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LETTER TO THE EDITOR

Integrable Hamiltonian N -body problems of goldfish type featuring N arbitrary functions

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A simple application of a neat formula relating the time evolution of the N zeros of a (monic) time-dependent polynomial of degree N in the complex variable w to the time evolution of its N coefficients allows to identify integrable Hamiltonian N -body problems in the plane featuring N arbitrary functions, the equations of motions of which are of Newtonian type: accelerations equal forces nonlinearly dependent on the coordinates of the N particle. The motions generally take place in the complex z -plane, or, equivalently, in the Cartesian xy -plane with $z = x + iy$. It is also easy to identify qualitative features of special subclasses of these models, for instance cases in which all the motions starting from an arbitrary real set of initial data are confined and multiply periodic. It is also indicated how to generate from these models hierarchies of analogous models with analogous properties.

Keywords: integrable many-body Hamiltonian systems in the plane, integrable many-body systems with Newtonian (accelerations equal forces) equations of motion

1. Introduction

The main tool used in this short paper are the nonlinear relations—by definition, algebraic—among the N coefficients of a monic polynomial of degree N in the (complex) variable w and its N zeros. The approach based on these relations allowed over time to identify several dynamical systems solvable by algebraic operations, including N -body problems characterized by Newtonian equations of motion (“accelerations equal forces”) [1]. These systems generally feature Newtonian equations of motion of the following goldfish type:

$$\ddot{z}_n = \sum_{\ell=1, \ell \neq n}^N \left(\frac{2 \dot{z}_n \dot{z}_\ell}{z_n - z_\ell} \right) + F_n(\vec{z}) , \quad (1.1)$$

where N is an arbitrary positive integer ($N \geq 2$), the complex numbers $z_n \equiv z_n(t)$ are the coordinates of N point-particles moving in the complex z -plane, $\vec{z} \equiv \vec{z}(t)$ is a N -vector of components $z_n \equiv z_n(t)$, the real independent variable t is time, superimposed dots indicate time-differentiations, and the nonlinear functions $F_n(\vec{z})$ are appropriately defined.

Remark 1.1. For the origin of the “goldfish” terminology see [2]; for examples of the use of this terminology see [3]; and note that, while in this paper we mainly confine our consideration to N -body problems of goldfish type characterized by equations of motion such as (1.1), the class of goldfish

type models is more general than (1.1), possibly featuring in their-right hand sides functions $F_n(\vec{z}, \dot{\vec{z}})$ rather than $F_n(\vec{z})$ (this is indeed the case for the *hierarchies* of analogous models tersely mentioned below, see Section 5).

These developments were until recently mainly restricted to the consideration of *nonlinear* evolutions satisfied by the *zeros* of a time-dependent polynomial the *coefficients* of which evolve according to *linear* systems of Ordinary Differential Equations (ODEs). Recently a convenient way to relate *directly* the time-evolution of the *zeros* of a time-dependent polynomial to the time-evolution of its *coefficients* has been noted [4], and this development has allowed the identification and investigation of several new *solvable* many-body problems characterized by the time-evolution of the *zeros* of polynomials the *coefficients* of which evolve in a *nonlinear* but *solvable/integrable* manner [4–6]. In the present paper we show how this development can be employed to identify *integrable Hamiltonian N -body problems of goldfish type*, see (1.1), with the forces $F_n(z)$ explicitly expressed in terms of N *arbitrary functions* of appropriate combinations of the N particle coordinates $z_n \equiv z_n(t)$ (see below). The motions take place in the *complex z -plane*, or, equivalently, in the Cartesian xy -plane with $z = x + iy$.

Notation 1.1. The basic building blocks underlining our findings are time-dependent *monic* polynomials of arbitrary order N ($N \geq 2$),

$$P_N(w; \vec{c}(t), \underline{z}(t)) = w^N + \sum_{m=1}^N [c_m(t) w^{N-m}] = \prod_{n=1}^N [w - z_n(t)] ; \quad (1.2)$$

here and hereafter the *complex* variable w is the argument of the polynomial, *indices* such as n, m run throughout from 1 to N , the N -vector $\vec{c}(t)$ features the N *coefficients* $c_m(t)$ of the polynomial (1.2) as its N components, $\underline{z}(t)$ denotes the *unordered* set of the N *zeros* $z_n(t)$ of the polynomial (1.2) (but see **Remark 1.2** below), and we generally assume all these variables to be *complex*. We instead assume the independent variable t (“time”) to be *real*. We generally focus on *generic* polynomials the *coefficients* and *zeros* of which are *generic complex* numbers, the *zeros* being *all different among themselves*, $z_n(t) \neq z_m(t)$ if $n \neq m$. Hereafter we often omit the *explicit* indication of the independent variable t when this can be done to streamline the presentation without causing confusion. Note that the notation $P_N(w; \vec{c}, \underline{z})$ is somewhat redundant, since this monic polynomial of degree N in w can be identified by assigning *either* its N *coefficients* c_m *or* its N *zeros* z_n ; indeed the N coefficients c_m can be expressed in terms of the N zeros z_n via the standard formula

$$c_m(t) = (-1)^m \sum_{1 \leq n_1 < n_2 < \dots < n_m \leq N} [z_{n_1}(t) z_{n_2}(t) \cdots z_{n_m}(t)] , \quad (1.3a)$$

so that

$$c_1(t) = - [z_1(t) + z_2(t) + \dots + z_N(t)] , \quad (1.3b)$$

$$\begin{aligned} c_2(t) = & z_1(t) z_2(t) + z_1(t) z_3(t) + \dots + z_1(t) z_N(t) \\ & + z_2(t) z_3(t) + z_2(t) z_4(t) + \dots + z_2(t) z_N(t) + \dots \\ & + z_{N-2}(t) z_{N-1}(t) + z_{N-2}(t) z_N(t) + z_{N-1}(t) z_N(t) , \end{aligned} \quad (1.3c)$$

and so on. On the other hand, while the assignment of the N coefficients c_m determines *uniquely*—up to permutations (but see **Remark 1.2** below)—the N zeros z_n , of course *explicit* formulas in terms of *elementary* functions (including *radicals*) expressing the zeros of a polynomial of degree N in terms of its *coefficients* are generally *only* available for $N \leq 4$. Finally let us note that hereafter formulas featuring the N -vector \underline{z} —see for instance (1.1)—are meant to be valid for *any* assignment of the N components z_n of this N -vector as one of the $N!$ permutations of the N components of the *unordered* set \underline{z} (but see the following **Remark 1.2**).

Remark 1.2. The statement according to which the set $\underline{z}(t)$ of the N zeros of the polynomial $P_N(w; c(t), \underline{z}(t))$, see (1.2), is *unordered* must be qualified according to the following observation, highly relevant to the following developments: if the time evolution of the zeros $z_n(t)$ is *continuous* in t —as it shall generally be in the following treatment—then the ordering of the N zeros $z_n(t)$ (i. e., the assignment of the value of the *integer* index n in its range from 1 to N to the function $z_n(t)$) can only be arbitrarily assigned at one specific time—for instance at the *initial* time $t = 0$ —remaining thereby unambiguously assigned for *all* time due to the continuous evolution over time of the functions $z_n(t)$.

In the following Section 2 we report and tersely discuss our main finding, which is then proven and further discussed in the following Section 3. In Section 4 a class of N -body problems of goldfish type, see (1.1), are displayed, still featuring N largely *arbitrary* functions while having the property that *all* its motions starting from *real* initial data (or *complex conjugate* pairs of such data: see below) are *multiply periodic*. A terse Section 5 outlines possible future developments.

2. Results

Proposition 2.1. *The N -body problem characterized by the following system of N coupled nonlinear Newtonian equations of motion of goldfish type, see (1.1), is Hamiltonian and integrable:*

$$\ddot{z}_n = \sum_{\ell=1, \ell \neq n}^N \left(\frac{2 \dot{z}_n \dot{z}_\ell}{z_n - z_\ell} \right) - \left[\prod_{\ell=1, \ell \neq n}^N (z_n - z_\ell) \right]^{-1} \sum_{m=1}^N \left[f_m(c_m) (z_n)^{N-m} \right]. \quad (2.1a)$$

Here the N quantities $z_n \equiv z_n(t)$ are the coordinates of the N particles moving in the complex z -plane, the N functions $c_m \equiv c_m(t)$ are expressed in terms of the coordinates $z_n \equiv z_n(t)$ by the formulas (1.3), while the N functions $f_m(c_m)$ can be arbitrarily assigned.

Moreover the initial-value problem—to compute the N functions $z_n(t)$ for all time $t > 0$ from the $2N$ given initial data $z_n(0), \dot{z}_n(0)$ —can be solved by algebraic operations (including changes of variables from the coefficients to the zeros of a polynomial of degree N such as (1.2)) and quadratures. The procedure to do so is detailed in the following Section 3.

For $N = 2$ this system, (2.1a), of 2 coupled nonlinear Newtonian equations of motion of goldfish type reads as follows:

$$\ddot{z}_n = -(-1)^n (z_1 - z_2)^{-1} [2 \dot{z}_1 \dot{z}_2 - f_1(-z_1 - z_2) z_n - f_2(z_1 z_2)], \quad n = 1, 2. \quad (2.1b)$$

For $N = 3$ this system, (2.1a), of 3 coupled nonlinear Newtonian equations of motion of *goldfish* type is given by (1.1) with

$$F_n(z) = - \left[\prod_{\ell=1, \ell \neq n}^3 (z_n - z_\ell) \right]^{-1} \left[f_1(-z_1 - z_2 - z_3) (z_n)^2 + f_2(z_1 z_2 + z_2 z_3 + z_3 z_1) z_n + f_3(z_1 z_2 z_3) \right], \quad n = 1, 2, 3. \quad (2.1c)$$

3. Proof

The proof of **Proposition 2.1** is actually quite easy. The starting point is the *identity* (for a proof see, if need be, [4])

$$\ddot{z}_n = \sum_{\ell=1, \ell \neq n}^N \left(\frac{2 \dot{z}_n \dot{z}_\ell}{z_n - z_\ell} \right) - \left[\prod_{\ell=1, \ell \neq n}^N (z_n - z_\ell) \right]^{-1} \sum_{m=1}^N \left[\ddot{c}_m (z_n)^{N-m} \right], \quad (3.1)$$

relating the time-evolution of the N zeros $z_n(t)$ and the N coefficients $c_m(t)$ of any time-dependent polynomial such as (1.2).

Now assume that the N functions $c_m \equiv c_m(t)$ satisfy the N (decoupled) Newtonian equations

$$\ddot{c}_m = f_m(c_m); \quad (3.2a)$$

it is then plain, see (3.1), that the N functions $z_n \equiv z_n(t)$ satisfy the system of ODEs (2.1a). **Proposition 2.1** is thereby proven. Indeed this implies that the solution of the initial-value problem for the system of ODEs (2.1a) is yielded by the following procedure. *Step (i)*: from the initial data $z_n(0)$, $\dot{z}_n(0)$ compute the corresponding functions $c_m(0)$, $\dot{c}_m(0)$ (via the formulas (1.3) and their time derivatives). *Step (ii)*: solve the *Newtonian* equations of motion (3.2a) with these initial data $c_m(0)$, $\dot{c}_m(0)$, obtaining thereby the functions $c_m(t)$ for all time $t > 0$. *Step (iii)*: the solutions $z_n(t)$ of the system of ODEs (2.1a) are then provided by the N zeros of the polynomial (1.2) with coefficients $c_m(t)$. And of course—since the *decoupled* Newtonian equations (3.2a) are of course *Hamiltonian* and *integrable*—and of course *solvable by quadratures*—all these properties also hold—up to *algebraic* operations—for the equations of motion (2.1a). Indeed the *Hamiltonian* equations of motion corresponding to the *Newtonian* equations of motion (3.2a) read of course as follows:

$$\dot{c}_m = p_m, \quad \dot{p}_m = f_m(c_m), \quad (3.2b)$$

where the N quantities $p_m \equiv p_m(t)$ are of course, in the *Hamiltonian* context, the *canonical momenta* conjugated to the *canonical coordinates* $c_m \equiv c_m(t)$. But then the *change of variables* from the N *canonical coordinates* $c_m \equiv c_m(t)$ of these standard *Hamiltonian* equations of motion to the *Hamiltonian* equations corresponding to the *Newtonian* equations of motion (2.1a) satisfied, in an *Hamiltonian* context, by the N new *canonical coordinates* $z_n \equiv z_n(t)$, besides being *algebraic*—see (1.2) and (1.3)—clearly corresponds to a *canonical transformation*, because it does not involve the *canonical momenta* (and it is moreover time-independent); implying that the goldfish system of equations of motion (2.1a) is as well *Hamiltonian* and *integrable*.

Remark 3.1. If it is assumed that the N , *a priori arbitrary*, functions $f_m(c_m)$ are *entire*, then clearly the only possible source of singularity in the evolution of the goldfish model (1.1) are *particle collisions*, i. e. the possibility that, at some time t_c during the time-evolution, two or more particle

coordinates coincide, say $z_n(t_c) = z_\ell(t_c)$ with $n \neq \ell$, implying a blow-up of the right-hand side of (2.1a). This is *not* a generic event for motions of the goldfish model (1.1) taking place in the *complex* plane and starting from a generic set of *complex* initial data $z_n(0), \dot{z}_n(0)$; while it is likely to happen for motions taking place on the *real* axis of the *complex* plane—as it might be the case when the functions $f_m(c_m)$ are all *real* functions of *real* arguments and the set of initial data $z_n(0)$ and $\dot{z}_n(0)$ include only *real* numbers (implying that the initial data $c_m(0)$ and $\dot{c}_m(0)$ are *all real*, see (1.2) and (1.3)). Note that such collisions generally imply that the *identities* of the colliding particles after the collision are *undetermined*; and generally in the case of motions taking place, before the collision, on the *real* axis, the collision causes the colliding particles to jump *off* the *real* axis, into the *complex* plane, in opposite orthogonal directions.

4. An example

In this Section 4 we display a subclass of N -body problems of *goldfish* type—see (2.1)—featuring the property that *all* the corresponding motions starting from *arbitrary real* (or *complex conjugate*: see below) initial data $z_n(0), \dot{z}_n(0)$ are *confined* and *multiply periodic*.

Let us assume for instance that the N functions $f_m(c_m)$ read as follows,

$$f_m(c_m) = (a_m - c_m) \exp[g_m(c_m)] , \quad (4.1)$$

where the N parameters a_m are N *arbitrary real* numbers and the N functions $g_m(c_m)$ are N *arbitrary real entire* functions of their arguments c_m (such that $g_m(\pm\infty) \geq 0$). It is then plain—see (3.2a)—that the time evolution of the quantities $c_m(t)$ corresponding to *any* assignment of *real* initial data $c_m(0), \dot{c}_m(0)$ takes place on the *real* axis of the *complex* c -plane and consists of periodic oscillations around the equilibria $c_m = a_m$ with some periods T_m (depending of course on the initial data and on the functions (4.1)):

$$c_m(t + T_m) = c_m(t), \quad \dot{c}_m(t + T_m) = \dot{c}_m(t) . \quad (4.2)$$

Hence—focussing now on the goldfish N -body model (2.1a) with (4.1)—for *any* assignment of initial data $z_n(0), \dot{z}_n(0)$ of the N particle coordinates which only include *real* numbers or *complex conjugate* pairs so that the corresponding initial data $c_m(0)$ and $\dot{c}_m(0)$ —see (1.2), (1.3) and the time-derivatives of (1.3)—are *all real*, it is plain that the corresponding time evolution of the coordinates $z_n(t)$ is *confined* and *multiply periodic*, because these coordinates are then the N *zeros* of a time-dependent polynomial, see (1.2), which is itself *multiply periodic*. These evolutions may of course take place on the *real* axis or in the *complex* plane (but then always featuring *pairs* of particles with *complex conjugate* coordinates), and may or may not feature particle collisions, see **Remark 3.1**. It is also plain that there are *open* sets of *real* initial data $z_n(0), \dot{z}_n(0)$ yielding *nonsingular* motions with the N particles remaining for *all* time confined to the *real* axis.

5. Outlook

It is plain that the approach employed in this paper provides the possibility to identify a large universe of new *integrable/solvable* N -body problems of goldfish type; it is for this reason that we felt the need to highlight this development in this paper, in spite of the fact that the finding reported herein is simply related to an approach already used in several previous papers [4–6]. Let us also note—without going into a detailed presentation—that the development reported herein opens the

possibility to generate *hierarchies* of such models having analogous features, as implied by the detailed treatment provided in [6].

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