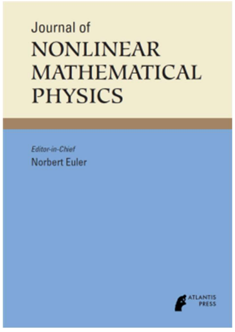

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Meromorphic and formal first integrals for the Lorenz system

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The Lorenz system
$$\dot{x}=\sigma(y-x),\ \ \dot{y}=rx-y-xz,\ \ \dot{z}=-\beta z+xy,$$
is completely integrable with two functional independent first integrals when $\sigma=0$ and β,r arbitrary. In this paper, we study the integrability of the Lorenz system when σ,β,r take the remaining values. For the case of $\sigma\beta\neq 0$, we consider the non-existence of meromorphic first integrals for the Lorenz system, and show that it is not completely integrable with meromorphic first integrals, and furthermore, if $2\sqrt{(\sigma+1)^2+4\sigma(r-1)/\beta}$ is not an odd number, then it also dose not admit any meromorphic first integrals and is not integrable in the sense of Bogoyavlensky. For the case of $\sigma\neq 0,\beta=0$, we study the existence of formal first integrals and present a necessary condition of the Lorenz system processing a time-dependent formal first integral in the form of $\Phi(x,y,z)\exp(\lambda t)$.

Keywords: Meromorphic first integrals; formal first integrals; Lorenz system.

2000 Mathematics Subject Classification: 70H06, 34M03, 34M15

1. Introduction

Consider the Lorenz system

$$\dot{x}=-\sigma x+\sigma y,\quad \dot{y}=rx-y-xz,\quad \dot{z}=-\beta z+xy, \tag{1.1}$$

where σ, r and β are parameters and the dot denotes derivative with respect to the time t. This system was first derived in [13] to describe atmospheric convection and was the first chaotic example showing the existence of the strange attractor. It is also well known for a variety of problems in lasers, dynamos, thermosyphons, brushless DC motors, electric circuits, chemical reactions and forward osmosis.

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The Lorenz system (1.1) has been widely investigated from the dynamical point of view, for more details please see the monograph [25]. Another important topic related to the Lorenz system (1.1) is the integrability. Kus [12] and Gupta [9] proposed a direct method to find some non-autonomous first integrals of the form $P(x,y,z)e^{\alpha t}$ with $P(x,y,z) \in \mathbb{C}[x,y,z]$ and $\alpha \in \mathbb{R}$. Later, by means of the weight homogeneous polynomials and the Darboux integrability theory, Zhang in his paper [29] characterized all the exponential factors and the Darboux first integrals of (1.1). Llibre and Valls in [15] studied the formal and analytic first integrals of the Lorenz system (1.1). Ballesteros *et al.* [4] applied the Poisson-Lie approach to the integrability of Lorenz system and obtained new integrable deformations for Lorenz system. For more details, see [10, 14, 27] and references quoted therein.

The existence of first integrals for differential equations plays a fundamental role in the qualitative theory of differential equations. In particular, for a given 3-dimensional differential system, if it admits a first integral, we can reduce the dimension of the considered system by one. The non-existence of first integrals can allow us to expect that the considered system is complex and admits a chaotic behavior. Therefore, it is an important issue to recognize the values of the parameters of given differential systems where they admit first integrals. However, there is no general way to deal with this problem. Many scholars have developed a lot of methods to study the existence of first integrals of the differential equations such as Ziglin analysis, Morales-Ramis theory, the Painlev*é* analysis, Darboux theory of integrability and the Carleman embedding procedure. See [1,6,17,24] and the references therein.

Let U be an open set in $\mathbb{C}^3$. A non-constant function $\Phi(x,y,z) \in C^1(\mathrm{U},\mathbb{C})$ is called a first integral of system (1.1) if it stays constant along all solution curves $(x(t),y(t),z(t))$ of system (1.1), i.e.,

$$(-\sigma x+\sigma y)\frac{\partial \Phi}{\partial x}+(rx-y-xz)\frac{\partial \Phi}{\partial y}+(-\beta z+xy)\frac{\partial \Phi}{\partial z}\equiv 0.$$

Similarly, we can also define a non-constant differentiable function $\Psi(x,y,z,t)$ as a time-dependent first integral of system (1.1) if $\Psi(x,y,z,t)$ satisfies

$$(-\sigma x+\sigma y)\frac{\partial \Psi}{\partial x}+(rx-y-xz)\frac{\partial \Psi}{\partial y}+(-\beta z+xy)\frac{\partial \Psi}{\partial z}+\frac{\partial \Psi}{\partial t}\equiv 0.$$

If the first integral $\Phi(x,y,z)$ or $\Psi(x,y,z,t)$ is a formal power series in (x,y,z) or a meromorphic function with respect to (x,y,z), then it is called a formal first integral or meromorphic first integral of (1.1), respectively.

The aim of this paper is to make a complete investigation to the integrability and non-integrability of the Lorenz system (1.1), which is well known to show different dynamical properties according to different values of parameters. We will mainly divide our study into three cases: $\sigma = 0$; $\sigma\beta \neq 0$; and $\sigma \neq 0, \beta = 0$.

The case when $\sigma = 0$ is trivial, since system (1.1) can be reduced to a linear system. Moreover, as pointed out by Llibre and Valls [15], system (1.1) with $\sigma = 0$ can also be viewed as a completely integrable system with two functionally independent first integrals

$$\Phi_1 = x \quad \text{and} \quad \Phi_2 = F_1(x,y,z)\exp\left(-2\arctan\frac{F_2(x,y,z)}{F_3(x)}\right),$$

where

$$\begin{aligned}F_1 =& x(r^2x^3-(1+\beta)rx^2y+bxy^2+x^3y^2+\beta(\beta-1)rxz\\&-2rx^3z-\beta(\beta-1)yz+(1-\beta)x^2yz+bxz^2+x^3z^2),\\F_2 =&\frac{(\beta-1)(rx-y)+(\beta+1)xz-2x^2y}{(\beta+1)((r-z)x-y)F_3},\\F_3 =&\sqrt{\frac{4(\beta+x^2)}{\beta+1}-1}.\end{aligned}$$

Our first result deals with the Lorenz system (1.1) with $\sigma\beta\neq 0$. In this case, we investigate the non-integrability of (1.1). Unlike Hamiltonian systems where the definition of the integrability is well-defined in the sense of Liouville, there is no universal definition of integrability for general dynamical systems. In our work, we focus on two kinds of notions of the integrability, complete integrability and B-integrability(see section 2 for the definitions and comments on two notions). Let Γ be the flow generated by the particular solution $(x,y,z)=(0,0,e^{-\beta t})$ of system (1.1). The first main result of the paper is the following.

Theorem 1.1. *Assume that $\sigma\beta\neq 0$. Then system* (1.1) *is not completely integrable with meromorphic first integrals in a neighborhood of Γ. Moreover, if*

$$\frac{2\sqrt{(\sigma+1)^2+4\sigma(r-1)}}{|\beta|}$$

is not an odd number, then system (1.1) *does not possess any meromorphic first integral and is not B-integrable in the meromorphic category.*

We note that Theorem 1.1 characterizes first integrals for system (1.1) with $\sigma\beta\neq 0$ in the category of meromorphic functions, and its proof is based on the differential Galois theory. This method has its roots in the work of Morales-Ruiz, Ramis, Simó, Baider, Churchill, Rod and Singer [3, 7, 22–24]. Roughly speaking, their results show that if Hamiltonian system with n degrees of freedom admits n meromorphic first integrals which are in involution and independent, then the identity component of the differential Galois group of the normal variational equations along a non-equilibrium solution must be commutative. Many scholars applied Morales-Ramis theory to large numbers of practical models, see for instance [8, 21, 26]. For general dynamical systems, the authors in [18, 19] presented some results which reveal the relation between the meromorphic first integrals and the differential Galois group of normal variational equations, which will be used to prove Theorem 1.1.

Next we shall study system (1.1) with $\sigma\neq 0, \beta=0$.

Theorem 1.2. *Assume that $\sigma\neq 0, \beta=0$, $m:=\sqrt{(\sigma-1)^2+4\sigma r}\neq 0$, and $(\sigma+1)/m\notin\mathbb{Q}\cap(-1,1)$. Then following statements hold.*

(**a**) *System* (1.1) *has one and only one formal first integral in the sense of functional independence.*

(**b**) *If* (1.1) *has one time-dependent formal first integral of the form* $\Phi(x,y,z)\exp(-\lambda t)$*, then*

$$\lambda = \frac{m}{2}(\alpha_1 - \alpha_2) - \frac{(\sigma+1)}{2}(\alpha_1 + \alpha_2). \tag{1.2}$$

(**c**) *If* (1.1) *has one time-dependent formal first integral of the form* $\Phi(x,y,z)\exp(-\lambda t)$*, and* $(\sigma+1)/m \notin \mathbb{Q}$*, then there exists a positive integer* $k \in \mathbb{N}$*, such that* $\lambda = -k(\sigma+1)$*.*

The outline of this paper is as follows. In Section 2, we present some preliminary results that will be used later. The proof of Theorem 1.1, 1.2 are given in Sections 3 and 4, respectively.

2. Preliminaries

Consider the following n-dimensional system

$$\frac{dx}{dt} = F(x), \quad x \in \mathscr{M}, \;\; t \in \mathbb{C}, \tag{2.1}$$

where $F(x)$ is an n-dimensional vector-valued analytic function, and $\mathscr{M}$ is an n-dimensional complex analytic manifold.

Definition 2.1. We say that the system (2.1) is completely integrable if it admits $n-1$ functionally independent first integrals $\Phi_1, \cdots, \Phi_{n-1}$.

Definition 2.2. We say that the system (2.1) is is B-integrable if it admits k functionally independent first integrals $\Phi_1, \cdots, \Phi_k$ and $(n-k)$ vector fields $w_1 = F, \cdots, w_{n-k}$ such that

$$[w_i, w_j] = 0, \;\text{ and }\; w_j[\Phi_i] = 0, \;\text{ for }\; 1 \le i \le k, \;\; 1 \le j \le n-k.$$

Assume system (2.1) is completely integrable. Then the orbits of this system are contained in the curves

$$\{x | \Phi_1(x) = c_1, \cdots, \Phi_{n-1}(x) = c_{n-1}, \;\text{ where }\; c_1, \cdots, c_{n-1} \in \mathbb{C}\}.$$

Hence, its general solutions can be obtained analytically in an "explicit" way. Furthermore, the topological structure of a completely integrable system is also simple. Llibre *et al.* [16] proved that a completely integrable system is orbitally equivalent to a linear differential system.

The notion of B-integrability was introduced by Bogoyavlensky [5]. It can be shown that if a system (2.1) is B-integrable, then it is integrable by quadrature. By definition, we can easily see that B-integrability is a weaker definition of complete integrability. Actually, B-integrability is also a natural generalization of the Liouvillian integrability from Hamiltonian systems to general dynamical systems, see [5] for details.

Let $\psi(t)$ be a non-equilibrium solution of system (2.1). The variational equations along the phase curve Γ of $\psi(t)$ read

$$\frac{d\xi}{dt} = A(t)\xi, \;\; A(t) = \left.\frac{\partial F}{\partial x}\right|_{x=\psi(t)}, \;\; \xi \in T_\Gamma \mathscr{M}, \tag{2.2}$$

where $T_\Gamma \mathscr{M}$ is the vector bundle of $T\mathscr{M}$ restricted on Γ. Further, denoting by $N = T_\Gamma \mathscr{M}/\Gamma$ the normal bundle, and using the nature projection $\pi : T_\Gamma \mathscr{M} \to N$, the variational equations can be reduced to the normal variational equations

$$\dot{\eta} = \pi_*(T(F)(\pi^{-1}\eta)), \quad \eta \in N. \tag{2.3}$$

Now we proceed to consider the differential Galois group of (2.3). Roughly speaking, the differential Galois group of (2.3) is a matrix group $G \subset GL(n-1,\mathbb{C})$ acting on the fundamental solutions of (2.3) such that it dose not change polynomial and differential relations between them, for more details see [24]. We also recall the identity component G^0 of G is the unique connected component containing the identity, which is a normal subgroup of G with finite index.

The main ingredients for the proofs of theorem 1.1 are the following theorems, which characterise the relation between the existence of meromorphic first integrals of (2.1) and the property of the identity component of the differential Galois group associated with (2.3), and give the necessary conditions for the complete integrability and B-integrability of (2.1), respectively.

Theorem 2.1. *[2] Assume that system (2.1) is B-integrable in the meromorphic category in a neighbourhood of a phase curve Γ. Then the identity component of the differential Galois group of the normal variational equations (2.3) along Γ is Abelian.*

Theorem 2.2. *[18,19] Assume that system* (2.1) *has $m (1 \le m < n)$ functionally independent meromorphic first integrals in a neighborhood of Γ. Then the Lie algebra $\mathscr{G}$ of the differential Galois group G of equations (2.2) has m meromorphic invariants, and the identity component G^0 of G has at most $(n-m-1)(n-1)$ generators, i.e.,*

$$G^0 = \{(e^{\mathfrak{X}_1 t_1}\cdot e^{\mathfrak{X}_2 t_2}\cdots e^{\mathfrak{X}_k t_k})^s \mid (t_1,\cdots,t_k) \in \tilde{\mathscr{V}} \subset \mathbb{C}^k, s \in \mathbb{N}\},$$

where $\{\mathfrak{X}_1,\cdots,\mathfrak{X}_k\}$ is a basis of $\mathscr{G}$ with $k \le (n-m-1)(n-1)$, $\tilde{\mathscr{V}}$ is a neighborhood of the original element in $\mathbb{C}^l$. Especially,

(1) *If $m = n-1$, i.e., system* (2.1) *is completely integrable, then $\mathscr{G} = \{0\}, G^0 = \{\mathbf{1}\}$, where* **1** *denote the identity element of G.*

(2) *If $m = n-2$, then $\mathscr{G}, G^0$ have at most $n-1$ generators.*

(3) *If $n = 3$ and $m = 1$, then $\mathscr{G}, G^0$ are solvable.*

In general, for a n-th($n \ge 2$) order linear differential equations, it is very difficult to compute the corresponding differential Galois group in most cases. However, for $n = 2,3$, Kovacic [11] and Singer [28] have proposed an algorithm to get the Galois group completely. In what follows, we list some results of second order differential equation by Kovacic [11].

Let us consider the following second order linear differential equation

$$\frac{d^2\xi}{dx^2} + a(x)\frac{d\xi}{dx} + b(x)\xi = 0, \quad a(x), b(x) \in \mathbb{C}(x), \tag{2.4}$$

where $\mathbb{C}(x)$ stands for the field consisting of all rational functions on the complex plane $\mathbb{C}$. Making a well-known change of variable

$$\xi = \chi \exp\left(-\frac{1}{2}\int_{x_0}^{x} a(s)ds\right),$$

we can eliminate the term involving $d\xi/dx$ and obtain the reduced form of (2.4)

$$\frac{d^2\chi}{dx^2} = r(x)\chi, \quad r(x) = \frac{a^2}{4} + \frac{1}{2}\frac{da}{dx} - b. \tag{2.5}$$

The differential Galois group G of system (2.5), belonging to $SL(2,\mathbb{C})$, can be classified with the following four cases [11].

Theorem 2.3. *One and only one of the following four cases can occur.*

Case 1. G is conjugate to a triangular group. In this case, (2.5) *has a solution of the form* $e^{\int \omega}$ *with* $\omega \in \mathbb{C}(x)$.

Case 2. G is not of case 1, but is conjugate to a subgroup of

$$D = \{\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} | c \in \mathbb{C}, c \neq 0\} \cup \{\begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} | c \in \mathbb{C}, c \neq 0\}.$$

In this case, (2.5) *has a solution of the form* $e^{\int \omega}$ *with* ω *algebraic over* $\mathbb{C}(x)$ *of degree* 2 *and case 1 does not hold.*

Case 3. G is not of case 1 or case 2, but is a finite group. In this case, all solutions of (2.5) *are algebraic over* $\mathbb{C}(x)$.

Case 4. $G = SL(2,\mathbb{C})$. *Then* (2.5) *is not integrable in Liouville sense.*

Recall that the order of the poles of $r(x) = p(x)/q(x)$, where polynomials $p(x)$ and $q(x)$ are relatively prime, is the multiplicity of the zero of $q(x)$, and the order of r at ∞ is $\deg q(x) - \deg p(x)$. Kovacic [11] presented a complete algorithm to analysis which cases the differential Galois group G of system (2.5) falls into, see Appendix A for the Kovacic's algorithm.

The next result [11] provides some necessary conditions for cases 1-3.

Theorem 2.4. *The following conditions are necessary for the respective types of Theorem 2.3 to hold.*

Case 1. Every pole of r must have even order or else have order 1. *The order of r at* ∞ *must be even or else be great than two.*

Case 2. r must have at least one pole such that either has odd order greater than 2 *or else has order two.*

Case 3. The order of a pole of r cannot exceed 2 *and the order of r at* ∞ *must be at least two. If the partial fraction expansion of r is*

$$r = \sum_i \frac{\alpha_i}{(x-\beta_i)^2} + \sum_i \frac{\gamma_j}{x-d_i}$$

then

$$\sqrt{1+4\alpha_i} \in \mathbb{Q}, \quad \sum_j \gamma_j = 0, \quad \sqrt{\sum_i \alpha_i + \sum_i \gamma_j d_i} \in \mathbb{Q}.$$

It should be pointed out that the identity component G^0 in case 1-3 are solvable, but in case 4 is not, see Lemma 2.2 in [24]. Form the proof of Theorem 2.4 by Kovacic, we also see that the necessary condition for case 3 is based on the assumption that G is a finite group and admits an algebraic solution, which has nothing to do with the assumption that case 1,2 do not hold. Hence, if the necessary condition for case 3 evaluates to false, then G is not a finite group.

3. The case of $\sigma\beta \neq 0$

Suppose $\sigma\beta \neq 0$, then $(x,y,z) = (0,0,e^{-\beta t})$ is a non-equilibrium solution of system (1.1). Let Γ be the flow generated by this solution. Taking $(x,y,z) = (\xi, \eta, \zeta + e^{-\beta t})$ in (1.1) and neglecting quadratic terms of (ξ,η,ζ), we obtain the variational equations along Γ

$$\begin{pmatrix}\dot{\xi}\\ \dot{\eta}\\ \dot{\zeta}\end{pmatrix}=\begin{pmatrix}-\sigma & \sigma & 0\\ r-e^{-\beta t} & -1 & 0\\ 0 & 0 & -\beta\end{pmatrix}\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix}. \tag{3.1}$$

Then, the equations for variables (ξ,η) form a closed subsystem

$$\begin{pmatrix}\dot{\xi}\\ \dot{\eta}\end{pmatrix}=\begin{pmatrix}-\sigma & \sigma\\ r-e^{-\beta t} & -1\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \tag{3.2}$$

which is the so-called normal variational equations along Γ. A straightforward computation yields a second-order equation

$$\ddot{\xi}+(\sigma+1)\dot{\xi}-\sigma(r-1-e^{-\beta t})\xi=0, \tag{3.3}$$

which is equivalent to system (3.2). In order to transform (3.3) into an equation with rational coefficients, we apply a variable change $\tau=e^{-\beta t}$ and obtain

$$\xi''+\frac{\beta-\sigma-1}{\beta\tau}\xi'-\frac{\sigma(r-\tau-1)}{\beta^2\tau^2}\xi=0, \tag{3.4}$$

where the prime denotes the derivative with respect to τ. Further, under the transform $\xi(\tau)=\chi(\tau)\tau^{\frac{1+\sigma-\beta}{2\beta}}$, system (3.4) can be rewritten as

$$\chi''=r(\tau)\chi,\ \ r(\tau)=\frac{(\sigma+1)^2-\beta^2+4\sigma(r-1)-4\sigma\tau}{4\beta^2\tau^2}. \tag{3.5}$$

The following lemma plays a key role in our first result.

Lemma 3.1. *Suppose $\sigma\beta\neq 0$. Then the following statements hold.*
(1) The differential Galois group G of (3.5) is not finite.
(2) The identity component G^0 of G is solvable if and only if

$$\frac{2\sqrt{(\sigma+1)^2+4\sigma(r-1)}}{|\beta|}$$

is an odd number.

Proof. (1) Obviously, $r(\tau)$ has two singular points 0 and ∞ with order 2 and 1, respectively. In view of Theorem 2.4, the necessary conditions for case 1,3 cannot hold. Therefore, the differential Galois group G of (3.5) is not finite.

(2) Note that in case 1-3 the identity component G^0 of G are solvable, but in case 4 is not. Based on the discussion above, we see that G^0 is solvable if and only if it belongs to case 2 in Theorem 2.3. In what follows, we apply the Kovacic's algorithm to (3.5) step by step to get the necessary and sufficient condition such that case 2 holds. We only consider the case $\beta>0$, the case $\beta<0$ can be dealt similarly.

Step 1 Some simple calculations lead to

$$E_0 = \{2, 2 \pm \frac{2\sqrt{(\sigma+1)^2+4\sigma(r-1)}}{\beta}\} \cap \mathbb{Z}, \quad E_\infty = \{1\}.$$

Step 2 If $2\sqrt{(\sigma+1)^2+4\sigma(r-1)}/\beta$ is not an odd number, then $d \triangleq \frac{1}{2}(e_\infty - e_0)$ is not a non-negative integer for any $e_0 \in E_0$ and $e_\infty \in E_\infty$, which means case 2 cannot happen. If $2\sqrt{(\sigma+1)^2+4\sigma(r-1)}/\beta$ is an odd number, then there exists only one option

$$e_0 = 2 - \frac{2\sqrt{(\sigma+1)^2+4\sigma(r-1)}}{\beta}, \quad e_\infty = 1,$$

such that $d = \frac{1}{2}(e_\infty - e_0) \in \mathbb{N} \cup \{0\}$. Hence, we obtain $\theta = \frac{1-2d}{2\tau}$ and progress to the next step.

Step 3 Let $P(\tau) = \sum\limits_{k=0}^{d} a_k \tau^k$, where

$$a_d = 1, \quad a_{k-1} = -\frac{f(k,d)}{g(k,d)} a_k, \quad k = d, d-1, ..., 1,$$
$$f(k,d) = k(k-1)(k-2) + \frac{3(1-2d)}{2} k(k-1) + d(2d-1)k, \quad g(k,d) = \frac{4\sigma}{\beta^2}(k-1-d),$$

then polynomial $P(\tau)$ satisfies

$$P''' + 3\theta P'' + (3\theta^2 + 3\theta' - 4r)P' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')P = 0,$$

i.e.,

$$P''' + \frac{3(1-2d)}{2\tau} P'' + (\frac{d(2d-1)}{\tau^2} + \frac{4\sigma}{\beta^2 \tau})P' - \frac{4d\sigma}{\beta^2 \tau^2} P = 0.$$

It follows that case 2 holds.

Summarizing the above discussion, when $\beta > 0$, we see that the differential Galois group of (3.5) belongs to case 2 if and only if $2\sqrt{(\sigma+1)^2+4\sigma(r-1)}/\beta$ is not an odd number. The proof is completed. □

Remark 3.1. One can also get the proof of Lemma 3.1 by observing that (3.3) can be transformed into the Whittaker equation for which their differential Galois group is studied, see the reference [24] for details.

Proof of Theorem 1.1. Let us observe that the differential Galois group of (3.2) can be viewed as a normal subgroup of the differential Galois group of (3.5), see Theorem 2.5 in [24]. So, by Lemma 3.1, the differential Galois group of (3.2) is also not finite. In addition, the identity component G^0 is a normal subgroup of G with finite index, so G^0 is the trivial subgroup $\{\mathbf{1}\}$ if and only if G is a finite group [24]. Therefore, the identity component G^0 of (3.2) is not $\{\mathbf{1}\}$ and the Lorenz system (1.1) is not completely integrable with meromorphic first integrals by Theorem 2.2.

Similarly, if

$$\frac{2\sqrt{(\sigma+1)^2+4\sigma(r-1)}}{|\beta|}$$

is not an odd number, according to Lemma 3.1, the identity component of differential Galois group of (3.5) as well as (3.2) is not solvable. Note that an Abelian group is a solvable group. Therefore, the identity component of differential Galois group of (3.2) is also not Abelian. In view of Theorem 2.1 and 2.2, we can trivially complete Theorem 1.1. □

Remark 3.2. Let $\beta = 8/3$, $\sigma = 10$ and $r = 28$, then it is not difficult to calculate that

$$\frac{2\sqrt{(\sigma+1)^2+4\sigma(r-1)}}{|\beta|} = \frac{3\sqrt{1201}}{8} \tag{3.6}$$

is not an odd number, by Theorem 1.1, we conclude that system (1.1) admits no meromorphic first integrals. On the other hand, it is well known that the Lorenz system (1.1) has a strange attractor [25] under the above conditions. This fact can be seen as a new evidence of the connection between the chaos and the non-integrability.

Remark 3.3. If $\sigma\beta \neq 0$ and $2\sqrt{(\sigma+1)^2+4\sigma(r-1)}/\beta$ is an odd number, then system (1.1) has at most one meromorphic first integral in sense of functional independence. Specifically, let $\beta = 2\sigma = 1$, $r = 0$, one can find a meromorphic first integral

$$\Phi(x) = \frac{2(x^2-z)^2}{y^2-1+2x^2z},$$

which means system (1.1) has one and only one meromorphic first integral $\Phi(x)$ in sense of functional independence.

4. The case of $\sigma \neq 0, \beta = 0$

Assume $\sigma \neq 0$ and $\beta = 0$. We make the change of variables

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & \frac{\sigma}{m} & 0 \\ \frac{\sigma+\lambda_1}{\sigma} & \frac{\sigma+\lambda_2}{m} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

and transform system (1.1) to

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 + a x_1 x_3 + b x_2 x_3, \\ \dot{x}_2 = \lambda_2 x_2 + c x_1 x_3 + d x_2 x_3, \\ \dot{x}_3 = e x_1^2 + f x_1 x_2 + g x_2^2, \end{cases} \tag{4.1}$$

where

$$\begin{aligned} &\lambda_1 = \frac{-(\sigma+1)-m}{2}, \ \ \lambda_2 = \frac{-(\sigma+1)+m}{2}, m = \sqrt{(\sigma-1)^2+4\sigma r}, \\ &a = \frac{\sigma}{m}, \ \ b = \frac{\sigma^2}{m^2}, c = -1, d = -\frac{\sigma}{m}, \\ &e = \frac{\sigma+\lambda_1}{\sigma}, \ \ f = \frac{\sigma-1}{m}, \ \ g = \frac{(\sigma+\lambda_2)\sigma}{m^2}. \end{aligned}$$

Lemma 4.1. *Assume that* $\beta = 0$, $m \neq 0$ *and* $(\sigma+1)/m \notin \mathbb{Q} \cap (-1,1)$. *Then system (1.1) has at most one formal first integral.*

Proof. For the singularity $(0,0,0)$, system (1.1) has the eigenvalues λ_1, λ_2, $\lambda_3 = 0$. By Theorem 1 in [30], we know the number of functionally independent formal first integrals of system (1.1) is no more than the number of linearly independent elements of the set

$$\mathscr{G} = \left\{(k_1,k_2,k_3) \in (\mathbb{Z}^+)^3 : k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3 = 0, k_1 + k_2 + k_3 \geq 1\right\}.$$

Clearly, as $m \neq 0$ and $(\sigma+1)/m \notin \mathbb{Q}\cap(-1,1)$, the elements in $\mathscr{G}$ have the form $(k_1,k_2,k_3) = (0,0,m)$ with $m \in \mathbb{Z}^+$. It follows that system (1.1) has at most one formal first integral in the sense of functionally independence. □

Now let us introduce two linear partial differential operators

$$\begin{aligned}
\mathscr{L} &= \lambda_1 x_1 \frac{\partial}{\partial x_1} + \lambda_2 x_2 \frac{\partial}{\partial x_2},\\
\mathscr{L}_\lambda &= \lambda_1 x_1 \frac{\partial}{\partial x_1} + \lambda_2 x_2 \frac{\partial}{\partial x_2} - \lambda,\\
\mathscr{H} &= -(ax_1x_3 + bx_2x_3)\frac{\partial}{\partial x_1} - (cx_1x_3 + dx_2x_3)\frac{\partial}{\partial x_2} - (ex_1^2 + fx_1x_2 + gx_2^2)\frac{\partial}{\partial x_3},
\end{aligned}$$

then one can regard $\mathscr{L}$ and $\mathscr{L}_\lambda$ as linear maps from H_{k+1} into H_{k+1}, and $\mathscr{H}$ as a linear map from H_k into H_{k+1}, where $H_k \subset \mathbb{C}[x_1,x_2,x_3]$ denote the linear space formed by the homogeneous polynomials of degree k.

Lemma 4.2. *Assume that $\beta = 0$, $m \neq 0$ and $(\sigma+1)/m \notin \mathbb{Q}\cap(-1,1)$. Then* Img $\mathscr{H} \subseteq$ Img $\mathscr{L}$.

Proof. Let

$$\mathscr{S} = \left\{P | P = \sum_{|\alpha|=k+1} C_{k+1,\alpha} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3},\ \ \alpha_1 + \alpha_2 > 0\right\},$$

where $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in (\mathbb{Z}^+)^3$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $C_{k+1,\alpha} \in \mathbb{C}$, $l \in \mathbb{N}/\{0\}$. Then $\mathscr{S}$ is a subset of H_{k+1}, and Img $\mathscr{H} \subseteq \mathscr{S}$. On the other hand, according to the assumption of as $m \neq 0$ and $(\sigma+1)/m \notin \mathbb{Q}\cap(-1,1)$, it is easy to check that

$$\lambda_1\alpha_1 + \lambda_2\alpha_2 \neq 0$$

for any $\alpha_1, \alpha_2 \in \mathbb{Z}^+$ with $\alpha_1 + \alpha_2 > 0$. Now for any element of $\mathscr{S}$

$$P = \sum_{|\alpha|=k+1} C_{k+1,\alpha} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3},$$

set

$$\widetilde{P} = \sum_{|\alpha|=k+1} \widetilde{C}_{k+1,\alpha} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3},$$

where $\widetilde{C}_{k+1,\alpha} = 0$ if $\alpha_1 + \alpha_2 = 0$ and $\widetilde{C}_{k+1,\alpha} = C_{k+1,\alpha}/(\lambda_1\alpha_1 + \lambda_2\alpha_2)$ if $\alpha_1 + \alpha_2 > 0$, then $\widetilde{P} \in H_{k+1}$ and $\mathscr{L}[\widetilde{P}] = P$, which means $\mathscr{S} \subseteq$ Img $\mathscr{L}$. Therefore, Img $\mathscr{H} \subseteq \mathscr{S} \subseteq$ Img $\mathscr{L}$. □

Proof of Theorem 1.2. (**a**) Note that if (1.1) admits a formal first integral $\widetilde{\Phi}(x,y,z)$, then we can rewrite $\widetilde{\Phi} = \Phi(x_1,x_2,x_3)$ in the variables (x_1,x_2,x_3) and Φ must be a formal first integral of (4.1) , and vice versa. Hence, we need only to study the existence and uniqueness of formal first integrals for system (4.1).

Let $\Phi = \Phi_l + \Phi_{l+1} + \cdots$ with

$$\Phi_k = \sum_{|\alpha|=k} C_{k,\alpha} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}, \quad k = l, l+1, \cdots,$$

and $\Phi_l \not\equiv 0$.

By the definition of first integrals, Φ is a first integral of (4.1) if and only if

$$(\lambda_1 x_1 + a x_1 x_3 + b x_2 x_3)\frac{\partial \Phi}{\partial x_1} + (\lambda_2 x_1 + c x_1 x_3 + d x_2 x_3)\frac{\partial \Phi}{\partial x_2} + (e x_1^2 + f x_1 x_2 + g x_2^2)\frac{\partial \Phi}{\partial x_3} = 0,$$

i.e.,

$$\mathscr{L}[\Phi] = \mathscr{H}[\Phi],$$

or equivalently

$$\mathscr{L}[\Phi_l] = 0, \tag{4.2}$$

$$\mathscr{L}[\Phi_{j+1}] = \mathscr{H}[\Phi_j], \quad j = l, l+1, \cdots. \tag{4.3}$$

The next thing to do is to show that the equations (4.2)-(4.3) are formally solvable. In fact, (4.2) has a solution $\Phi_l^* = x_3^l$. By Lemma 4.2, $\mathscr{H}[\Phi_l^*] \in \mathrm{Img}\ \mathscr{H} \subseteq \mathrm{Img}\ \mathscr{L}$, so we can get $\Phi_{l+1}^* \in H_{l+1}$ such that $\mathscr{L}[\Phi_{l+1}^*] = \mathscr{H}[\Phi_l^*]$. By induction, we could get Φ_j^*, $j = l, l+1, \cdots$, such that (4.3) hold. Let $\Phi^* = \Phi_l^* + \Phi_{l+1}^* \cdots$, then Φ^* is a formal first integral of (4.1). Moreover, by Lemma 4.1, the proof is completed.

(**b**) Assume that (1.1) has a time-dependent formal first integral of the form $\tilde{\Phi}(x,y,z)e^{-\lambda t}$, then (4.1) has a corresponding formal time-dependent formal first integral of the form $\Phi(x_1,x_2,x_3)e^{-\lambda t}$, i.e,

$$\mathscr{L}_\lambda[\Phi] = \mathscr{H}[\Phi]. \tag{4.4}$$

Expanding $\Phi(x_1,x_2,x_3)$ as

$$\Phi = \Phi_l + \Phi_{l+1} + \cdots,$$

with $\Phi_k = \sum_{|\alpha|=k} C_{k,\alpha} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$, $k = l, l+1, \cdots$, and $\Phi_l \not\equiv 0$.

Comparing the lowest order terms of (4.4) yields

$$\mathscr{L}_\lambda[\Phi_l] = 0$$

or

$$\sum_{|\alpha|=l} C_{l,\alpha}(\lambda_1\alpha_1 + \lambda_2\alpha_2 - \lambda) x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} = 0.$$

Hence, $\lambda = \lambda_1\alpha_1 + \lambda_2\alpha_2$ has to be fulfilled for any nonzero coefficient $C_{l,\alpha}$, namely,

$$\lambda = \frac{\sqrt{(\sigma-1)^2 + 4\sigma r}}{2}(\alpha_1 - \alpha_2) - \frac{(\sigma+1)}{2}(\alpha_1 + \alpha_2). \tag{4.5}$$

This relation implies (1.2).

(**c**) Repeating the same arguments, we get (4.5) and claim that $(\alpha_1,\alpha_2)\in(\mathbb{Z}^+)^2$ satisfying $\lambda=\lambda_1\alpha_1+\lambda_2\alpha_2$ is unique. In deed, suppose there exists $(\tilde\alpha_1,\tilde\alpha_2)\in(\mathbb{Z}^+)^2$ such that $(\tilde\alpha_1,\tilde\alpha_2)\neq(\alpha_1,\alpha_2)$ and $\lambda=\lambda_1\tilde\alpha_1+\lambda_2\tilde\alpha_2$. Then, we have

$$\lambda_1(\tilde\alpha_1-\alpha_1)+\lambda_2(\tilde\alpha_2-\alpha_2)=0,$$

or

$$\frac{\sigma+1}{\sqrt{(\sigma-1)^2+4\sigma r}}=\frac{\tilde\alpha_2-\alpha_2-\tilde\alpha_1+\alpha_1}{\tilde\alpha_2-\alpha_2+\tilde\alpha_1-\alpha_1}\in\mathbb{Q},$$

which contradicts assumptions. Without loss of generality, we set $\Phi_l=x_1^{\alpha_1}x_2^{\alpha_2}x_3^{l-\alpha_1-\alpha_1}$.

Next, identifying the homogenous component of degree $l+1$ of (4.4) yields

$$\mathscr{L}_\lambda[\Phi_{l+1}]=\mathscr{H}[\Phi_l]. \tag{4.6}$$

With some calculations, we obtain

$$\begin{aligned}\mathscr{H}[\Phi_l]=&-(a\alpha_1+d\alpha_2)x_1^{\alpha_1}x_2^{\alpha_2}x_3^{l-\alpha_1-\alpha_1+1}\\&-b\alpha_1x_1^{\alpha_1-1}x_2^{\alpha_2+1}x_3^{l-\alpha_1-\alpha_1+1}\\&-c\alpha_2x_1^{\alpha_1+1}x_2^{\alpha_2-1}x_3^{l-\alpha_1-\alpha_1+1}\\&-e\alpha_3x_1^{\alpha_1+2}x_2^{\alpha_2}x_3^{l-\alpha_1-\alpha_1-1}\\&-f\alpha_3x_1^{\alpha_1+1}x_2^{\alpha_2+1}x_3^{l-\alpha_1-\alpha_1-1}\\&-g\alpha_3x_1^{\alpha_1}x_2^{\alpha_2+2}x_3^{l-\alpha_1-\alpha_1-1}.\end{aligned}$$

Using the same argument as in the proof of statement (a), the left side of (4.6) has no term $x_1^{\alpha_1}x_2^{\alpha_2}x_3^{l-\alpha_1-\alpha_1+1}$ and so

$$a\alpha_1+d\alpha_2=\frac{\sigma}{m}(\alpha_1-\alpha_2)=0. \tag{4.7}$$

Taken together with (4.5) and (4.7), we have $\alpha_1=\alpha_2:=\alpha$ and

$$-\frac{\lambda}{\sigma+1}=\alpha\in\mathbb{N}.$$

□

Remark 4.1. Consider the case of $\sigma=\frac{1}{3},\beta=0,r=2$, it is easy to verify the conditions of Theorem 1.2. In this case, we can set $\lambda=-\frac{4}{3}$. Then, the Lorenz system may have a formal integrals of motion of the form $\Phi(x,y,z)\exp(\frac{4}{3}t)$. In fact, system (1.1) admits the time-dependent first integral

$$I=\big(-x^2+\frac{1}{3}y^2+\frac{2}{3}xy+x^2z-\frac{3}{4}x^4\big)e^{\frac{4}{3}t}.$$

Remark 4.2. The statement (**a**) in Theorem 1.2 can also be derived from Theorem 1 in [20]. However, the criterion we present is more elementary and crucial to the proof of the statement (**b**, **c**) in Theorem 1.2. Moreover, we can adopt the same procedure as in the proof of Theorem 1.2 to other nonlinear differential systems which have a fixed point with some single multiply zero eigenvalues and other $\mathbb{N}$-independent nonzero eigenvalues. Then we can obtain similar results about the existence of formal first integrals of them.

Remark 4.3. In statement (b), we show that if $-\lambda/(\sigma+1)\in\mathbb{N}$, then there is no term $x_1^{\alpha}x_2^{\alpha}x_3^{l+1-2\alpha}$ in $\mathscr{H}[\Phi_l]$. Furthermore, some tedious calculations lead to
the coefficient of $x_1^{\alpha_1}x_2^{\alpha_2}x_3^{l+2-\alpha_1-\alpha_1}$ in $\mathscr{H}[\Phi_{l+1}]$

$$\frac{bc\alpha}{-\lambda_1+\lambda_2}+\frac{bc\alpha}{\lambda_1-\lambda_2}=0,$$

the coefficient of $x_1^{\alpha_1}x_2^{\alpha_2}x_3^{l+3-\alpha_1-\alpha_1}$ in $\mathscr{H}[\Phi_{l+2}]$

$$\frac{bc\alpha(\alpha+1)}{(-\lambda_1+\lambda_2)^2}\big(a(\alpha-1)+d(\alpha+1)\big)+\frac{bc\alpha(\alpha+1)}{(\lambda_1-\lambda_2)^2}\big(a(\alpha+1)+d(\alpha-1)\big)=0,$$

the coefficient of $x_1^{\alpha_1}x_2^{\alpha_2}x_3^{l+4-\alpha_1-\alpha_1}$ in $\mathscr{H}[\Phi_{l+3}]$

$$\frac{bc\alpha(\alpha+1)}{(-\lambda_1+\lambda_2)^3}\big(a(\alpha-1)+d(\alpha+1)\big)^2+\frac{bc\alpha(\alpha+1)}{(\lambda_1-\lambda_2)^3}\big(a(\alpha+1)+d(\alpha-1)\big)^2=0.$$

This leads us to conjecture that there is no term $x_1^{\alpha_1}x_2^{\alpha_2}x_3^{k-\alpha_1-\alpha_1}$ in $\mathscr{H}[\Phi_k], k=l,l+1,\cdots$, which means (4.6) are formal solved and (4.1) has a time-dependent formal first integral.

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Appendix A. Kovacic's Algorithm

Let r be the complex rational function that defines the second order linear differential equation $\xi''=r\xi$. Let $\mathscr{P}$ be the set containing ∞ and the poles of r.

A1. Kovacic's Algorithm of Case 1

Step 1. For each $c\in\mathscr{P}$ we define $[\sqrt{r}]_c,\ \alpha_c^{\pm}$ as follows:

(a) If c is a pole of order 1, then $[\sqrt{r}]_c=0,\ \alpha_c^{+}=\alpha_c^{-}=1$.

(b) If c is a pole of order 2, then $[\sqrt{r}]_c=0,\ \alpha_c^{\pm}=\frac{1}{2}\pm\frac{1}{2}\sqrt{1+4b}$, where b is coefficients of $1/(x-c)^2$ in the partial fraction expansion for r.

(c) If c is a pole of order $2\nu\geq 4$, then $[\sqrt{r}]_c=\frac{a}{(x-c)^{\nu}}+\cdots+\frac{d}{(x-c)^2}$ of negative order part of the Laurent series expansion of $\sqrt{r}$ at c, $\alpha_c^{\pm}=\frac{1}{2}(\pm\frac{b}{a}+\nu)$, where b is the coefficient of $1/(x-c)^{\nu+1}$ in r minus the coefficient of $1/(x-c)^{\nu+1}$ in $[\sqrt{r}]_c^2$.

(d) If the order of r at ∞ is >2, then $[\sqrt{r}]_\infty=0,\ \alpha_\infty^{+}=0,\alpha_\infty^{-}=1$.

(e) If the order of r at ∞ is 2, then $[\sqrt{r}]_\infty=0,\ \alpha_\infty^{\pm}=\frac{1}{2}\pm\frac{1}{2}\sqrt{1+4b}$, where b is coefficients of $1/x^2$ in the Laurent series expansion of r at ∞.

(f) If ∞ is a pole of order $-2\nu \leq 0$, then $[\sqrt{r}]_\infty = ax^\nu + \cdots + d$ of positive order part of the Laurent series expansion of $\sqrt{r}$ at ∞, $\alpha_\infty^\pm = \frac{1}{2}(\pm\frac{b}{a} - \nu)$, where b is the coefficient of $x^{\nu-1}$ in r minus the coefficient of $x^{\nu-1}$ in $[\sqrt{r}]_\infty^2$.

Step 2. Let $d = \sum_{c\in\mathscr{P}} t(c)\alpha_c^{s(c)}$, where $s(c) \in \{+,-\}$ for any $c \in \mathscr{P}$, $t(\infty) = +1$ and $t(c) = -1$ for any $c \in \mathscr{P}\setminus\{\infty\}$. If d is a non-negative integer, then let $w = \sum_{c\in\mathscr{P}}(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}) + s(\infty)[\sqrt{\infty}]$, otherwise, the family is discarded. If no families remain under consideration, case 1 of Theorem 5. cannot happen.

Step 3. For each family retained from step 2, we search for a monic polynomial P of degree d such that the equation $P'' + 2wP' + (w' + w^2 - r)P = 0$ holds. If such a polynomial exists, then $\xi = Pe^{\int w}$ is a solution of $\xi'' = r\xi$. If no such polynomial is found for any family retained from Step 2, case 1 of Theorem 2.3 cannot happen.

A2. Kovacic's Algorithm of Case 2

Step 1. For each $c \in \mathscr{P}$ we define E_c as follows:

(a) If c is a pole of order 1, then $E_c = \{4\}$.

(b) If c is a pole of order 2, then $E_c = \{2 + k\sqrt{1+4b}, k = 0, \pm 2\} \cap \mathbb{Z}$, where b is coefficients of $1/(x-c)^2$ in the partial fraction expansion for r.

(c) If c is a pole of order $\nu > 2$, then $E_c = \{\nu\}$.

(d) If the order of r at ∞ is > 2, then $E_\infty = \{0, 2, 4\}$.

(e) If the order of r at ∞ is 2, then $E_\infty = \{2 + k\sqrt{1+4b}, k = 0, \pm 2\} \cap \mathbb{Z}$, where b is coefficients of $1/x^2$ in the Laurent series expansion of r at ∞.

(f) If ∞ is a pole of order $\nu < 2$, then$E_\infty = \{\nu\}$.

Step 2. Let $d = \frac{1}{2}\sum_{c\in\mathscr{P}} t(c)e_c$, where $e_c \in E_c$ for any $c \in \mathscr{P}$, $t(\infty) = +1$ and $t(c) = -1$ for any $c \in \mathscr{P}\setminus\{\infty\}$. If d is a non-negative integer, then let $\theta = \frac{1}{2}\sum_{c\in\mathscr{P}} \frac{e_c}{x-c}$, otherwise, the family is discarded. If no families remain under consideration, case 2 of Theorem 5. cannot happen.

Step 3. For each family retained from step 2, we search for a monic polynomial P of degree d such that the equation $P''' + 3\theta P'' + (3\theta^2 + 3\theta' - 4r)P' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')P = 0$ holds. If such a polynomial exists, let $\phi = \theta + \frac{P'}{P}$ and let w be a solution of the equation $w^2 + \phi w + (\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r) = 0$, then $\xi = e^{\int w}$ is a solution of $\xi'' = r\xi$. If no such polynomial is found for any family retained from Step 2, case 2 of Theorem 2.3 cannot happen.

A3. Kovacic's Algorithm of Case 3

Step 1. For each $c \in \mathscr{P}$ we define E_c as follows:

(a) If c is a pole of order 1, then $E_c = \{12\}$.

(b) If c is a pole of order 2, then $E_c = \{6 + \frac{12k}{n}\sqrt{1+4b}, k = 0, \pm 1, \pm 2, \cdots, \pm\frac{n}{2}\} \cap \mathbb{Z}$, where b is coefficients of $1/(x-c)^2$ in the partial fraction expansion for r.

(c) $E_\infty = \{6 + \frac{12k}{n}\sqrt{1+4b}, k = 0, \pm 1, \pm 2, \cdots, \pm\frac{n}{2}\} \cap \mathbb{Z}$, where b is coefficients of $1/x^2$ in the Laurent series expansion of r at ∞.

Step 2. Let $d = \frac{n}{12}\sum_{c\in\mathscr{P}} t(c)e_c$, where $e_c \in E_c$ for any $c \in \mathscr{P}$, $t(\infty) = +1$ and $t(c) = -1$ for any $c \in \mathscr{P}\setminus\{\infty\}$. If d is a non-negative integer, then let $\theta = \frac{n}{12}\sum_{c\in\mathscr{P}} \frac{e_c}{x-c}, S = \prod_{c\in\mathscr{P}\setminus\{\infty\}}(x-c)$, otherwise, the family is discarded. If no families remain under consideration, case 3 of Theorem 5. cannot happen.

Step 3. For each family retained from step 2, we search for a monic polynomial P of degree d such that the recursive equations

$$P_n = P,$$

$$P_{i-1} = -SP_i + ((n-i)S' - S\theta)P_i - (n-i)(i+1)S^2 rP_{i+1};(\ \ i = n, n-1, \cdots, 0\ \ ),$$

with $P_{-1} \equiv 0$ holds. If such a polynomial exists, let w be a solution of the equation $\sum_{i=0}^{n} \frac{S^i P_i}{(n-i)!} w^i = 0$, then $\xi = e^{\int w}$ is a solution of $\xi'' = r\xi$. If no such polynomial is found for any family retained from Step 2, case 3 of Theorem 2.3 cannot happen.

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