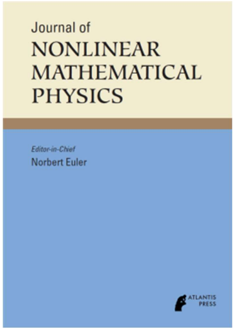

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

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To cite this article: José Ferńandez Núñez, Wifredo García Fuertes, Askold M. Perelomov (2018) Generating functions for characters and weight multiplicities of irreducible $\mathfrak{sl}(4)$-modules, Journal of Nonlinear Mathematical Physics 25:4, 618–632, DOI: https://doi.org/10.1080/14029251.2018.1503436

To link to this article: https://doi.org/10.1080/14029251.2018.1503436

Published online: 04 January 2021

Journal of Nonlinear Mathematical Physics, Vol. 25, No. 4 (2018) 618–632

Generating functions for characters and weight multiplicities of irreducible $\mathfrak{sl}(4)$-modules

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Received 22 January 2018

Accepted 14 May 2018

Generating functions for the characters of the irreducible representations of simple Lie algebras are rational functions where both the numerator and denominator can be expressed as polynomials in the characters corresponding to the fundamental weights. They encode much information on the representation theory of the algebra, but their explicit expressions are in general very complicated. In fact, it seems that rank three is the highest rank tractable. In this paper, we use a method based on the quantum Calogero-Sutherland model to compute the full generating function for the characters of irreducible modules over the complex Lie algebra $\mathfrak{sl}(4)$, and exploit this result to obtain also generating functions giving the multiplicities of some low order weights in all representations. We have applied the same method to compute the generating function for the characters of the modules the other rank three simple Lie algebras, but in these cases the full expressions are very long and appear only in the arXiv version of the paper (arXiv:1705.03711 [math-ph]). Nevertheless, when the generating functions are limited to some particular subsets of characters, the results are quite simple and we present them here.

Keywords: Calogero-Sutherland models; Generating functions; Lie algebra representation theory.

2000 Mathematics Subject Classification: 81R12, 35Q40, 17B10, 17B80, 05A15

1. Introduction

Generating functions for characters are very useful tools for the study of representations of Lie algebras. They were introduced and shown to be rational functions in [21], and since then different procedures for computing these objects and extracting practical information from them have been developed, see for instance [17] and references therein. We have recently shown how the theory of the quantum integrable Calogero-Sutherland models can be used to obtain the generating functions [5] in an efficient manner, and we have obtained, proceeding on this basis, several results regarding characters and weight multiplicities of the representations of the rank two simple Lie algebras [6,7]. Compared to other schemes, the approach based on the Calogero-Sutherland model is advantageous in that cumbersome combinatorial recipes involving the Weyl group are bypassed. This makes the method a rather convenient one to be applied to any simple Lie algebra.

Nevertheless, an unavoidable fact which is independent of the method employed is that, as the rank increases, the final result for the full generating function of characters becomes quickly exceedingly complicated. Thus, in general, for higher rank algebras only the generating functions for some restricted sets of characters, typically with only one or two non-vanishing Dynkin indices, are tractable without falling into an excessive clumsiness. Apart from rank two algebras, the only exception seems to be the case of the complex Lie algebra $\mathfrak{sl}(4)$, in which the Weyl orbits of the fundamental weights, the Weyl formula for dimensions, and the Calogero-Sutherland Hamiltonian, are still quite simple, these circumstances suggesting that the whole generating function for characters is not too unwieldy to be computed. As far as we know, this generating function has not been explicitly written down in previous works. It seems thus worth to use the method of [5] to obtain it and to explore some other results which can be deduced from it.

Results of this type are interesting in themselves, but also due to the fact that $\mathfrak{sl}(4) \simeq \mathfrak{o}(6)$ is a Lie algebra with applications to important physical systems. Let us mention, among several others, the grand unified Pati-Salam model with gauge group $SU(2)_L \times SU(2)_R \times SU(4)$ [23], which has been much investigated owing to the fact that it fits well with the string or M-theory framework and it is also suitable to explain phenomenological issues like neutrino oscillations or baryogenesis [22]; the effective $SU(4)$ hadronic models which, in spite of the large breaking of $SU(4)$ flavor symmetry, can be used to study the phenomenology of charmed particles, see for instance [10]; the $SU(4)$-Kondo effect, due to the interplay between spin and orbital electronic degrees of freedom, which has recently aroused remarkable interest [11] because of its role in condensed matter settings such as quantum dots, carbon nanotubes or nanowires; or the AdS_5/CFT_4 string-gauge equivalence [15], in which $SU(4)$ is the R-symmetry in the supersymmetric quantum field theory side of the duality and $SO(6)$ is the symmetry of the effective IIB gauged supergravity on the string side.

2. The generating function for characters

This section is devoted to the computation of the generating function for the characters of the irreducible representations of $\mathfrak{sl}(4)$ by means of the approach developed in [5]. This approach needs a background on Calogero-Sutherland models [1,26], which is succinctly explained in Subsection 2.1. The computation of the generating function is done in Subsection 2.2.

2.1. *Review of the theory of quantum Calogero-Sutherland models*

Calogero-Sutherland models are a class of mechanical systems which enjoy the existence of a complete set of mutually commuting integrals of motion and are, therefore, integrable systems in the sense of Liouville. The first analysis of a system of this kind was performed by Calogero [1] who studied, from the quantum standpoint, the dynamics on the infinite line of a set of particles interacting pairwise by rational plus quadratic potentials, and found that the problem was exactly solvable. Soon afterwards, Sutherland [26] arrived to similar results for the quantum problem on the circle, this time with trigonometric interaction; and later Moser [16] proved, in terms of Lax pairs, that the classical counterparts of these models also enjoyed integrability. The identification of the general scope of these discoveries came with the work of Olshanetsky and Perelomov [18], who realized that it is possible to associate models of this kind to all the root systems of the simple Lie algebras, and that all these models are integrable, both in the classical and the quantum framework, for interactions of the type rational (or inverse-square), q^{-2}; rational+quadratic, $q^{-2}+\omega^2 q^2$; trigonometric, $\sin^{-2} q$; hyperbolic, $\sinh^{-2} q$; and the most general, given by the Weierstrass elliptic function $\mathscr{P}(q)$. Nowadays, there is a widespread interest in this kind of integrable systems, and many mathematical and physical applications for them have been found, see for instance [2].

Here, we limit ourselves to review the most salient features of the trigonometric model which are useful for our present purposes and refer the reader to [19] for a more detailed treatment. Let $\mathscr{A}$ be a complex simple Lie algebra of rank r. As is well known [20], the roots $\alpha_1, \alpha_2, \ldots, \alpha_r$ and fundamental weights $\lambda_1, \lambda_2, \ldots, \lambda_r$ of $\mathscr{A}$ can be conveniently represented by elements of a vector space V whose dimension is r or $r+1$ depending on the algebra. The Hamiltonian of Calogero-Sutherland model associated to $\mathscr{A}$ has the form

$$H = \frac{1}{2}p^2 + U(q)$$

where the coordinates q and momenta p are elements of V. The potential term is

$$U(q) = \sum_{\alpha\in\mathscr{R}^+} \kappa_\alpha(\kappa_\alpha - 1)\sin^{-2}\langle\alpha, q\rangle,$$

where $\mathscr{R}^+$ is the set of positive roots of $\mathscr{A}$ and $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product on V. The constants κ_α must be chosen in such a way that the couplings $g_\alpha^2 = \kappa_\alpha(\kappa_\alpha - 1)$ are equal for roots of equal length. It turns out that the energy eigenfunctions of this Hamiltonian depend on r quantum numbers $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ and are of the form $\Psi_{\mathbf{m}}^\kappa = \Psi_0^\kappa \cdot \Phi_{\mathbf{m}}^\kappa$, where

$$\Psi_0^\kappa(q) = \prod_{\alpha\in\mathscr{R}^+} \sin^{\kappa_\alpha}\langle\alpha, q\rangle$$

is the wave function of the ground state and the $\Phi_{\mathbf{m}}^\kappa$ are solutions of the related Schrödinger equation

$$\Delta^\kappa \Phi_{\mathbf{m}}^\kappa(q) = \varepsilon(\mathbf{m};\kappa)\,\Phi_{\mathbf{m}}^\kappa(q)\,, \tag{2.1}$$

where Δ^κ is the linear differential operator

$$\Delta^\kappa = -\frac{1}{2}\sum_{j=1}^{l} \partial_{q_j}^2 - \sum_{\alpha\in\mathscr{R}^+} \langle\alpha,\alpha\rangle\,\kappa_\alpha \cot\langle\alpha, q\rangle\,\langle\alpha, \partial_q\rangle$$

and the eigenvalues are

$$\varepsilon(\mathbf{m};\kappa) = 2\langle\lambda + 2\rho(\kappa), \lambda\rangle \tag{2.2}$$

for $2\rho(\kappa) = \sum_{\alpha\in\mathscr{R}^+} \kappa_\alpha \alpha$ and λ the highest weight $\lambda = m_1\lambda_1 + m_2\lambda_2 + \cdots + m_r\lambda_r$ defined by $\mathbf{m}$.

The most relevant fact for us is that if we tune all coupling constants κ_α to one, the eigenfunctions of this Schrödinger operator Δ^κ are precisely the characters $\boldsymbol{\chi}_{\mathbf{m}}$ of the irreducible representations of the algebra [19]

$$\Phi_{\mathbf{m}}^1(q) = \boldsymbol{\chi}_{\mathbf{m}}(q) = \sum_w n_w e^{2i\langle\lambda_j, q\rangle}, \tag{2.3}$$

where the sum extends to all weights w entering in the representation and n_w is the multiplicity of the weight w. This comes about as follows. Although the potential vanishes for $\kappa_\alpha \to 1$, there is a remnant of the interaction in that, to take the limit consistently, we have to choose fermionic boundary conditions ensuring that the wave functions are zero when $\sin\langle\alpha,q\rangle = 0$ for any positive root. As a consequence, the wave function $\Psi^1_{\mathbf{m}}$ is given by a Weyl-alternating sum of free-particle exponentials which turns out to coincide exactly with the numerator of the Weyl character formula. The ground state wave function, on the other hand, can be rewritten as the denominator of the Weyl formula, and the $\Phi^1_{\mathbf{m}}$ are the characters of the irreducible modules of the Lie algebra thereby. (The particles are free also for $\kappa_\alpha = 0$, but in that case bosonic boundary conditions are appropriate and the $\Phi^0_{\mathbf{m}}$ are the monomial symmetric functions associated to the root system; the $\Phi^\kappa_{\mathbf{m}}$ for other values of the couplings are systems of orthogonal polynomials which interpolate between the monomial symmetric functions and the characters.)

Thus, we can obtain the characters by solving the second order differential equation

$$\Delta^1 \boldsymbol{\chi}_{\mathbf{m}}(q) = \varepsilon(\mathbf{m};1)\boldsymbol{\chi}_{\mathbf{m}}(q). \tag{2.4}$$

Furthermore, if we change variables and describe the dynamical system by means of the characters $z_k = \boldsymbol{\chi}_{\lambda_k}$ of the fundamental representations R_{λ_k}, $k = 1,2,\dots,r$, the differential operator Δ^1 takes the form

$$\Delta^1_z = \sum_{j,k=1}^{r} a_{jk}(z)\partial_{z_j}\partial_{z_k} + \sum_{j=1}^{r} b_j(z)\partial_{z_j}, \tag{2.5}$$

with $a_{jk}(z)$ and $b_j(z)$ polynomials in the z_k with integer coefficients, and the Schrödinger equation can be solved by iterative methods [24]. This operator can be given an explicit form taking into account that:

- $b_j(z) = \Delta^1_z z_j = \varepsilon(0,\dots,1^{(j},\dots,0;1)z_j$, and
- $\Delta^1_z(z_j z_k) = 2a_{jk}(z) + b_j(z)z_k + b_k(z)z_j$,

while $z_j z_k$ is the character of the tensor product $R_{\lambda_j} \otimes R_{\lambda_k}$. Hence, knowing all the quadratic Clebsch-Gordan series of the algebra we will be able to determine the $a_{jk}(z)$ coefficients. For more explicit details, see for instance [4] and references therein. Once we know the definite expression for the operator Δ^1_z, it is possible to compute the characters $\boldsymbol{\chi}_{\mathbf{m}}$ as polynomials in the z-variables by solving the Schrödinger equation (2.1) in a recursive way.

2.2. *Computation of the generating function for* $\mathfrak{sl}(4)$

For the particular case of $\mathfrak{sl}(4)$, the model can be interpreted as describing the quantum dynamics of a system of four particles moving on a circle. The one-dimensional coordinates are assembled into the four-tuple $q = (q_1,q_2,q_3,q_4)$ with the constraint that the center of mass is fixed at the origin, i.e., $q_1+q_2+q_3+q_4 = 0$. The particles interact through a pairwise potential of trigonometric form whose strength is governed by a single coupling constant κ. The eigenfunctions $\Phi^\kappa_{\mathbf{m}}$ and the eigenvalues $\varepsilon(\mathbf{m};\kappa)$ are indexed by the 3-tuples of non-negative integers $\mathbf{m} = (m_1,m_2,m_3)$ –the quantum numbers,– and $m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3$ are the highest weights of the irreducible representations of the algebra, with $\lambda_1,\lambda_2,\lambda_3$ being the fundamental weights. The set of independent Weyl-invariant variables z_1,z_2,z_3, namely the characters of the three fundamental representations R_{λ_k} of $\mathfrak{sl}(4)$, are related to the q-variables by

$$\begin{aligned} z_1 &= \boldsymbol{\chi}_{1,0,0} = x_1 + \frac{x_2}{x_1} + \frac{x_3}{x_2} + \frac{1}{x_3}\,, \\ z_2 &= \boldsymbol{\chi}_{0,1,0} = x_2 + \frac{x_3}{x_1} + \frac{x_2}{x_1 x_3} + \frac{x_1 x_3}{x_2} + \frac{x_1}{x_3} + \frac{1}{x_2}\,, \\ z_3 &= \boldsymbol{\chi}_{0,0,1} = x_3 + \frac{x_2}{x_3} + \frac{x_1}{x_2} + \frac{1}{x_1}\,, \end{aligned} \tag{2.6}$$

where $x_j = e^{2\mathrm{i}\langle \lambda_j, q\rangle}$, the fundamental weights λ_j of $\mathfrak{sl}(4)$ given as four-tuples in the standard way. Higher order characters are polynomials in the z-variables with integer coefficients. The change of variables from the coordinates on the circle to the fundamental characters, see [9], leads to the following form Δ_z^1 of the Hamiltonian for coupling $\kappa = 1$ in terms of the z-variables:

$$\begin{aligned} \Delta_z^1 = \frac{1}{2}\big[&(3z_1^2 - 8z_2)\partial_{z_1}^2 + (4z_2^2 - 8z_1z_3 - 16)\partial_{z_2}^2 + (3z_3^2 - 8z_2)\partial_{z_3}^2 + (4z_1z_2 - 24z_3)\partial_{z_1}\partial_{z_2} \\ &+ (2z_1z_3 - 32)\partial_{z_1}\partial_{z_3} + (4z_2z_3 - 24z_1)\partial_{z_2}\partial_{z_3} + 15z_1\partial_{z_1} + 20z_2\partial_{z_2} + 15z_3\partial_{z_3}\big]; \end{aligned} \tag{2.7}$$

the eigenvalues (2.2), on the other hand, are

$$\varepsilon(\mathbf{m};1) = \frac{1}{2}(3m_1^2 + 4m_2^2 + 3m_3^2 + 4m_1m_2 + 2m_1m_3 + 4m_2m_3 + 12m_1 + 16m_2 + 12m_3). \tag{2.8}$$

In this setup, as explained in [5], the generating function for characters

$$G(t_1,t_2,t_3;z_1,z_2,z_3) = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty} t_1^{m_1}t_2^{m_2}t_3^{m_3}\boldsymbol{\chi}_{m_1,m_2,m_3}(z_1,z_2,z_3) \tag{2.9}$$

is a rational function

$$G(t_1,t_2,t_3;z_1,z_2,z_3) = \frac{N(t_1,t_2,t_3;z_1,z_2,z_3)}{D(t_1,t_2,t_3;z_1,z_2,z_3)} \tag{2.10}$$

that can be obtained by applying (2.4) to the generating function, which yields the differential equation

$$(\Delta_t - \Delta_z^1)G(t_k;z_k) = 0, \tag{2.11}$$

where the differential operator Δ_t is what $\varepsilon(\mathbf{m};1)$ turns into after making the substitution $m_i \to t_i\partial_{t_i}$ in (2.8). The computation goes through the following four steps:

Step (i): The denominator of the generating function is

$$D(t_1,t_2,t_3;z_1,z_2,z_3) = D_1 \times D_2 \times D_3\,, \tag{2.12}$$

where $D_i = \prod_j (1 - t_i x_1^{n_{j1}} x_2^{n_{j2}} x_3^{n_{j3}})$ with the product extended to all the weights $\sum_{k=1}^{3} n_{jk}\lambda_k$ entering in the Weyl orbit of the fundamental representation R_{λ_i}. Looking at (2.6) and bearing in mind that the fundamental representations of $\mathfrak{sl}(4)$ contain one single Weyl orbit, the result follows. After changing variables back from the x_j to the z_k, it can be written as

$$\begin{aligned} D_1 &= 1 - t_1 z_1 + t_1^2 z_2 - t_1^3 z_3 + t_1^4 \,, \\ D_2 &= 1 + t_2^6 - (t_2 + t_2^5) z_2 + (t_2^2 + t_2^4)(z_1 z_3 - 1) + t_2^3 (2 z_2 - z_1^2 - z_3^2) \,, \\ D_3 &= 1 - t_3 z_3 + t_3^2 z_2 - t_3^3 z_1 + t_3^4 \,. \end{aligned} \tag{2.13}$$

Step (ii): The Weyl formula for dimensions gives for the representation R_λ, where $\lambda = \sum_i m_i \lambda_i$,

$$\dim R_\lambda = \frac{1}{12}(m_1+1)(m_2+1)(m_3+1)(m_1+m_2+2)(m_2+m_3+2)(m_1+m_2+m_3+3). \tag{2.14}$$

To obtain the generating function for dimensions $E(t_1,t_2,t_3)$ it suffices to perform the change $m_i \to t_i \partial_{t_i}$ in this formula and to apply the resulting differential operator to $\prod_{i=1}^3 (1-t_i)^{-1}$. One finds in this way that

$$E(t_1,t_2,t_3) = \frac{P(t_1,t_2,t_3)}{(1-t_1)^4 (1-t_2)^6 (1-t_3)^4} \,, \tag{2.15}$$

where

$$\begin{aligned} P(t_1,t_2,t_3) = {} & 1 - 4t_1t_2 - 4t_2t_3 - t_1t_3 - t_2^2 + t_1^2 t_2 + t_2 t_3^2 + 4t_1 t_2^2 + 4t_2^2 t_3 + 6t_1t_2t_3 - t_1^2 t_2^3 \\ & - t_2^3 t_3^2 - 4t_1^2 t_2^2 t_3 - 4t_1 t_2^2 t_3^2 - 6t_1 t_2^3 t_3 + t_1 t_2^4 t_3 + t_1^2 t_2^2 t_3^2 + 4t_1 t_2^3 t_3^2 + 4t_1^2 t_2^3 t_3 - t_1^2 t_2^4 t_3^2 \,. \end{aligned}$$

A further simplification is possible, but we have written $E(t_1,t_2,t_3)$ in such a way that the denominator comes from the substitution in (2.12) of the fundamental characters z_1, z_2 and z_3 by their dimensions.

Step (iii): We will compute the numerator $N(t_1,t_2,t_3;z_1,z_2,z_3)$ of the generating function of characters (2.10) by tentatively assuming that it contains only the terms appearing in the numerator $P(t_1,t_2,t_3)$ of (2.15), now with coefficients depending of the z-variables. Under such hypothesis, we can expand N/D as a series in the t-variables and compare with the right-hand side of (2.9), so that we will be able to fix the coefficients in $N(t_1,t_2,t_3;z_1,z_2,z_3)$ provided that the expressions of the characters of some $\mathfrak{sl}(4)$-modules with small values of m_i (in fact, $m_i \leq 4$ for the case at hand) are known. To obtain these is not a difficult task: as we have said, they are polynomials in the z-variables and, given the simple structure of the Hamiltonian Δ_z^1 (2.7), they can be computed by recursively solving the eigenvalue equation (2.1)[a]. Thus we obtain

$$\begin{aligned} N(t_1,t_2,t_3;z_1,z_2,z_3) = {} & 1 - z_3 t_1 t_2 - z_1 t_2 t_3 - t_1 t_3 - t_2^2 + t_1^2 t_2 + t_2 t_3^2 + z_1 t_1 t_2^2 \\ & + z_3 t_2^2 t_3 + z_2 t_1 t_2 t_3 - t_1^2 t_2^3 - t_2^3 t_3^2 - z_1 t_1^2 t_2^2 t_3 - z_3 t_1 t_2^2 t_3^2 \\ & - z_2 t_1 t_2^3 t_3 + t_1 t_2^4 t_3 + t_1^2 t_2^2 t_3^2 + z_1 t_1 t_2^3 t_3^2 + z_3 t_1^2 t_2^3 t_3 - t_1^2 t_2^4 t_3^2 \,. \end{aligned} \tag{2.16}$$

Step (iv): We need to be sure that the conjecture to limit the number of unknown coefficients in step (iii) is correct. For this purpose, we have to verify that $G = N/D$ (2.10) does indeed satisfy the differential equation (2.11). This is a matter of directly plugging (2.10) into (2.11) and doing the derivatives. In this way, one can check that (2.11) is fulfilled. Thus (2.10) with (2.12), (2.13), and (2.16) is the correct generating function for characters of the irreducible modules of $\mathfrak{sl}(4)$.

[a]To compute the characters, or to check other results of the paper, see the Mathematica notebooks attached as ancillary files to the preprint arXiv:1705.03711.

3. Generating functions for weight multiplicities

Once we have the generating function for characters, it is possible to use it to obtain some other results. Let us consider, in particular, generating functions of the form

$$A_{n_1,n_2,n_3}(t_1,t_2,t_3) = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty} \mu_{m_1,m_2,m_3}(n_1,n_2,n_3)\, t_1^{m_1} t_2^{m_2} t_3^{m_3}\,, \tag{3.1}$$

where $\mu_{m_1,m_2,m_3}(n_1,n_2,n_3)$ is the multiplicity of the weight $n_1\lambda_1+n_2\lambda_2+n_3\lambda_3$ in the representation $R_{m_1\lambda_1+m_2\lambda_2+m_3\lambda_3}$ of $\mathfrak{sl}(4)$. The way in which these generating functions can be computed is described in [6,7], see also [3]: after expressing $G(t_1,t_2,t_3;z_1,z_2,z_3)$ in the x-variables by means of (2.6), they are given by the triple integral

$$A_{n_1,n_2,n_3}(t_1,t_2,t_3) = \frac{1}{(2\pi i)^3}\oint dx_3 \oint dx_1 \oint dx_2 \frac{G(t_1,t_2,t_3;x_1,x_2,x_3)}{x_1^{1+n_1} x_2^{1+n_2} x_3^{1+n_3}}\,, \tag{3.2}$$

where the integration contours are along the unit circles on the complex x_1, x_2 and x_3-planes. We will give here explicit expressions of $A_{n_1,n_2,n_3}(t_1,t_2,t_3)$ for $n_1+n_2+n_3 \leq 2$. In all these cases, the integrations (3.2), which are readily performed by means of the residue theorem, go along the same pattern. First, the integral in x_2 acquires contributions from poles arising at $x_2 = t_1$, $x_2 = t_1x_3$, $x_2 = t_3x_1$ and $x_2 = t_2x_1x_3$; then, for the integral in x_1 there are poles at $x_1 = t_3$, $x_1 = t_1^2x_3$, $x_1 = t_2x_3$ and $x_1 = t_2^2x_3^{-1}$; finally, the poles contributing to the last integral are located at $x_3 = t_1$, $x_3 = t_2t_3$, $x_3 = t_3^3$, $x_3 = t_2^{3/2}$ and $x_3 = t_2^{-3/2}$, except for the case $(n_1,n_2,n_3) = (2,0,0)$, where an additional pole at $x_3 = 0$ occurs. After the residues are evaluated, we find the final results for (3.1) in the form

$$A_{n_1,n_2,n_3}(t_1,t_2,t_3) = \frac{N_{n_1,n_2,n_3}(t_1,t_2,t_3)}{D_0(t_1,t_2,t_3)}\,, \tag{3.3}$$

where the denominator is in all cases

$$D_0(t_1,t_2,t_3) = (1-t_1^4)(1-t_3^4)(1-t_1t_3)^2(1-t_1^2t_2)(1-t_2t_3^2)(1-t_2^2)^2$$

and the numerators are given in the Appendix.

Explicit formulas for the weight multiplicities of the representations of simple Lie algebras are in general difficult to obtain and are known only in particular cases, see for instance the recent paper [14], devoted to study this subject for the so-called fundamental string representations of the classical algebras. Regarding this point, we should mention that a possible application of the generating function $A_{n_1,n_2,n_2}(t_1,t_2,t_3)$ is to obtain closed formulas for the multiplicities $\mu_{m_1,m_2,m_3}(n_1,n_2,n_3)$ by proceeding as done for rank two algebras in [7] (for an alternative approach, applied also to these algebras, see [25]). Nevertheless, in the case of $\mathfrak{sl}(4)$ the expressions given above are somewhat complicated and the procedure turns out to be considerably cumbersome, as are indeed other approaches: see for instance [27], or [13] for a recent computation of $\mu_{m_1,m_2,m_3}(0,0,0)$. Thus, we have studied the case of the real weights in the previous list by means of the Kostant multiplicity formula [12], see [8] for a pedagogic exposition,

$$\mu_{m_1,m_2,m_3}(n_1,n_2,n_3) = \sum_{w\in W}(-1)^w \mathscr{K}\left[w(\sum_{i=1}^{3}(m_i+1)\lambda_i) - \sum_{i=1}^{3}(n_i+1)\lambda_i\right],$$

where W is the Weyl group and $\mathscr{K}[\sum_{i=1}^{3} k_i\alpha_i] \equiv \mathscr{K}[k_1,k_2,k_3]$ is the Kostant partition function for $\mathfrak{sl}(4)$. This function gives the number of different ways in which a vector of the root lattice can be expressed as a linear combination of the positive roots with non-negative integer coefficients. The generating function for $\mathscr{Z}[k_1,k_2,k_3]$ is

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty} t_1^{k_1}t_2^{k_2}t_3^{k_3}\,\mathscr{Z}[k_1,k_2,k_3] = \frac{1}{(1-t_1)(1-t_2)(1-t_3)(1-t_1t_2t_3)(1-t_1t_2)(1-t_2t_3)}\,.$$

Thus, $\mathscr{Z}[k_1,k_2,k_3]$ is symmetric under interchange of k_1 and k_3, and its expression for $k_1 \le k_3$ can eventually found to be

$$\begin{aligned}
6\,\mathscr{Z}[k_1,k_2,k_3] &= (k_2+1)(k_2+2)(k_2+3)\,,\\
6\,\mathscr{Z}[k_1,k_2,k_3] &= (k_1+1)(k_1+2)(3k_2-2k_1+3)\,,\\
6\,\mathscr{Z}[k_1,k_2,k_3] &= (k_1+1)(k_1+2)(3k_3-k_1+3)\,,\\
6\,\mathscr{Z}[k_1,k_2,k_3] &= (k_2-k_3+1)(k_2-k_3+2)(2k_2-3k_1+k_3+3)\\
&\quad-(k_1-k_2+k_3)(3k_1+3k_3-12k_2+2k_1^2+2k_3^2-k_2^2-k_1k_2-2k_1k_3-k_2k_3-11)\,,
\end{aligned}$$

for, respectively, the cases

$$\begin{array}{ll}
\text{i)}\ \ k_1 \ge k_2; & \text{ii)}\ \ k_1 < k_2, \quad k_3 \ge k_2;\\
\text{iii)}\ \ k_3 < k_2, \quad k_1 \le k_2 - k_3; & \text{iv)}\ \ k_3 < k_2, \quad k_1 > k_2 - k_3.
\end{array}$$

With this, and taking advantage of the symmetry under $m_1 \leftrightarrow m_3$ to state the results only for the case $m_1 \le m_3$, one finds the following formulas:

- $\mu_{m_1,m_2,m_3}(0,0,0) \neq 0$ only if $m_3 - m_1 = 2m_2 + 4p$ with p integer, and in this case

$$\begin{aligned}
\mu_{m_1,m_2,m_3}(0,0,0) &= a(m_1,m_2,m_3) && \text{if } p \ge 0\,,\\
8\,\mu_{m_1,m_2,m_3}(0,0,0) &= (m_1+1)[b(m_1,m_2,m_3)+8] && \text{if } p < 0\,;
\end{aligned}$$

- $\mu_{m_1,m_2,m_3}(0,1,0) \neq 0$ only if $m_3 - m_1 = 2(m_2-1) + 4p$ with p integer, and in this case

$$\begin{aligned}
\mu_{m_1,m_2,m_3}(0,1,0) &= a(m_1,m_2,m_3) - 2(m_1+1)\delta_{p,0} && \text{if } p \ge 0\,,\\
8\,\mu_{m_1,m_2,m_3}(0,1,0) &= (m_1+1)[b(m_1,m_2,m_3)+4] && \text{if } p < 0\,;
\end{aligned}$$

- $\mu_{m_1,m_2,m_3}(1,0,1) \neq 0$ only if $m_3 - m_1 = 2m_2 + 4p$ with p integer, and in this case

$$\begin{aligned}
\mu_{m_1,m_2,m_3}(1,0,1) &= a(m_1,m_2,m_3) - (m_1+1)\delta_{p,0} && \text{if } p \ge 0\,,\\
8\,\mu_{m_1,m_2,m_3}(1,0,1) &= (m_1+1)b(m_1,m_2,m_3) && \text{if } p < 0\,;
\end{aligned}$$

- $\mu_{m_1,m_2,m_3}(0,2,0) \neq 0$ only if $m_3 - m_1 = 2(m_2-2) + 4p$ with p integer, and in this case

$$\begin{aligned}
\mu_{m_1,m_2,m_3}(0,2,0) &= a(m_1,m_2,m_3) - 2(m_1+1)(\delta_{p,1}+3\delta_{p,0})\\
&\quad+\delta_{p,1}\delta_{m_2,0}+\delta_{p,0}\delta_{m_2,2} && \text{if } p \ge 0\,,\\
8\,\mu_{m_1,m_2,m_3}(0,2,0) &= (m_1+1)[b(m_1,m_2,m_3)-8]+8\delta_{m_1,m_3} && \text{if } p < 0\,,
\end{aligned}$$

where

$$a(m_1,m_2,m_3) = \frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2)\,,$$
$$b(m_1,m_2,m_3) = 4\left[(m_2+1)(m_3+1)-1\right]-(m_1-m_3)^2.$$

The derivation of these expressions from the Kostant multiplicity formula is a laborious process: the Weyl group of $\mathfrak{sl}(4)$ has order 24, and hence there are many different cases which must be separately considered and then assembled together. Thus, to give a detailed description of the proof of these results is pretty tedious. Nevertheless, once they are written down, the generating functions for multiplicities given above provide a practical way to check that they are correct. In each case, with the help of a program for symbolic computations, it is easy to expand the generating function as a Taylor series in t-variables up to some high order and to subtract from this expansion the corresponding series built with the $\mu_{m_1,m_2,m_3}(n_1,n_2,n_3)$ coefficients. One then finds that the difference is zero, as it should be. This application illustrates one the benefits of working out explicit formulas like (2.10) or (3.3): despite their awkward appearance, they are considerably useful tools to check at once a number of other results concerning the representations of the algebra.

4. Generating function for the characters of real representations

The generating function obtained in Section 2 collects together the characters of all irreducible representations of $\mathfrak{sl}(4)$. It can be of interest to have also generating functions for particular subsets of characters. The simplest examples are the generating functions for characters with only one or two non-vanishing Dynkin indices, which follow directly from (2.10) when the appropriate t-variables are taken to vanish. A more interesting distinction is between the characters of complex and real representations, the latter being those with highest weight symmetric under interchange of z_1 and z_3, i.e., of the form $\boldsymbol{\chi}_{m_1,m_2,m_1}$. The general four-step procedure used in Section 2 can be also applied to construct the generating function for characters of this type,

$$G_{\mathbf{R}}(t_1,t_2;z_1,z_2,z_3) = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} t_1^{m_1}t_2^{m_2}\boldsymbol{\chi}_{m_1,m_2,m_1}(z_1,z_2,z_3),$$

as follows:

Step (i): Assuming that the generating function $G_{\mathbf{R}}$ is rational, the denominator is now

$$D_{\mathbf{R}}(t_1,t_2;z_1,z_2,z_3) = D_{13}\times D_2\,, \tag{4.1}$$

where the weights entering in D_{13} are those in the Weyl orbit $R_{\lambda_1+\lambda_3}$. These can be read from the corresponding monomial symmetric function

$$M_{\lambda_1+\lambda_3} = x_1x_3+\frac{x_1x_2}{x_3}+\frac{x_1^2}{x_2}+\frac{x_2x_3}{x_1}+\frac{x_2^2}{x_1x_3}+\frac{x_3^2}{x_2}+\text{c.c.}\,,$$

and lead to the expression

$$\begin{aligned} D_{13} = {} & 1+t_1^{12}-(t_1+t_1^{11})(z_1z_3-4)+(t_1^2+t_1^{10})(z_1^2z_2-2z_2^2-4z_1z_3+z_2z_3^2+10)\\ & -(t_1^3+t_1^9)(z_1^4-7z_1^2z_2+8z_2^2+13z_1z_3+z_1z_2^2z_3-7z_2z_3^2+z_3^4-20)\\ & +(t_1^4+t_1^8)d_{48}+(t_1^5+t_1^7)d_{57}+t_1^6d_6\,, \end{aligned}$$

where

$$\begin{aligned} d_{48} &= z_2^4 - 3z_1^4 + 18z_1^2z_2 - 16z_2^2 - 24z_1z_3 + z_1^3z_2z_3 - 8z_1z_2^2z_3 - z_1^2z_3^2 + 18z_2z_3^2 + z_1z_2z_3^3 \\ &- 3z_3^4 + 31\,, \\ d_{57} &= 4z_2^4 - 6z_1^4 + 29z_1^2z_2 - 24z_2^2 - z_1^2z_2^3 - 34z_1z_3 + 5z_1^3z_2z_3 - 19z_1z_2^2z_3 - 2z_1^2z_3^2 + 29z_2z_3^2 \\ &- z_2^3z_3^2 - z_1^3z_3^3 + 5z_1z_2z_3^3 - 6z_3^4 + 40\,, \\ d_6 &= z_1^2z_2^2z_3^2 - 7z_1^4 + 34z_1^2z_2 - 28z_2^2 - 2z_1^2z_2^3 + 6z_2^4 - 40z_1z_3 + 6z_1^3z_2z_3 - 24z_1z_2^2z_3 + 34z_2z_3^2 \\ &- 2z_2^3z_3^2 - 2z_1^3z_3^3 + 6z_1z_2z_3^3 - 7z_3^4 + 44\,. \end{aligned}$$

Step (ii): For real representations, the dimensions (2.14) are

$$\dim R_{m_1\lambda_1+m_2\lambda_2+m_1\lambda_3} = \frac{1}{12}(m_1+1)^2(m_2+1)(m_1+m_2+2)^2(2m_1+m_2+3)\,.$$

Given this formula, we can proceed as in Section 2 to shape the generating function for dimensions. It turns out to be

$$E_{\mathbf{R}}(t_1,t_2) = \frac{(1-t_1)^6(1-t_2)P_{\mathbf{R}}(t_1,t_2)}{(1-t_1)^{12}(1-t_2)^6},$$

where

$$\begin{aligned} P_{\mathbf{R}}(t_1,t_2) &= 1 + 9t_1 + 9t_1^2 + t_1^3 + t_2 - 17t_1t_2 - 39t_1^2t_2 - 5t_1^3t_2 + 5t_1t_2^2 \\ &+ 39t_1^2t_2^2 + 17t_1^3t_2^2 - t_1^4t_2^2 - t_1t_2^3 - 9t_1^2t_2^3 - 9t_1^3t_2^3 - t_1^4t_2^3\,. \end{aligned}$$

Step (iii): We next compute the numerator $N_{\mathbf{R}}(t_1,t_2;z_1,z_2,z_3)$ of $G_{\mathbf{R}}$ by provisionally assuming that the only non-vanishing coefficients correspond to the monomials appearing in the numerator of $E_{\mathbf{R}}(t_1,t_2)$. After using the eigenvalue equation (2.1) to figure out the real characters needed, we get

$$\begin{aligned} N_{\mathbf{R}} &= 1 + t_1^9 - t_2^2 + t_1t_2^4 - t_1^{10}t_2^2 + t_1^{10}t_2^4 + 3(t_1 + t_1^8 + t_1^2t_2^4 + t_1^9t_2^4) + n_1(t_1t_2 + t_1^9t_2^3) \\ &+ n_2(t_1t_2^2 + t_1^9t_2^2) - z_2(t_1^9t_2 + t_1t_2^3) - n_3(t_1^2 + t_1^7 + t_1^3t_2^4 + t_1^8t_2^4) + n_4(t_1^2t_2 + t_1^8t_2^3) \\ &+ n_5(t_1^2t_2^2 + t_1^8t_2^2) - 2z_2(t_1^8t_2 + t_1^2t_2^3) + n_6(t_1^3 + t_1^6 + t_1^4t_2^4 + t_1^7t_2^4) + n_7(t_1^3t_2 + t_1^7t_2^3) \\ &+ n_8(t_1^3t_2^2 + t_1^7t_2^2) + n_9(t_1^7t_2 + t_1^3t_2^3) + n_{10}(t_1^4 + t_1^5 + t_1^5t_2^4 + t_1^6t_2^4) + n_{11}(t_1^4t_2 + t_1^6t_2^3) \\ &+ n_{12}(t_1^4t_2^2 + t_1^6t_2^2) + n_{13}(t_1^6t_2 + t_1^4t_2^3) + n_{14}(t_1^5t_2 + t_1^5t_2^3) + n_{15}t_1^5t_2^2\,, \end{aligned}$$

where the coefficients are

$$
\begin{aligned}
n_1 &= z_2 - z_1^2 - z_3^2\,,\\
n_2 &= 2z_1z_3 - 4\,,\\
n_3 &= z_2^2 - 6\,,\\
n_4 &= 2z_1z_2z_3 - 3z_1^2 + 2z_2 - 3z_3^2\,,\\
n_5 &= z_2^2 - z_1^2z_2 + 6z_1z_3 - z_2z_3^2 - 9\,,\\
n_6 &= z_1^2z_2 - 3z_2^2 - 2z_1z_3 + z_2z_3^2 + 10\,,\\
n_7 &= 8z_1z_2z_3 - 5z_1^2 + 2z_2 - z_2^3 - z_1^3z_3 - 5z_3^2 - z_1z_3^3\,,\\
n_8 &= z_1^4 - 5z_1^2z_2 + 4z_2^2 + 14z_1z_3 - 5z_2z_3^2 + z_3^4 - 16\,,\\
n_9 &= z_2^3 - z_1^2 - 2z_2 - z_3^2\,,\\
n_{10} &= 2z_1^2z_2 - 4z_2^2 - 2z_1z_3 - z_1^2z_3^2 + 2z_2z_3^2 + 12\,,\\
n_{11} &= 10z_1z_2z_3 - 6z_1^2 + 2z_2 - 2z_2^3 - z_1^3z_3 - 6z_3^2 - z_1z_3^3\,,\\
n_{12} &= 2z_1z_2^2z_3 + 2z_1^4 - 10z_1^2z_2 + 7z_2^2 + 22z_1z_3 - z_1^2z_3^2 - 10z_2z_3^2 + 2z_3^4 - 22\,,\\
n_{13} &= 4z_1z_2z_3 - 3z_1^2 - 2z_2 - z_1^2z_2^2 + 2z_2^3 - 3z_3^2 - z_2^2z_3^2\,,\\
n_{14} &= 8z_1z_2z_3 - 5z_1^2 - z_1^2z_2^2 - z_1^3z_3 - 5z_3^2 + z_1^2z_2z_3^2 - z_2^2z_3^2 - z_1z_3^3\,,\\
n_{15} &= 4z_1z_2^2z_3 + 3z_1^4 - 12z_1^2z_2 + 8z_2^2 + 24z_1z_3 - z_1^3z_2z_3 - 12z_2z_3^2 - z_1z_2z_3^3 + 3z_3^4 - 24\,.
\end{aligned}
$$

Step (iv): There only remains to find out if

$$G_{\mathbf{R}}(t_1,t_2;z_1,z_2,z_3) = \frac{N_{\mathbf{R}}(t_1,t_2;z_1,z_2,z_3)}{D_{\mathbf{R}}(t_1,t_2;z_1,z_2,z_3)} \tag{4.2}$$

solves the differential equation

$$(\Delta_t^{\mathbf{R}} - \Delta_z^1)G_{\mathbf{R}}(t_1,t_2;z_1,z_2,z_3) = 0\,,$$

where the explicit form of $\Delta_t^{\mathbf{R}}$ is derived from (2.8) in the usual way:

$$\Delta_t^{\mathbf{R}} = 4t_1^2\partial_{t_1}^2 + 2t_2^2\partial_{t_2}^2 + 4t_1t_2\partial_{t_1}\partial_{t_2} + 16t_1\partial_{t_1} + 10t_2\partial_{t_2}.$$

The result of this checking is positive and we can thus conclude that (4.2) is the generating function we were seeking for.

5. Conclusions and outlook

The technique for computing generating functions for characters of irreducible modules over simple Lie algebras introduced in [5] has by now been used to obtain a variety of results concerning characters and weight multiplicities in the case of rank two algebras in [6] and [7] and to study the case of the rank three algebra $\mathfrak{sl}(4) \simeq \mathfrak{o}(6)$ in the present paper. These works have made obvious the versatility and usefulness of the method, which enabled us to present a number of results with potential applicability in mathematics and mathematical physics. It seems, however, that if we insist in computing the generating functions in full generality, the algebra considered in this paper is the highest rank one in which the formulas obtained through this approach are kept under a reasonable size.

Thus, for instance, we have computed the generating functions of irreducible characters also for the remaining algebras of rank three, $\mathfrak{o}(7)$ and $\mathfrak{sp}(6)$, but the results are exceedingly complicated, with respectively 311 and 315 terms in the numerator, and with coefficients that in many cases are long expressions in z-variables. Therefore, for these algebras it is better to limit the treatment to some particular sets of characters, and we present here only a few of the simplest results. In the standard notation in which α_3 is the root of unequal length, the generating function for the characters of the representations $R_{m_1\lambda_1}$ and $R_{m_3\lambda_3}$ of $\mathfrak{o}(7)$ are, respectively,

$$\frac{1+t_1}{1+t_1^6-(t_1+t_1^5)(z_1-1)+(t_1^2+t_1^4)(z_2-z_1+1)-t_1^3(z_3^2-2z_2-2)} \tag{5.1}$$

and

$$\frac{1-t_3^2}{1+t_3^8-(t_3+t_3^7)z_3+(t_3^2+t_3^6)(z_1+z_2)-(t_3^3+t_3^5)z_1z_3+t_3^4(z_1^2+z_3^2-2z_2-1)}\,. \tag{5.2}$$

The denominator of the generating function for the representations $R_{m_1\lambda_1+m_3\lambda_3}$ is the product of the denominators of (5.1) and (5.2) and the numerator is

$$1+t_1-t_3^2-t_1^3t_3^2+t_1^2t_3^4+t_1^3t_3^4+t_1(1+t_1)t_3^2z_1-t_1t_3(1+t_1t_3^2)z_3\,.$$

The analogous of (5.1) and (5.2) for $\mathfrak{sp}(6)$ are, respectively, given by

$$\frac{1}{1+t_1^6-(t_1+t_1^5)z_1+(t_1^2+t_1^4)(z_2+1)-t_1^3(z_1+z_3)} \tag{5.3}$$

and

$$\frac{1-t_3^4+t_3z_1-t_3^3z_1}{1+t_3^8+(t_3+t_3^7)(z_1-z_3)+(t_3^2+t_3^6)(z_2^2-2z_1z_3)+(t_3^3+t_3^5)(z_1^2-2z_2-1)(z_1-z_3)+t_3^4C}\,, \tag{5.4}$$

with $C=z_1^2+z_1^4-4z_1^2z_2+2z_2^2+2z_1z_3+z_3^2-2$. In the case of representations $R_{m_1\lambda_1+m_3\lambda_3}$ the denominator is the product of the denominators of (5.3) and (5.4) and the numerator is

$$\begin{aligned}1-t_1^3t_3-t_3^4+t_1^3t_3^5+t_3(1+t_1^2-t_1^3t_3-t_3^2+t_1t_3^3+t_1^3t_3^3)z_1\\ -t_1t_3(1+t_1t_3)(1+t_3^2)z_2-t_1t_3^2(1+t_1t_3)z_3+t_1t_3^2(t_1+t_3)z_1^2.\end{aligned}$$

Thus, the approach based in Calogero-Sutherland model can used to deal with other higher rank classical Lie algebras, or to the exceptional ones, but for these applications it is convenient to select characters of some special types, like those above, in order to keep the results under a manageable size.

Appendix

We give here the form of the numerators $N_{n_1,n_2,n_3}(t_1,t_2,t_3)$ of the generating functions for weight multiplicities for the cases $n_1+n_2+n_3\le 2$. The cases not explicitly written arise through the change

$t_1 \leftrightarrow t_3$ on the appropriate numerator.

$$\begin{aligned}
N_{0,0,0} = {} & 1 + 2t_1^2t_2 + t_1^4t_2^2 + t_1t_3 + t_1^3t_2t_3 + t_1t_2^2t_3 - 2t_1^5t_2^2t_3 - t_1^3t_2^3t_3 + t_1^2t_3^2 + 2t_2t_3^2 \\
& - 2t_1^4t_2t_3^2 - 4t_1^2t_2^2t_3^2 + t_1^6t_2^2t_3^2 - 2t_1^4t_2^3t_3^2 + t_1^3t_3^3 + t_1t_2t_3^3 - t_1^5t_2t_3^3 - 2t_1^3t_2^2t_3^3 - t_1t_2^3t_3^3 \\
& + t_1^5t_2^3t_3^3 + t_1^3t_2^4t_3^3 - 2t_1^2t_2t_3^4 + t_2^2t_3^4 - 4t_1^4t_2^2t_3^4 - 2t_1^2t_2^3t_3^4 + 2t_1^6t_2^3t_3^4 + t_1^4t_2^4t_3^4 - t_1^3t_2t_3^5 \\
& - 2t_1t_2^2t_3^5 + t_1^5t_2^2t_3^5 + t_1^3t_2^3t_3^5 + t_1^5t_2^4t_3^5 + t_1^2t_2^2t_3^6 + 2t_1^4t_2^3t_3^6 + t_1^6t_2^4t_3^6\,,
\end{aligned}$$

$$\begin{aligned}
N_{1,0,0} = {} & t_1 + 2t_1^3t_2 + t_1t_2^2 + t_1^2t_3 + 2t_2t_3 - t_1^4t_2t_3 - t_1^2t_2^2t_3 - t_1^4t_2^3t_3 + t_1^3t_3^2 + t_1t_2t_3^2 - t_1^5t_2t_3^2 \\
& - 3t_1^3t_2^2t_3^2 - t_1t_2^3t_3^2 - t_1^5t_2^3t_3^2 + t_3^3 + t_2^2t_3^3 - 3t_1^4t_2^2t_3^3 - 2t_1^2t_2^3t_3^3 + 2t_1^6t_2^3t_3^3 + t_1^4t_2^4t_3^3 \\
& - 2t_1^3t_2t_3^4 - 2t_1t_2^2t_3^4 - t_1^5t_2^2t_3^4 + t_1^5t_2^4t_3^4 - t_1^4t_2t_3^5 - 2t_1^2t_2^2t_3^5 + t_1^6t_2^2t_3^5 + t_1^4t_2^3t_3^5 + t_1^6t_2^4t_3^5 \\
& + t_1^3t_2^2t_3^6 + 2t_1^5t_2^3t_3^6 + t_1^3t_2^4t_3^6\,,
\end{aligned}$$

$$\begin{aligned}
N_{0,1,0} = {} & t_1^2 + t_2 + t_1^4t_2 + t_1^2t_2^2 + t_1^3t_3 + 2t_1t_2t_3 - t_1^5t_2t_3 - t_1^3t_2^2t_3 - t_1^5t_2^3t_3 + t_3^2 + t_2^2t_3^2 \\
& - 4t_1^4t_2^2t_3^2 - 3t_1^2t_2^3t_3^2 + t_1^6t_2^3t_3^2 + t_1t_3^3 - t_1t_2^2t_3^3 - t_1^5t_2^2t_3^3 + t_1^5t_2^4t_3^3 + t_2t_3^4 - 3t_1^4t_2t_3^4 \\
& - 4t_1^2t_2^2t_3^4 + t_1^6t_2^2t_3^4 + t_1^6t_2^4t_3^4 - t_1t_2t_3^5 - t_1^3t_2^2t_3^5 - t_1t_2^3t_3^5 + 2t_1^5t_2^3t_3^5 + t_1^3t_2^4t_3^5 + t_1^4t_2^2t_3^6 \\
& + t_1^2t_2^3t_3^6 + t_1^6t_2^3t_3^6 + t_1^4t_2^4t_3^6\,,
\end{aligned}$$

$$\begin{aligned}
N_{2,0,0} = {} & t_1^2 + 2t_1^4t_2 + t_1^2t_2^2 + t_2^3 - t_1^4t_2^3 + t_1^3t_3 + 2t_1t_2t_3 - t_1^5t_2t_3 - t_1^3t_2^2t_3 - t_1^5t_2^3t_3 + t_1^4t_3^2 \\
& + t_1^2t_2t_3^2 - t_1^6t_2t_3^2 + 3t_2^2t_3^2 - 6t_1^4t_2^2t_3^2 - 4t_1^2t_2^3t_3^2 + 2t_1^6t_2^3t_3^2 - t_2^4t_3^2 + t_1^4t_2^4t_3^2 + t_1t_3^3 - t_1t_2^2t_3^3 \\
& - t_1^5t_2^2t_3^3 + t_1^5t_2^4t_3^3 + 2t_2t_3^4 - 4t_1^4t_2t_3^4 - 4t_1^2t_2^2t_3^4 + t_1^6t_2^2t_3^4 - t_2^3t_3^4 + t_1^4t_2^3t_3^4 + t_1^6t_2^4t_3^4 - t_1t_2t_3^5 \\
& - t_1^3t_2^2t_3^5 - t_1t_2^3t_3^5 + 2t_1^5t_2^3t_3^5 + t_1^3t_2^4t_3^5 + t_3^6 - t_1^4t_3^6 - t_1^2t_2t_3^6 + t_1^6t_2t_3^6 - 2t_2^2t_3^6 + 3t_1^4t_2^2t_3^6 \\
& + 2t_1^2t_2^3t_3^6 + t_2^4t_3^6\,,
\end{aligned}$$

$$\begin{aligned}
N_{1,1,0} = {} & t_1^3 + t_1t_2 + t_1^5t_2 + t_1^3t_2^2 + t_1t_2^3 - t_1^5t_2^3 + t_1^4t_3 + 2t_1^2t_2t_3 - t_1^6t_2t_3 + 2t_2^2t_3 - 3t_1^4t_2^2t_3 \\
& - 2t_1^2t_2^3t_3 + t_1^6t_2^3t_3 + t_1t_3^2 - 3t_1^5t_2^2t_3^2 - 2t_1^3t_2^3t_3^2 - t_1t_2^4t_3^2 + t_1^5t_2^4t_3^2 + t_1^2t_3^3 + 2t_2t_3^3 - 2t_1^4t_2t_3^3 \\
& - 3t_1^2t_2^2t_3^3 + t_1^6t_2^2t_3^3 + t_1^6t_2^4t_3^3 - 2t_1^5t_2t_3^4 - 3t_1^3t_2^2t_3^4 - 2t_1t_2^3t_3^4 + 2t_1^5t_2^3t_3^4 + t_1^3t_2^4t_3^4 + t_3^5 - t_1^4t_3^5 \\
& - 2t_1^2t_2t_3^5 + t_1^6t_2t_3^5 - t_2^2t_3^5 + t_1^6t_2^3t_3^5 + t_1^4t_2^4t_3^5 - t_1t_2^2t_3^6 + 2t_1^5t_2^2t_3^6 + 2t_1^3t_2^3t_3^6 + t_1t_2^4t_3^6\,,
\end{aligned}$$

$$\begin{aligned}
N_{1,0,1} = {} & t_1^4 + 2t_1^2t_2 + t_2^2 + t_1t_3 + t_1^3t_2t_3 + t_1t_2^2t_3 - 2t_1^5t_2^2t_3 - t_1^3t_2^3t_3 + t_1^2t_3^2 + 2t_2t_3^2 - 2t_1^4t_2t_3^2 \\
& - 3t_1^2t_2^2t_3^2 - 2t_1^4t_2^3t_3^2 - t_1^2t_2^4t_3^2 + t_1^6t_2^4t_3^2 + t_1^3t_3^3 + t_1t_2t_3^3 - t_1^5t_2t_3^3 - 2t_1^3t_2^2t_3^3 - t_1t_2^3t_3^3 + t_1^5t_2^3t_3^3 \\
& + t_1^3t_2^4t_3^3 + t_3^4 - t_1^4t_3^4 - 2t_1^2t_2t_3^4 - 3t_1^4t_2^2t_3^4 - 2t_1^2t_2^3t_3^4 + 2t_1^6t_2^3t_3^4 + t_1^4t_2^4t_3^4 - t_1^3t_2t_3^5 - 2t_1t_2^2t_3^5 \\
& + t_1^5t_2^2t_3^5 + t_1^3t_2^3t_3^5 + t_1^5t_2^4t_3^5 + t_1^6t_2^2t_3^6 + 2t_1^4t_2^3t_3^6 + t_1^2t_2^4t_3^6\,,
\end{aligned}$$

$$
\begin{aligned}
N_{0,2,0} = {} & t_1^4 + t_1^2 t_2 + t_1^6 t_2 + t_2^2 + t_1^2 t_2^3 - t_1^6 t_2^3 + t_1^5 t_3 + 2t_1^3 t_2 t_3 - t_1^7 t_2 t_3 + 2t_1 t_2^2 t_3 - 3t_1^5 t_2^2 t_3 \\
& - 2t_1^3 t_2^3 t_3 + t_1^7 t_2^3 t_3 + t_1^2 t_3^2 + t_2 t_3^2 - t_1^4 t_2 t_3^2 - t_1^2 t_2^2 t_3^2 - 2t_1^6 t_2^2 t_3^2 + t_2^3 t_3^2 - 3t_1^4 t_2^3 t_3^2 - 3t_1^2 t_2^4 t_3^2 \\
& + 3t_1^6 t_2^4 t_3^2 + t_1^3 t_3^3 + 2t_1 t_2 t_3^3 - 2t_1^5 t_2 t_3^3 - 3t_1^3 t_2^2 t_3^3 + t_1^7 t_2^2 t_3^3 - 2t_1 t_2^3 t_3^3 + 2t_1^5 t_2^3 t_3^3 + 2t_1^3 t_2^4 t_3^3 \\
& - t_1^7 t_2^4 t_3^3 + t_3^4 - t_1^4 t_3^4 - t_1^2 t_2 t_3^4 - t_1^6 t_2 t_3^4 - 3t_1^4 t_2^2 t_3^4 - 3t_1^2 t_2^3 t_3^4 + 3t_1^6 t_2^3 t_3^4 + t_1^4 t_2^4 t_3^4 + t_1 t_3^5 \\
& - t_1^5 t_3^5 - 2t_1^3 t_2 t_3^5 + t_1^7 t_2 t_3^5 - 3t_1 t_2^2 t_3^5 + 2t_1^5 t_2^2 t_3^5 + 2t_1^3 t_2^3 t_3^5 - t_1^7 t_2^3 t_3^5 + t_1^5 t_2^4 t_3^5 + t_2 t_3^6 \\
& - t_1^4 t_2 t_3^6 - 2t_1^2 t_2^2 t_3^6 + 3t_1^6 t_2^2 t_3^6 - t_2^3 t_3^6 + 3t_1^4 t_2^3 t_3^6 + 3t_1^2 t_2^4 t_3^6 - 2t_1^6 t_2^4 t_3^6 - t_1 t_2 t_3^7 + t_1^5 t_2 t_3^7 \\
& + t_1^3 t_2^2 t_3^7 - t_1^7 t_2^2 t_3^7 + t_1 t_2^3 t_3^7 - t_1^5 t_2^3 t_3^7 - t_1^3 t_2^4 t_3^7 + t_1^7 t_2^4 t_3^7 \, ,
\end{aligned}
$$

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