysis [9]. It is based on those singularities of the general solution which depend on the initial conditions. The second approach applies to the class of equations, to which the BDE belongs, investigated by Roger Liouville [20], and in the third method the differential order of the BDE is reduced by using Lie symmetries.

In our case, the first method cannot apply, since a necessary condition of its applicability is the existence of a relative integer n such that $C = -1 + 1/n$ [9], but this condition never happens.

However a useful property is the form invariance of equation (1.1) [24]. Indeed, the one-parameter point transformation φ_a ,

$$T = \frac{at}{t+a}, h = \left(\frac{t}{T}\right)^{1/(1+C)} H, \gamma = \left(\frac{t}{T}\right)^{-(3+4C)/(1+C)} \Gamma, a \text{ arbitrary}, \quad (1.2)$$

only changes the acceleration,

$$\frac{d^2H}{dT^2} = B \Gamma(T) - \frac{C}{H} \left(\frac{dH}{dT}\right)^2. \quad (1.3)$$

The inverse of φ_a is φ_{-a} and the composition of two such transformations does not generate a new transformation, since $\varphi_b \varphi_a = \varphi_{ab/(a+b)}$. In particular, a power-law acceleration $\gamma \sim t^n$ is mapped to another power-law $\gamma \sim t^{-\alpha-n}$, with

$$\alpha = \frac{3+4C}{1+C} = 4 - \frac{1}{1+C}, \quad (1.4)$$

the fixed point of such a map being $\gamma \sim t^{-\alpha/2}$, which will indeed be encountered below.

The paper is organized as follows. In section 2, we first recall the existing analytic solutions of the BDE. In section 3, we build a Hamiltonian description of the dissipative equation (1.1) and, for specific t -dependences of the acceleration $\gamma(t)$, find an invariant of the resulting Hamiltonian system. In section 4, for selected accelerations $\gamma(t)$, we lower the differential order by one unit.

2. Existing solutions

Some solutions of Eq. (1.1) for the RTI are already known in four cases, the first three have been derived using Lie symmetries [4, 18, 24]. Throughout the paper, t_0 denotes some time unit.

- (1) For a constant acceleration $\gamma = \gamma_0$, the BDE is autonomous and admits the first integral

$$\gamma = \gamma_0 : K = h^{2C} \left[\left(\frac{dh}{dt}\right)^2 - \frac{2B\gamma_0}{1+2C} h \right]. \quad (2.1)$$

The BDE is then invariant under the two symmetries ∂_t and $t\partial_t + 2h\partial_h$ (the subscripts represent the partial derivatives), which allows us to obtain its general solution in the implicit

form [4] [14, formula 3.194]

$$\gamma = \gamma_0, \quad t = t_1 + \frac{h^{1+C}}{(1+C)\sqrt{\varepsilon K}} {}_2F_1\left(\frac{1}{2}, \frac{1+C}{1+2C}, \frac{2+3C}{1+2C}, -\frac{2B\gamma_0 h^{1+2C}}{(1+2C)\varepsilon K}\right), \quad \varepsilon = \text{sign}(K), \quad (2.2)$$

in which ${}_2F_1$ is the hypergeometric function of Gauss, and the two arbitrary constants are t_1 and K .

This expression yields the asymptotic behavior of the height as $t \rightarrow +\infty$

$$h \sim \frac{B\gamma_0}{2(1+2C)} t^2. \quad (2.3)$$

Such a time-dependence has already been found earlier by Neuvazhaev [26].

This quadratic law is consistent with the fact that in a constant acceleration field the free fall distance scales as t^2 . This result is not really surprising because the friction term $-(C/h)(dh/dt)^2$ scales as t^0 and the BDE becomes a free fall equation.

In a constant acceleration field γ_0 without friction ($C = 0$), the general solution of Eq. (1.1) behaves like $h \sim (B\gamma_0/2) t^2$, and the comparison with (2.3) shows, however, that the actual solution accounts for friction effects. We could have expected a balance between the acceleration term $B\gamma_0$ and the friction term $-(C/h)(dh/dt)^2$, i.e. $h \sim [B\gamma_0/(4C)] t^2$, in the asymptotic evolution. Nevertheless, Eq. (2.3) proves that this condition is not fulfilled and, instead, the balance between the three terms (acceleration, friction and inertia d^2h/dt^2) is achieved in the BDE.

- (2) For the power law acceleration $\gamma \sim t^{-\alpha}$, the general solution is simply obtained from (2.2) by action of the transformation (1.2) to generate the case $\Gamma = \gamma_0$ in (1.3).

As expected, for large time t , the behavior $h \sim t^2$ is no longer valid. The actual asymptotic solution is obtained by noting that $\lim_{t \rightarrow +\infty} T = a$. As a consequence, T and H are bounded with $H \rightarrow H(a)$. Using (1.2), we get $h \sim (t/a)^{1/(1+C)} H(a)$, i.e.

$$h \sim t^{1/(1+C)}, \quad t \rightarrow +\infty. \quad (2.4)$$

This behavior is actually very surprising for the Rayleigh–Taylor mixing since such a power law is relevant to the Richtmyer–Meshkov instability (no acceleration or impulsive acceleration when a shock wave front reaches an interface). Indeed, for a zero acceleration in (1.1), the BDE reduces to

$$\frac{d^2h}{dt^2} = -\frac{C}{h} \left(\frac{dh}{dt}\right)^2, \quad (2.5)$$

and the general solution is

$$\gamma = 0: \quad h_{\text{RM}} = K(t - t_1)^{1/(1+C)} \quad (2.6)$$

where the subscript RM stands for Richtmyer–Meshkov and where K and t_1 are the two constants of integration. Asymptotically (large time t), we get

$$h_{\text{RM}} \sim t^{1/(1+C)}, \quad t \rightarrow +\infty. \quad (2.7)$$

As a consequence, the physical interpretation of (2.4) is as follows: for $n = -\alpha$, the acceleration $\gamma(t)$ decreases very quickly with time and the mixing régime is not driven by the RTI

at large time. Instead, the system experiences the Richtmyer–Meshkov instability (RMI) asymptotically because $\gamma \rightarrow 0$ for $t \rightarrow +\infty$.

Finally, as in the linear stage of the RMI we have a ballistic growth of the mixing zone, we get $h \sim t$. However, (2.4) shows that in the nonlinear régime, h increases more slowly than linearly in t (the exponent $1/(1+C)$ is always smaller than 1) and the observed deceleration is produced by the friction term $-(C/h)(dh/dt)^2$.

- (3) For a general power law acceleration $\gamma \sim t^n$ with an arbitrary real exponent n , the BDE (1.1) only admits the symmetry $t\partial_t + (2+n)h\partial_h$, which allows its reduction to an Abel equation [4],

$$\begin{cases} \gamma = \gamma_0 \left(\frac{t}{t_0}\right)^n, & I = \left(\frac{t}{t_0}\right)^{-n-2} h, & J = (1+C) \left(\frac{t dh}{h dt} - n - 2\right), \\ IJ dJ + (1+C) \left[\left(J + (1+C)n + \frac{3+4C}{2} \right)^2 - \frac{1}{4} - (1+C) \frac{B\gamma_0 t_0^2}{I} \right] dI = 0. \end{cases} \quad (2.8)$$

Since this Abel ODE is in general not integrable [7, 25, 30, 32], this only defines the zero-parameter scaling solution,

$$\begin{cases} \gamma = \gamma_0 \left(\frac{t}{t_0}\right)^n, \\ h = h_0 \left(\frac{t}{t_0}\right)^{n+2}, & h_0 = \frac{B\gamma_0 t_0^2}{(n+2)[n(1+C) + 1 + 2C]}, & (n+2) \left(n + \frac{1+2C}{1+C} \right) \neq 0. \end{cases} \quad (2.9)$$

In these relations, the exponent n must satisfy two major constraints. First, as the mixing zone thickness increases with time, the exponent n should obey $n+2 > 0$. Second, the coefficient h_0 has to be positive and, since $n > -2$, we get

$$n > -\frac{1+2C}{1+C} = -2 + \frac{1}{1+C}. \quad (2.10)$$

As a consequence, one expects the solution $h \sim t^{n+2}$ to be valid for n either positive or negative, provided its value is larger than the lower bound given by Eq. (2.10). For $n > 0$, the acceleration grows with time and the thickness of the mixing zone increases. For the case $n < 0$, although the acceleration vanishes for large time, the thickness of the mixing zone increases too. In both cases, the mixing is driven by the RTI.

Moreover, for the lower bound (2.10), we have $n+2 = 1/(1+C)$ and the time dependence in (2.4) and (2.9) coincides. According to (2.7), this specific time variation of the acceleration produces the same asymptotic mixing rate for the RTI than for the RMI. Finally, for $n = 0$ we have $h_0 = B\gamma_0 t_0^2/[2(1+2C)]$ and therefore $h = B\gamma_0 t^2/[2(1+2C)]$. This solution corresponds to (2.3) and we conclude that the particular solution (2.9) provides the correct asymptotic behavior of the RTI for a constant acceleration.

If the acceleration decreases with t faster than prescribed by the lower bound value (2.10), the solution $h \sim t^{n+2}$ does not happen anymore. According to our explanations, we might expect that the asymptotic solution of the BDE be given by (2.7). This prediction actually arises for $\gamma \sim t^{-\alpha}$ [see item (2) above]: the value $n = -\alpha = -2 - [(1+2C)/(1+C)]$ is always smaller than -2 and is therefore below the lower value (2.10), and Eq. (2.4) shows that $h(t)$ behaves like $t^{1/(1+C)}$ instead of $t^{n+2} = t^{-(1+2C)/(1+C)}$ that decreases with time.

In the next item, this prediction is also proven to hold for another negative value of n .

- (4) For the value of n which leaves the behaviour $\gamma \sim t^n$ invariant under the map φ_a , i.e. $n = -\alpha/2$, the Abel equation (2.8) becomes linear in J^2 and there exists a first integral

$$n = -\frac{\alpha}{2}, \gamma = \gamma_0 \left(\frac{t}{t_0}\right)^n, K = \left[J^2 - \frac{1}{4} - 2\frac{(1+C)^2 B\gamma_0 t_0^2}{1+2C} I \right] I^{2(1+C)}. \quad (2.11)$$

Moreover, when K vanishes, the first order ODE for $h(t)$ can be integrated. Indeed, the change of function

$$(h, t) \rightarrow (H, T) : h = TH, T = \left(\frac{t}{t_0}\right)^{n+2}, \quad (2.12)$$

maps the first integral to

$$T^2 \left(\frac{dH}{dT}\right)^2 - H(H - H_0) = 0, H_0 = -8B\gamma_0 t_0^2 \frac{(1+C)^2}{1+2C}, \quad (2.13)$$

an equation linearizable by derivation. This ODE has two kinds of solutions [6]: the so-called singular solution $H = H_0$, $h = H_0(t/t_0)^{2-\alpha/2}$, which must be rejected because it is never solution of the BDE, and the general solution [24],

$$H = (c_1 T^{-1/2} + c_2 T^{1/2})^2, c_1 c_2 = \frac{H_0}{4}, \frac{c_1}{c_2} \text{ arbitrary}, \quad (2.14)$$

i.e.

$$n = -\frac{\alpha}{2}, \gamma = \gamma_0 \left(\frac{t}{t_0}\right)^n, h = \left[c_1 + c_2 \left(\frac{t}{t_0}\right)^{n+2} \right]^2, c_1 c_2 = \frac{H_0}{4}. \quad (2.15)$$

The latter solution depends on the arbitrary parameter c_1/c_2 but it never reduces to the previous solution (2.9).

The exponent $n + 2$ in (2.15) is always positive and hence, for large time t , the constant c_1 can be neglected in (2.15) showing that h grows like $t^{2(n+2)}$, which is,

$$h \sim t^{1/(1+C)}. \quad (2.16)$$

Interestingly, the behavior (2.4) is recovered.

This result is not surprising because the exponent $n = -\alpha/2$, which can be written as the sum $n = -2 + 1/(1+C) - 1/[2(1+C)]$ is therefore smaller than the lower bound (2.10).

We conclude that, although the acceleration decreases with t more slowly for $n = -\alpha/2$ than for $n = -\alpha$, the function $\gamma(t)$ vanishes rapidly enough for the system to exhibit a mixing zone driven by the RMI instead of the RTI.

3. Method of Roger Liouville for second order equations

The approach developed by R. Liouville [20] applies to the class of equations

$$\frac{d^2 h}{dt^2} + a_3(h, t) \left(\frac{dh}{dt}\right)^3 + 3a_2(h, t) \left(\frac{dh}{dt}\right)^2 + 3a_1(h, t) \frac{dh}{dt} + a_0(h, t) = 0, \quad (3.1)$$

whose property is to be form invariant under the point transformation

$$(t, h) \rightarrow (T, H) : t = F(T, H), h = G(T, H). \quad (3.2)$$

By determining the invariants of (3.1) under the transformation (3.2), one may be able to integrate. A nice account of this method can be found in [3], to which we refer for the notation. In the case of (1.1), one obtains

$$L_1 = CB\gamma(t)h^{-2}, L_2 = 0, v_5 = 0, w_1 = 0, i_2 = C(2+C)B\gamma(t)h^{-3}, \quad (3.3)$$

therefore (see e.g. [3, Lemma 1 page 458]) the ordinary differential equation (1.1) can be mapped to an ODE (3.1) in which $a_3 = a_2 = a_1 = 0$,

$$t = t, h = H^{1/(1+C)}, \frac{d^2H}{dt^2} - (1+C)B\gamma(t)H^{C/(1+C)} = 0. \quad (3.4)$$

Since this equation is independent of dH/dt , it can be interpreted as the Hamilton equation of a classical time-dependent Hamiltonian \mathbb{H} defined by

$$\mathbb{H}(q, p, t) = \frac{p^2}{2} + V(q, t), q = H, p = \frac{dq}{dt}, V = -\frac{(1+C)^2}{1+2C}B\gamma(t)q^{\frac{1+2C}{1+C}} \quad (3.5)$$

where q, p and V are respectively the position, linear momentum and potential. Under this transformation, equation (3.4) becomes a standard equation of motion

$$\frac{d^2q}{dt^2} + \frac{\partial}{\partial q} V(q, t) = 0, \quad (3.6)$$

for which we are going to look for an invariant of motion $\mathbb{I}(q, p, t)$.

The invariant obeys the equation $\partial\mathbb{I}/\partial t + p(\partial\mathbb{I}/\partial q) - (\partial V/\partial q)(\partial\mathbb{I}/\partial p) = 0$ and since \mathbb{H} is quadratic in p , it is natural to look for $\mathbb{I}(q, p, t)$ also quadratic in p . Such an invariant does exist,

$$\mathbb{I}(q, p, t) = (d_1t^2 + 2d_2t + d_3)\mathbb{H}(q, p, t) - (d_1t + d_2)qp + d_1\frac{q^2}{2}, (d_1, d_2, d_3) \text{ arbitrary}, \quad (3.7)$$

however only for specific accelerations $\gamma(t)$ defined by,

$$(d_1t^2 + 2d_2t + d_3)\frac{d\gamma}{dt} + (d_1t + d_2)\alpha\gamma = 0, \alpha = \frac{3+4C}{1+C}. \quad (3.8)$$

According to the values of the d_i 's, three types of t -dependence for $\gamma(t)$ come out,

$$\gamma = \begin{cases} \gamma_0 \left(\frac{t^2 - t_1^2}{t_0^2} \right)^{-\alpha/2} & (d_1 \neq 0), \\ \gamma_0 \left(\frac{t}{t_0} \right)^{-\alpha/2} & (d_1 = 0, d_2 \neq 0), \\ \gamma_0 & (d_1 = d_2 = 0, d_3 \neq 0), \end{cases} \quad (3.9)$$

where t_1^2 is an arbitrary real constant of any sign.

The first expression in (3.9) has been obtained by performing a time translation $t \rightarrow t - d_2/(2d_1)$ (which leaves the BDE invariant), and $t_1^2 = (d_2)^2/(4d_1^2) - d_3/d_1$.

This expression is an extension of the second case listed in section 2 and it is recovered for $t_1^2 = 0$.

The last two values correspond to the first and fourth cases listed in section 2 (the second value requires the shift $t \rightarrow t - d_3/d_2$).

Let us now show that one can recover these three values by another approach, and even find more general ones.

4. Method of Lie point symmetries

Given any partial differential equation $E(x, t, u(x, t), u_x, u_t, \dots) = 0$, a Lie point symmetry is a transformation

$$(x, t, u) \rightarrow (X, T, U) : X = F(x, t, u), T = G(x, t, u), U = H(x, t, u),$$

mapping a solution $u(x, t)$ to another solution $U(X, T)$.

Practically, instead of this finite transformation, one computes the infinitesimal transformation

$$X = x + \varepsilon \zeta(x, t, u), T = t + \varepsilon \xi(x, t, u), U = u + \varepsilon \eta(x, t, u), \quad (4.1)$$

associated to the infinitesimal point symmetry $S = \zeta \partial_x + \xi \partial_t + \eta \partial_u$ where the subscripts stand for the partial derivatives [28, 29].

In the case of (1.1), the assumption

$$T = t + \varepsilon \xi(t, h), H = h + \varepsilon \eta(t, h) \quad (4.2)$$

yields the set of determining equations for $\xi(t, h), \eta(t, h)$ [18]

$$\begin{cases} \xi_{hh} - (C/h)\xi_h = 0, \\ \eta_{hh} + (C/h)\eta_h - (C/h^2)\eta - 2\xi_{th} = 0, \\ \xi_{tt} + 3B\gamma(t)\xi_h - 2(C/h)\eta_t - 2\eta_{th} = 0, \\ \eta_{tt} - 2B\gamma(t)\xi_t + B\gamma(t)\eta_h - B\gamma'(t)\xi = 0. \end{cases} \quad (4.3)$$

where $\gamma'(t)$ is the time derivative of $\gamma(t)$.

Remark. The above assumption (4.2) for Lie point symmetries does not act on $\gamma(t)$, therefore it cannot detect the form invariance (1.2).

The system (4.3) is a linear overdetermined system for $\xi(t, h), \eta(t, h)$, solved as follows.

The first equation is an ODE for $\xi(h)$ (with t a parameter) having the type of Fuchs, whose general solution depends on two arbitrary functions of t ,

$$\xi = C_1(t) \frac{h^{1+C}}{1+C} + C_2(t). \quad (4.4)$$

The second equation is then an ODE for $\eta(h)$ (with t a parameter) of the type of Fuchs, whose general solution introduces two more arbitrary functions of t ,

$$\eta = C_3(t)h + C_4(t)h^{-C} + C_1'(t) \frac{h^{2+C}}{(1+C)^2} \quad (4.5)$$

where $C_1'(t)$ stands for the time derivative of $C_1(t)$.

Inserting these two expressions in the last two equations (4.3) puts constraints on $C_j(t)$ and $\gamma(t)$.

Finally, ξ and η depend on four arbitrary constants d_i [18]

$$\xi = d_1 t^2 + 2d_2 t + d_3, \quad \eta = \frac{d_1 t + d_2 + d_4}{1+C} h,$$

and $\gamma(t)$ must obey a first order linear ODE,

$$(d_1 t^2 + 2d_2 t + d_3) \frac{d\gamma}{dt} + [(d_1 t + d_2 + d_4)\alpha - 4d_4] \gamma = 0 \quad (4.6)$$

whose only fixed parameter is the positive constant α . This ODE is an extension of (3.8).

This four-dimensional algebra, generated by

$$X_1 = t^2 \partial_t + t h \partial_h, \quad X_2 = 2t \partial_t + h \partial_h, \quad X_3 = \partial_t, \quad X_4 = h \partial_h, \quad (4.7)$$

is decomposable into $\{X_1, X_2, X_3\} \oplus X_4$ and its nonzero commutators are,

$$[X_1, X_2] = -2X_1, \quad [X_1, X_3] = -X_2, \quad [X_2, X_3] = -2X_3. \quad (4.8)$$

Since the ODE (4.6) contains one more parameter than the similar ODE (3.8) resulting from the Hamiltonian structure, the set of values of $\gamma(t)$ is now larger,

$$\gamma = \begin{cases} \gamma_0 \left(\frac{t-t_1}{t_0} \right)^{n_1} \left(\frac{t+t_1}{t_0} \right)^{n_2}, & n_1 + n_2 = -\alpha, \quad n_1 - n_2 = \alpha \frac{t_2}{t_1} \quad (d_1 \neq 0), \\ \gamma_0 \left(\frac{t}{t_0} \right)^n & (d_1 = 0, \quad d_2 \neq 0), \\ \gamma_0 e^{\alpha t/t_3} & (d_1 = d_2 = 0, \quad d_3 \neq 0, \quad d_4 \neq 0), \\ \gamma_0 & (d_1 = d_2 = d_4 = 0, \quad d_3 \neq 0), \end{cases} \quad (4.9)$$

in which t_1, t_2, t_3 and n are arbitrary real constants, and time has been shifted like in (3.9) in order to derive the first three expressions.

The second and fourth values of $\gamma(t)$ yield the solutions already mentioned in section 2, but the first and the third ones are new.

The first value defines a new case of reduction to an Abel equation in which the invariants I and J are given by

$$\begin{cases} \frac{d\gamma}{\gamma dt} = -\alpha \frac{t-t_2}{t^2-t_1^2}, \quad t_2 \text{ and } t_1 = \text{arbitrary real constants}, \\ \gamma = \gamma_0 \left(\frac{t-t_1}{t_0} \right)^{n_1} \left(\frac{t+t_1}{t_0} \right)^{n_2}, \quad n_1 + n_2 = -\alpha, \quad n_1 - n_2 = \alpha t_2/t_1, \\ I = \left(\frac{t-t_1}{t_0} \right)^{-n_1-2} \left(\frac{t+t_1}{t_0} \right)^{-n_2-2} h, \\ J = \frac{1}{t_0} \left[(1+C)(t^2-t_1^2) \frac{dh}{h dt} - t - (3+4C)t_2 \right], \\ IJ \frac{dJ}{dI} + (1+C) \left[\left(J + (3+4C) \frac{t_2}{t_0} \right)^2 - \left(\frac{t_1}{t_0} \right)^2 - (1+C) \frac{B\gamma_0 t_0^2}{I} \right] = 0. \end{cases} \quad (4.10)$$

This solution contains the first expression of the list (3.9) as a special case for $n_1 = n_2 = -\alpha/2$.

The only closed form solution which this defines is,

$$h = \frac{(1+C)B\gamma_0 t_0^4}{(3+4C)^2 t_2^2 - t_1^2} \left(\frac{t-t_1}{t_0}\right)^{n_1+2} \left(\frac{t+t_1}{t_0}\right)^{n_2+2}, \quad (4.11)$$

and it depends on the two arbitrary constants t_1, t_2 .

Let us examine the asymptotic form of this solution. For large time ($t \gg t_1$), the acceleration and the height are

$$\gamma \sim t^{n_1+n_2}, \quad h \sim t^{n_1+n_2+4}, \quad t \rightarrow +\infty. \quad (4.12)$$

Obviously, this behavior does not belong to the class (2.9) with $n = n_1 + n_2$. However, with (1.4) and the constraint $n_1 + n_2 = -\alpha$, we obtain $h \sim t^{1/(1+C)}$ which not surprisingly is identical to (2.4). The RMI growth is again recovered.

Similarly, the third value in the list (4.9) also defines a reduction to an Abel equation

$$\begin{cases} \gamma = \gamma_0 e^{\alpha t/t_3}, \quad t_3 \neq 0, \\ I = e^{-\alpha t/t_3} h, \\ J = (1+C) \left(t_3 \frac{dh}{h dt} - \alpha \right), \\ IJ \frac{dJ}{dI} + (1+C) \left[(J+3+4C)^2 - (1+C) \frac{B\gamma_0 t_3^2}{I} \right] = 0, \end{cases} \quad (4.13)$$

and the particular solution

$$h = \frac{B\gamma_0 t_3^2}{\alpha^2(1+C)} e^{\alpha t/t_3}. \quad (4.14)$$

It is clear that this solution arises only for $t_3 > 0$: if $t_3 < 0$, the acceleration decreases exponentially with time and after a transient phase, h will follow the RMI law (2.4) instead of (4.14).

The three ODEs for $J(I)$ defined in (2.8), (4.10) and (4.13) are Abel equations of the second kind,

$$IJ \frac{dJ}{dI} + a \left[(J-b)^2 - c^2 - \frac{d}{I} \right] = 0, \quad (4.15)$$

for the respective values

$$a = 1+C, \quad d = (1+C)B\gamma_0 t_0^2, \quad c = \frac{1}{2}, \quad b = -(1+C)n - \frac{3+4C}{2}, \quad (4.16)$$

$$a = 1+C, \quad d = (1+C)B\gamma_0 t_0^2, \quad c = \frac{t_1}{t_0}, \quad b = -(3+4C)\frac{t_2}{t_0}, \quad (4.17)$$

$$a = 1+C, \quad d = (1+C)B\gamma_0 t_3^2, \quad c = 0, \quad b = -(3+4C). \quad (4.18)$$

The classical method to investigate the integrability of an Abel ODE is recalled in the Appendix A. In our case (4.15), the first two relative invariants of Roger Liouville evaluate to

$$\begin{aligned} s_3 &= \frac{2a^2 b}{27I^4} [a(b^2 - 9c^2)I - 9d(a-1)], \\ s_5 &= \frac{2a^3 b}{27I^7} [a^2(b^2 - 9c^2)(b^2 + 3c^2)I^2 - 3d(a-1)[(2b^2 a + 18c^2 a - 3b^2 + 3c^2)I + 3d(3a-2)]] \end{aligned} \quad (4.19)$$

Since the absolute invariant s_3^3/s_5^5 is generically nonconstant, there exists no change of variables making Eq. (4.15) separable.

The conditions $s_3 = s_5 = 0$ define two nongeneric cases, $b = 0$ and $(a = 1, b^2 = 9c^2)$, the second one being nonphysical. Moreover, the Maple package developed in [7], which knows all the Abel ODEs which have been integrated before the year 2000, fails to map (4.15) to one of the integrated equivalence classes, except in the unphysical case $d = 0$.

To summarize, the only cases when a first integral K is known to Eq. (4.15) are,

$$\left\{ \begin{array}{l} b = 0, a \neq 1/2 : K = \log I + \frac{\log[(2a-1)(J^2-c^2) - 2ad/I]}{2a}, \\ b = 0, a = 1/2 : K = \log I + \frac{(J^2-c^2)I}{d}, \\ a = 1, c^2 \neq b^2 : K = (c+b) \log[d + (b-c)I(J-b-c)] + (c-b) \log[d + (b+c)I(J-b+c)], \\ a = 1, c^2 = b^2 : K = \log[2bIJ + d] - 2\frac{b}{d}I(J-2b), \\ d = 0, c \neq 0 : K = \log I + \frac{(c+b) \log[J-b-c] + (c-b) \log[J-b+c]}{2ac}, \\ d = 0, c = 0 : K = \log I + \frac{1}{a} \log(J-b) - \frac{b}{a(J-b)}. \end{array} \right. \quad (4.20)$$

The first case mentioned in the list (4.20) corresponds to the solution (2.11).

Various assumptions extrapolating the three cases $b(a-1)d = 0$, such as

$$K = \log f(I) + k_+ \log[J + f_+(I)] + k_- \log[J + f_-(I)], \quad (4.21)$$

have failed to provide any new integrable case.

5. Conclusion. Physical interpretation

Six closed form solutions $(\gamma(t), h(t))$ of the BDE have been presented in this article: (2.2) and its sister solution for $\gamma = \gamma_0(t/t_0)^{-\alpha}$, (2.9), (2.15), (4.11), (4.14).

If one excludes the unit of time t_0 and the unit of acceleration γ_0 , the number of their arbitrary parameters is respectively two (t_1 and K), one (n), one (c_1/c_2), two (t_1, t_2) and one (t_3).

As far as we know, sister solution of (2.2) and solutions (2.15), (4.11) and (4.14) are new.

Moreover, there exist two values of $\gamma(t)$ allowing the BDE to reduce to a first order ODE of the type of Abel. The only hope to integrate this Abel equation (4.15) is to guess an integrating factor, necessarily outside the classes (A.7) already examined by Abel, Liouville and Appell [2].

In spite of its simplicity, solution (2.9) has been found to play a key role. It has helped to exhibit two families of solutions for the spatial extension $h(t)$ of the mixing zone when the acceleration behaves asymptotically ($t \rightarrow +\infty$) like $\gamma(t) \sim t^n$ where n is an arbitrary positive or negative exponent. For $n > 0$, the growth of the acceleration is monotonic with time and h increases like $h(t) \sim t^{n+2}$ for $t \rightarrow +\infty$. For $n < 0$, the acceleration decreases asymptotically with time, and for large negative values of n , equation (2.5) shows that the asymptotic solution does not depend anymore on n but is given by h_{RM} [see Eq. (2.7)]. In this paper, we claim that $h(t) \sim t^{n+2}$ describes an acceleration-driven mixing, *i.e.* a mixing of Rayleigh–Taylor type while $h(t) \sim t^{1/(1+C)}$ corresponds to an acceleration decreasing too fast with time in order to be able to drive the evolution of the mixing zone and the solution of the BDE corresponds to a Richtmyer–Meshkov mixing. The BDE exhibits therefore

two leading behaviors and the way each of them arises is explained as follows: first, we notice that for $n \in [-2, -(1+2C)/(1+C)]$, solution (2.9) is not physically valid since $h_0 < 0$ for $n \in]-2, -(1+2C)/(1+C)[$ and h_0 diverges for the two values $n = -2$ and $n = -(1+2C)/(1+C)$. Second, we show that for $n = -\alpha$ (this value is always below the range $[-2, -(1+2C)/(1+C)]$) and $n = -\alpha/2$ (this value is always in the range $[-2, -(1+2C)/(1+C)]$), the asymptotic solution is $h(t) \sim t^{1/(1+C)}$.

As a consequence, we conclude that the threshold value for n is $n_{th} = -(1+2C)/(1+C)$ (lower bound value (2.10)) : for $n < n_{th}$, the acceleration decreases quickly with time and the system experiences the RMI with an asymptotic solution given by $h(t) \sim t^{1/(1+C)}$; for $n > n_{th}$, the system is driven by the RTI with the asymptotic solution $h(t) \sim t^{n+2}$.

After submission of the present manuscript, we noticed that these two régimes had been already evidenced by Pandian, Swisher and Abarzhi under the name “acceleration-driven mixing” and “dissipation-driven mixing” for $n > n_{th}$ and $n < n_{th}$, respectively [31].

We notice that the RTI mixing occurs therefore also for $n_{th} < n < 0$ although $\gamma(t)$ decreases with time. Finally, for $n = n_{th}$, the two régimes collapse in a single one since $t^{n+2} = t^{1/(1+C)}$: the growth rates of the RTI and the RMI are the same. According to us, this claim is an important issue that would be worth being investigated through numerical simulations.

The solutions exhibited in this work provide an extension of the self-similar variable acceleration Rayleigh–Taylor (SSVART) flows studied by Llor [22,23] where the author uses an acceleration $\gamma(t) \sim t^n$ for $t > 0$ if $n > -2$, and $\gamma(t) \sim (-t)^n$ for $t < 0$ if $n < -2$ (such flows have been also considered earlier by Neuvazhaev for $n > -1$ [27]). In our case, the exponent $n = -2$ is not singular. For this value of n , the direct application of (2.9) leads to $h \sim t^{n+2} \sim t^0$ which of course is physically wrong although the BDE is satisfied. Actually, this special value of n satisfies $n < n_{th}$ and the correct asymptotic behavior is $h \sim t^{1/(1+C)}$.

The case $n = 0$ describes the standard RTI (*i.e.* constant acceleration) and, as expected, (2.3) is recovered from the particular solution (2.9). For a constant acceleration, $h(t)$ is usually written as $h(t) \sim \alpha\gamma t^2$ [13] and the comparison with (2.3) leads to the analytical value $\alpha = (B/2)/(1+2C)$. For $B = 1$, the case $C = 4$ leads to the smallest value $1/18 \approx 0.055$ for α in agreement with experiments whereas the value derived from numerical simulations is about twice smaller [13].

Finally, let us examine the case $n = -1$. This value is above n_{th} and one expects the asymptotic solution $h \sim h_0(t/t_0)$ where the constant h_0 is given by (2.9). Indeed, $d^2h/dt^2 = 0$ for this solution but the balance between $B\gamma(t)$ and $-(C/h)(dh/dt)^2$ is satisfied in the BDE.

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Appendix A. Abel equation

We recall in this Appendix how to integrate an Abel equation.

The most general Abel equation (which can always be assumed to be of the first kind),

$$-\frac{du}{dx} + a_3(x)u^3 + a_2(x)u^2 + a_1(x)u + a_0(x) = 0, \quad (\text{A.1})$$

is form-invariant under the mapping

$$(u, x) \rightarrow (U, X) : x = F(X), u = P(X)U(X) + Q(X), F'P \neq 0. \quad (\text{A.2})$$

The classical method to decide whether it is integrable or not has been introduced by Roger Liouville [21]. This first requires to compute its invariants under this mapping, which are rational functions of the a_j 's and their derivatives. The relative invariants of (A.1) have weight $2m + 1$, with m a positive integer,

$$s_3 = a_0 a_3^3 + \frac{1}{3} \left(\frac{2}{9} a_3^3 - a_1 a_2 a_3 + a_3 a_2' - a_2 a_3' \right), \quad (\text{A.3})$$

$$s_{2m+1} = a_3 s_{2m-1}' - (2m-1) s_{2m-1} \left(a_3' + a_1 a_3 - \frac{1}{3} a_2^2 \right), \quad (\text{A.4})$$

and the absolute invariants I_n are

$$I_1 = \frac{s_5^3}{s_3^5}, I_2 = \frac{s_3 s_7}{s_5^2}, I_3 = \frac{s_9}{s_3^3}, \dots \quad (\text{A.5})$$

If I_1 is constant, all other absolute invariants are also constant and there exists a mapping (A.2) making the transformed equation separable.

If I_1 is not constant, integrating is equivalent to finding an integrating factor $\mu(u, x)$ to (A.1). According to Abel, it is then more convenient to first put the equation under the form

$$u \frac{du}{dx} + p(x) + q'(x)u = 0, \quad (\text{A.6})$$

(whose advantage is to minimize the global degree in u and u'), then to make various assumptions for μ , such as those considered by Abel,

$$\begin{aligned} \log \mu &= (b_1(x)u + b_0(x))^{-1}, \quad \mu = (b_1(x)u + b_0(x))^n, \quad \mu = (u + b_1(x))^a (u + b_2(x))^b, \\ \log \mu &= b_3(x)u^3 + b_2(x)u^2 + b_1(x)u + b_0(x), \quad \mu = [7, \text{p. 215 Eq. (48)}]. \end{aligned} \quad (\text{A.7})$$

But the primary use of the invariants is to decide whether the equation belongs to the same equivalence class than one of the Abel ODEs previously integrated. An outstanding presentation of the current achievements can be found in [7], where the authors reviewed all the cases listed in the books [17], [25] [32], added a few ones and ordered them in different classes of equivalence *modulo* (A.2).

Let us mention a different approach [30] to try to integrate Abel ODEs. The authors split the single ODE into a system made of one Riccati ODE and another condition. If the Riccati ODE can be integrated, then this may lead to the integration of the Abel ODE. However, this method has not yet produced new integrated Abel equations.

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